題型01A: 投影, 鏡射, 旋轉　(原題型17C)

本章的題型有某些題目需用到第八章, 第十三章, 及附錄D. 最好是由第二章開始研讀.

０１A**01** 【 清大86資料[6] 】

(a) What is a unitary transformation ?　(2%)

(b) List three equivalent conditions which characterize the unitary transformation.　(3%)

(c) Categorize the unitary transformation in $\mathbb{R}^2$ (plane) and describe these transformations geometrically.

【解】(a) 對內積空間V,W, 及線性映射$T: V\to W$, 若

$$\forall u,v\in V,\quad \langle T(u),T(v)\rangle=\langle u,v\rangle$$

就稱T為unitary transformation.　(綜線CH13Sec1)

(b) 對內積空間V,W, 及線性映射$T: V\to W$, 下列各敘述等價:

(ｉ) $\forall u,v\in V,\quad \langle T(u),T(v)\rangle=\langle u,v\rangle$

(ii) $\forall v\in V,\quad \| T(v)\| = \| v\|$.

(iii) $\forall S\subseteq V$, 若S為正交單位集, 則$T[S]$為正交單位集.

(iv) $T^*\circ T=I$.

若T在正交單位基底的矩陣表示是A, 則前述條件等價於 $A^{\mathrm{H}}A=I$.

(c) 若T為$\mathbb{R}^2$上的unitary transformation,

令$A=\begin{bmatrix} a & c \\ b & d \end{bmatrix}$ 是T在標準基底的矩陣表示.

$$A^{\mathrm{T}}A=I \iff \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & c \\ b & d \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\iff a^2+b^2=1,\quad c^2+d^2=1,\quad ac+bd=0$$

由$a^2+b^2=1$ 可設 $a=\cos\theta, b=\sin\theta$.

由 $(\cos\theta)c+(\sin\theta)d=0$

解得 $c=-t\sin\theta, d=t\cos\theta$, $t\in\mathbb{R}$.

代入$c^2+d^2=1$ 得 $t^2=1$， 即$t=\pm 1$.

$$\therefore\quad A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \text{ 或 } A=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$$

當 $A=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$時，

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix},\quad A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}=\begin{bmatrix}\cos(\theta+\pi/2)\\ \sin(\theta+\pi/2)\end{bmatrix}$$

這在幾何上是$\mathbb{R}^2$ 沿逆時針方向的旋轉, 角度為θ .

當 $A=\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$時， 可對$A$做對角化:

$\det(A-kI)=k^2-1$, 求得A的特徵值為1, –1 .

先令$\eta=\theta/2$.

$$A-I=\begin{bmatrix}\cos\theta-1 & \sin\theta\\ \sin\theta & -\cos\theta-1\end{bmatrix}=\begin{bmatrix}-2\sin^2\eta & 2\sin\eta\cos\eta\\ 2\sin\eta\cos\eta & -2\cos^2\eta\end{bmatrix}$$

$$\sim\begin{bmatrix}\sin\eta & -\cos\eta\\ 0 & 0\end{bmatrix}$$

特徵值1的特徵子空間為 $\{\, t[\cos\eta, \sin\eta]^T \mid t\in\mathbb{R}\,\}$

$$A+I=\begin{bmatrix}\cos\theta+1 & \sin\theta \\ \sin\theta & -\cos\theta+1\end{bmatrix}=\begin{bmatrix}2\cos^2\eta & 2\sin\eta\cos\eta \\ 2\sin\eta\cos\eta & 2\sin^2\eta\end{bmatrix}$$

$$\sim\begin{bmatrix}\cos\eta & \sin\eta \\ 0 & 0\end{bmatrix}$$

特徵值–1的特徵子空間爲 $\{ t[\sin\eta, -\cos\eta]^T \mid t\in\mathbb{R} \}$

∴ T爲以 $\{ t[\cos(\theta/2), \sin(\theta/2)]^T \mid t\in\mathbb{R} \}$ 爲軸的鏡射.

０１A**02** 【 師大84資教[9] 】

Let L be the linear operator rotating each vector in the 2D space by an angle θ in the lockwise direction. Determinethe matrix A representing L.

【解】 $L([1,0]^T) = [\cos\theta, \sin\theta]^T$,

$L([0,1]^T) = [\cos(\theta+\pi/2), \sin(\theta+\pi/2)]^T = [-\sin\theta, \cos\theta]^T$

∴ $L([x, y]^T) = L(x[1,0]^T + y[0,1]^T)$

$= x[\cos\theta, \sin\theta]^T + y[-\sin\theta, \cos\theta]^T$ (線性條件)

$= [x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta]^T$.

