題型05C: 子空間的計算

０５Ｃ**０１**【 交大80資工[1](a)(ｉ, ii, iv) 】

True (T) or False (F): (1 for each)

(a) Suppose A is row equivalent to B, i.e., $A \overset{R}{\sim} B$.

ｉ. NS(A)＝NS(B)

ii. CS(A)＝CS(B)

iv. RS(A)＝RS(B)

【解】(a) (ｉ) True. 請參閱CH5定理20

(a) (ii) False. 反例如下:

$$取A=\begin{bmatrix}1\\2\end{bmatrix}, B=\begin{bmatrix}1\\0\end{bmatrix}$$

則 $A \overset{R}{\sim} B$

$$但\ CS(A)=\left\{ t\begin{bmatrix}1\\2\end{bmatrix} \middle| t為純量 \right\}$$

$$CS(B)=\left\{ t\begin{bmatrix}1\\0\end{bmatrix} \middle| t為純量 \right\}$$ (綜線CH5定義16①)

可知 CS(A)≠CS(B)

(a) (iv) True. 請參閱CH5定理17

０５Ｃ**０２**【 中央84資工[1](f) 】

(f) If matrices A and B are row equivalent then their column spaces are the same, but their row spaces may be different.

【解】(f) False.

例如: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ 列等價於 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

前者的column space 為 $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}$,

後者的column space 為 $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$,

並不相等.

本題正好講反了. 應該是

... then their row spaces are the same, but their column spaces may be different.

０５Ｃ**03**【 交大84資料[5] 】

Show that the row space of matrix AB is a subspace of the row space of matrix B .

【參考章節】綜線CH5定理21a.

【解】 $x \in \mathrm{RSP}(AB)$

$\Longrightarrow \exists\, u$ 使得 $x = u(AB)$ (CH5定理17)

$\Longrightarrow \exists\, u$ 使得 $x = (uA)B$

$\Longrightarrow x \in \mathrm{RSP}B$ (CH5定理17)

[另證]

對任意的k,

"AB的第k列" 就是 "(A的第k列)B", (CH2定理7①)

它是B的列的線性組合. (CH2定理7②)

所以"AB的第k列" 在RSPB之內.

所以拿AB的列做出的線性組合也都在RSPB之內. (RSPB的封閉性)

所以 $\mathrm{RSP}(AB) \subseteq \mathrm{RSP}(B)$.

０５Ｃ**04** 【 中央82資電[4] 】

Prove that

(a) The column space of matrix AB is contained in the column space of matrix A (5%)

(b) The left nullspace of matrix AB contains the left null space of matrix A. (5%)

【解】(a) $x\in$[column space of AB]

$\Longrightarrow \exists\, u$ 使得$x=(AB)u$ (綜線CH5定理17)

$\Longrightarrow \exists\, u$ 使得$x=A(Bu)$

$\Longrightarrow \exists\, w$ 使得$x=Aw$

$\Longrightarrow x\in$ [column space of A] (綜線CH5定理17)

(b) $x \in$ [left null space of A]

$\Longrightarrow xA=o$ (綜線CH5定義19)

$\Longrightarrow (xA)B=o$

$\Longrightarrow x(AB)=o$

$\Longrightarrow x\in$[left null space of AB] (綜線CH5定義19)

０５Ｃ**05** 【 交大81資工[3](b) 】

Let

$$A=\begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

(b) (3%) Suppose space V is spanned by the first two columns of A, and space W is spanned by the last two columns of A. Find a basis for their intersection $V\cap W$.

【參考章節】 綜合線性代數CH5範例23

【解】(b) 由題意:

$$V=\left\{ x\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\},$$

$$W=\left\{ p\begin{bmatrix}1\\0\\1\end{bmatrix}+q\begin{bmatrix}1\\2\\2\end{bmatrix} \;\middle|\; p,q\in\mathbb{R} \right\}.$$

考慮W內的向量 $w=p\begin{bmatrix}1\\0\\1\end{bmatrix}+q\begin{bmatrix}1\\2\\2\end{bmatrix}$,

$w\in V$

$\Longleftrightarrow$ $\exists x, y$ 使得 $x\begin{bmatrix}1\\-1\\0\end{bmatrix}+y\begin{bmatrix}-2\\3\\1\end{bmatrix}=w$

$\Longleftrightarrow$ x, y的方程式 $\begin{cases} x-2y=p+q \\ -x+3y=2q \\ y=p+2q \end{cases}$ 有解

以列運算試解前述方程式:

$$\left[\begin{array}{cc|c}1&-2&p+q\\-1&3&2q\\0&1&p+2q\end{array}\right]\sim\left[\begin{array}{cc|c}1&-2&p+q\\0&1&p+3q\\0&1&p+2q\end{array}\right]\sim\left[\begin{array}{cc|c}1&-2&p+q\\0&1&p+3q\\0&0&-q\end{array}\right]$$

$\therefore$ $w\in V$ $\Longleftrightarrow$ $q=0$ (綜線CH3定理10)

$\Longleftrightarrow$ $w=p\begin{bmatrix}1\\0\\1\end{bmatrix}$

$$\therefore\ V\cap W=\left\{p\begin{bmatrix}1\\0\\1\end{bmatrix}\ \middle|\ p\in\mathbb{R}\right\}$$

$$\therefore\ \text{取}\ \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}\ \text{爲}V\cap W\text{的基底。}$$

０５Ｃ**06**【 元智80工工[5] 】

Let $U=L(\ (1, 2, 3, 6), (4, -1, 3, 6), (5, 1, 6, 13)\ )$, $V=L(\ (1, -1, 1, 1), (2, -1, 4, 5)\ )$

(Here, $L(x_1, x_2, \ldots , x_r)$ stands for the linear–span by vectors $x_1, x_2, \ldots , x_r$)

Find a basis for $U\cap V$.

【解】考慮 $v=p(1,-1,1,1)+q(2,-1,4,5)\in V$:

$v\in U$

$\Longleftrightarrow$ $\exists x, y, z$使得$x(1,2,3,6)+y(4,-1,3,6)+z(5,1,6,13)=p(1,-1,1,1)+q(2,-1,4,5)$

$\Longleftrightarrow$ x, y, z的方程組

$$\begin{cases}x+4y+5z=p+2q\\2x-y+z=-p-q\\3x+3y+6z=p+4q\\6x+6y+13z=p+5q\end{cases}$$

有解 (p, q視爲已知數) .

以列運算試解前述方程式:

$$\left[\begin{array}{ccc|c} 1 & 4 & 5 & p+2q \\ 2 & -1 & 1 & -p-q \\ 3 & 3 & 6 & p+4q \\ 6 & 6 & 13 & p+5q \end{array}\right] \begin{array}{l}(-2)(-3)(-6)\\ \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 4 & 5 & p+2q \\ 0 & -9 & -9 & -3p-5q \\ 0 & -9 & -9 & -2p-2q \\ 0 & -18 & -17 & -5p-7q \end{array}\right] \begin{array}{l} \\ (-1)(-2) \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 4 & 5 & p+2q \\ 0 & -9 & -9 & -3p-5q \\ 0 & 0 & 0 & p+3q \\ 0 & 0 & 1 & p+3q \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 4 & 5 & p+2q \\ 0 & -9 & -9 & -3p-5q \\ 0 & 0 & 1 & p+3q \\ 0 & 0 & 0 & p+3q \end{array}\right]$$

$\therefore$ 方程式有解 $\Longleftrightarrow p+3q=0$ (綜線CH3定理10)

$\Longleftrightarrow p=-3q$

$\therefore$ $U\cap V=\{v\in V \mid v\in U\}=\{p(1,-1,1,1)+q(2,-1,4,5) \mid p=-3q\}$

$=\{-3q(1,-1,1,1)+q(2,-1,4,5) \mid q$為任意純量$\}$

$=\{q(-1,2,1,2) \mid q$為任意純量$\}$

【另解】考慮 $u=x(1,2,3,6)+y(4,-1,3,6)+z(5,1,6,13)\in U$

$u\in V$

$\Longleftrightarrow \exists\, p,q$ 使得$p(1,-1,1,1)+q(2,-1,4,5)=x(1,2,3,6)+y(4,-1,3,6)+z(5,1,6,13)$

$\Longleftrightarrow p,q$的方程組

$$\begin{cases} p+2q=x+4y+5z \\ -p-q=2x-y+z \\ p+4q=3x+3y+6z \\ p+5q=6x+6y+13z \end{cases}$$

有解 (x, y, z視為已知數) .

以列運算試解前述方程式:

$$\left[\begin{array}{cc|c} 1 & 2 & x+4y+5z \\ -1 & -1 & 2x-y+z \\ 1 & 4 & 3x+3y+6z \\ 1 & 5 & 6x+6y+13z \end{array}\right] \begin{array}{c}(1)(-1)(-1)\\ \\ \\ \\ \end{array} \sim \left[\begin{array}{cc|c} 1 & 2 & x+4y+5z \\ 0 & 1 & 3x+3y+6z \\ 0 & 2 & 2x-y+z \\ 0 & 3 & 5x+2y+8z \end{array}\right] \begin{array}{c} \\ (-2)(-3) \\ \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & x+4y+5z \\ 0 & 1 & 3x+3y+6z \\ 0 & 0 & -4x-7y-11z \\ 0 & 0 & -4x-7y-10z \end{array}\right] \begin{array}{c} \\ \\ (-1) \\ \\ \end{array} \sim \left[\begin{array}{cc|c} 1 & 2 & x+4y+5z \\ 0 & 1 & 3x+3y+6z \\ 0 & 0 & -4x-7y-11z \\ 0 & 0 & z \end{array}\right]$$

∴ 方程式有解 $\Longleftrightarrow \begin{cases} -4x-7y-11z=0 \\ z=0 \end{cases}$ (綜線CH3定理0)

$\Longleftrightarrow \begin{cases} -4x-7y=0 \\ z=0 \end{cases} \Longleftrightarrow \begin{cases} 4x+7y=0 \\ z=0 \end{cases}$

∴ $U\cap V=\{\, u\in U \mid u\in V \,\}$

$=\{\, x(1,2,3,6)+y(4,-1,3,6)+z(5,1,6,13) \mid 4x+7y=0,\ z=0 \}$

$=\{\, x(1,2,3,6)+y(4,-1,3,6)+z(5,1,6,13) \mid x=7t,\ y=-4t,\ z=0,\ t$爲任意純量$\}$

$=\{\, 7t(1,2,3,6)-4t(4,-1,3,6) \mid t$爲任意純量 $\}$

$=\{\, t(-9,18,9,18) \mid t$爲任意純量 $\}=\{\, s(-1,2,1,2) \mid s$爲任意純量 $\}$

０５C**０７**【 元智84工工Ｘ[2] 】

考慮多項式所成的向量空間. 令$V_3(\mathbb{R})=\langle 1, x, x \rangle$所生成的向量空間.

① 若$v_1=1+x^2$; $v_2=x^2-x$, $v_3=3-2x$; 試問$\langle v_1, v_2, v_3 \rangle$是否能生成$V_3(\mathbb{R})$

(即: $\langle v_1, v_2, v_3 \rangle \overset{?}{=} V_3(\mathbb{R})$) [證明之或給反例] (10%)

② 考慮$U=\langle 1+2x+x^3, 1-x-x^2 \rangle$, $V=\langle x+x^2-3x^3, 2+2x-2x^3 \rangle$

試求空間$U+V$的一組基底. (12%)

又問$\dim(U+V)$是否與$\dim(V_3(\mathbb{R}))$相同. (2%)

③ 求$U\cap V$的一組基底(15%) 與$\dim(U\cap V)$. (1%)

④ 於本題中, $\dim(U+V)+\dim(U\cap V)$是否等於$\dim(U)+\dim(V)$? (5%)

【解】① (請參閱題型06C)

利用isomorphic 的觀念,

將 $a+bx+cx^2$ 視爲 $[\,a \quad b \quad c\,]$, 將$V_3(\mathbb{R})$視爲$\mathbb{R}^{1\times 3}$.

v_1, v_2, v_3 分別視爲 $[\,1 \quad 0 \quad 1\,], [\,0 \quad -1 \quad 1\,], \quad [\,3 \quad -2 \quad 0\,]$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

$\therefore\ \dim\langle v_1, v_2, v_3\rangle = 3$ (綜線CH6定理23)

$\therefore\ \langle v_1, v_2, v_3\rangle = V(\mathbb{R})$ (綜線CH6定理22a)

② 將 $a+bx+cx^2+dx^4$ 視爲$[a \quad b \quad c \quad d\,]$, 將$V_4(\mathbb{R})$視爲$\mathbb{R}^{1\times 4}$.

$1+2x+x^3$ 視爲$[1 \quad 2 \quad 0 \quad 1]$, $1-x-x^2$視爲$[1 \quad -1 \quad -1 \quad 0]$.

$x+x^2-3x^3$ 視爲$[0 \quad 1 \quad 1 \quad -3]$, $2+2x-2x^3$ 視爲$[2 \quad 2 \quad 0 \quad -2]$.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -3 \\ 2 & 2 & 0 & -2 \end{bmatrix} \overset{\text{列運算}}{\sim \cdots\cdots \sim} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$\therefore\ U+V$的基底可取爲 $\{1-3x^3, x+2x^3, x^2-5x^3\}$

$\dim(U+V)=3=\dim V_3(\mathbb{R})$.

③ 仍將各多項式視爲列矩陣,

考慮V中的一般向量$v=p(0, 1, 1, -3)+q(2, 2, 0, -2)$:

$v\in U \iff \exists\, x, y$ 使$x(1, 2, 0, 1)+y(1, -1, -1, 0)=v$.

試解x,y的方程式 $\begin{cases} x+y=2q \\ 2x-y=p+2q \\ -y=p \\ x=-3p-2q \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 1 & 2q \\ 2 & -1 & p+2q \\ 0 & -1 & p \\ 1 & 0 & -3p-2q \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & p+2q \\ 2 & 0 & 2q \\ 0 & -1 & p \\ 1 & 0 & -3p-2q \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & p+2q \\ 0 & 0 & -2p-2q \\ 0 & -1 & p \\ 0 & 0 & -4p-4q \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & p+2q \\ 0 & -1 & p \\ 0 & 0 & p+q \\ 0 & 0 & p+q \end{array}\right]$$

$v\in U \iff$ 前述x, y的方程式有解 $\iff p+q=0 \iff q=-p$

$\therefore\ U\cap V=\{p(0,1,1,-3)-p(2,2,0,-2)\mid p\in\mathbb{R}\}=\{p(-2,-1,1,-1)\mid p\in\mathbb{R}\}$

$\therefore\ U\cap V$的基底可取為 $\{-2-x+x^2-x^3\}$,

$\dim(U\cap V)=1$.

④ $\because\ 1+2x+x^3, 1-x-x^2$ 線性獨立, $\therefore\ \dim U=2$,

$\because\ x+x^2-3x^3, 2+2x-2x^3$ 線性獨立, $\therefore\ \dim V=2$.

$\therefore\ \dim(U+V)+\dim(U\cap V)=4=\dim U+\dim V$.

０５Ｃ**08**【 台大82資工[6] 】

Let $V=M_{2\times 2}(F)$,

$$W_1=\left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} \in V \ :\ a, b, c\in F \right\},$$

and

$$W_2 = \left\{ \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} \in V \ : \ a, b \in F \right\},$$

(a) Prove that W_1 and W_2 are subspaces of V. (10%)

(b) Find the dimensions of W_1 , W_2 , $W_1 + W_2$, $W_1 \cap W_2$. (10%)

【解】(a) 對W_1 證明封閉性: (綜線CH5定理11)

$$\forall\, p, q \in F, \quad \forall \begin{bmatrix} a & b \\ c & a \end{bmatrix}, \begin{bmatrix} d & e \\ f & d \end{bmatrix} \in W_1,$$

$$p\begin{bmatrix} a & b \\ c & a \end{bmatrix} + q\begin{bmatrix} d & e \\ f & d \end{bmatrix} = \begin{bmatrix} pa+qd & pb+qe \\ pc+qf & pa+qd \end{bmatrix} \in W_1$$

對W_2 證明封閉性:

$$\forall\, p, q \in F, \quad \forall \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix}, \begin{bmatrix} 0 & c \\ -c & d \end{bmatrix} \in W_2,$$

$$p\begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} + q\begin{bmatrix} 0 & c \\ -c & d \end{bmatrix} = \begin{bmatrix} 0 & pa+qc \\ -(pa+qc) & pb+qd \end{bmatrix} \in W_2$$

(b)

$$\begin{bmatrix} a & b \\ c & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\therefore \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \text{ 生成 } W_1$$

而此集合顯然是線性獨立, 所以形成W_1 的基底.

$\therefore \quad \dim W_1 = 3$

$$\begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} = a\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\therefore \quad \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ 生成 W_2

而此集合顯然是線性獨立, 所以形成W_2 的基底.

$\therefore \quad \dim W_2 = 2$

$| \ W_1 \cap W_2 = \{ M \in W_2 \mid M \in W_1 \}$

$$= \left\{ \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} \in V : \ b = 0 \right\}, = \left\{ \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \in V : \ a \in F \right\},$$

顯然 $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ 是 $W_1 \cap W_2$ 的基底.

$\therefore \quad \dim(W_1 \cap W_2) = 1$

$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$ (綜線CH6定理25)

$= 3 + 2 - 1 = 4$