０１A**03** 【 交大82資科[1] 】

Find the matrix representations, with respect to the standard basis

$\{e_1, e_2\} = \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\}$, for the following linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$.

(A) Projection on the line $y=mx$.

(B) Reflection in the line $y=mx$.

(C) First make a rotation about the origin through the angle θ, then make a projection on the line $y=mx$.

【參考章節】(A),(B) CH1定義8,CH1習題13.1 (C) CH8範例31, CH8定理23③

【解】(A) 設所求之projection爲T_P ,

直線方程式$y=mx$表示一個$\mathbb{R}^2$ 的子空間$\{[x,y]^T \in \mathbb{R} \mid y^2=mx\}$

$\because \begin{bmatrix} 1 \\ m \end{bmatrix}$爲基底，

$\therefore$

$$T_P\begin{bmatrix} x \\ y \end{bmatrix} = \frac{\langle \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} 1 \\ m \end{bmatrix} \rangle}{\langle \begin{bmatrix} 1 \\ m \end{bmatrix}, \begin{bmatrix} 1 \\ m \end{bmatrix} \rangle} \begin{bmatrix} 1 \\ m \end{bmatrix} = \ldots$$ (綜線CH1定義8)

$$= \frac{1}{1+m^2}\begin{bmatrix} x+my \\ mx+m^2y \end{bmatrix} = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \quad [\,T_P\,] = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$$

(B) 設所求之reflection爲T_R,

$$\begin{bmatrix} x \\ y \end{bmatrix} + T_R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2\,T_P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$$ (綜線CH1習題13.1)

$$\therefore \quad T_R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 2\,T_P\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) - \begin{bmatrix} x \\ y \end{bmatrix} = \ldots = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \quad [\,T_R\,] = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}$$

(C) 設所求之rotation爲T_θ, 本小題所求之映射爲T.

$$\because \quad T_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

$$T_\theta \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \cos(\theta+\pi/2) \\ \sin(\theta+\pi/2) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

$$T_\theta \begin{bmatrix} x \\ y \end{bmatrix} = xT_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix} + yT_\theta \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$ (綜線CH7定義1)

$$= \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \quad [T_\theta] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$[T] = [T_P \circ T_\theta] = [T_P][T_\theta]$$ (綜線CH8定理23)

$$= \frac{1}{1+m^2} \begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \frac{1}{1+m^2} \begin{bmatrix} \cos\theta + m\sin\theta & -\sin\theta + m\cos\theta \\ m\cos\theta + m^2\sin\theta & -m\sin\theta + m^2\cos\theta \end{bmatrix}$$

０１A**04** 【 中央82資電[3] 】

Find the 2×2 linear transformation matrix that reflect every vector through the θ-line. (i.e., symmetric transformation to the line having slope tan(θ)).

【解】 令T代表此reflection, 對θ-line做鏡射,

角0應映到2θ，角π/2應映到θ–(π/2–θ)＝2θ–π/2 .

$$\therefore\quad T\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}\cos(2\theta)\\\sin(2\theta)\end{bmatrix}$$

$$T\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}\cos(2\theta-\pi/2)\\\sin(2\theta-\pi/2)\end{bmatrix}=\begin{bmatrix}\sin(2\theta)\\-\cos(2\theta)\end{bmatrix}$$

$$T\begin{bmatrix}x\\y\end{bmatrix}=xT\begin{bmatrix}1\\0\end{bmatrix}+yT\begin{bmatrix}0\\1\end{bmatrix}=x\begin{bmatrix}\cos(2\theta)\\\sin(2\theta)\end{bmatrix}+y\begin{bmatrix}\sin(2\theta)\\-\cos(2\theta)\end{bmatrix}$$

$$=\begin{bmatrix}\cos(2\theta)&\sin(2\theta)\\\sin(2\theta)&-\cos(2\theta)\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

∴ 所求矩陣爲 $\begin{bmatrix}\cos(2\theta)&\sin(2\theta)\\\sin(2\theta)&-\cos(2\theta)\end{bmatrix}$ (綜線CH7定理15)

【另解】令 m＝tanθ, 並令T代表此reflection

$\because$ $\begin{bmatrix}1\\m\end{bmatrix}$在θ-line上, $\therefore\ T\begin{bmatrix}1\\m\end{bmatrix}=\begin{bmatrix}1\\m\end{bmatrix}$

$\because$ $\begin{bmatrix}-m\\1\end{bmatrix}$與θ-line垂直, $\therefore\ T\begin{bmatrix}-m\\1\end{bmatrix}=-\begin{bmatrix}-m\\1\end{bmatrix}=\begin{bmatrix}m\\-1\end{bmatrix}$

$\forall\ \begin{bmatrix}x\\y\end{bmatrix}\in\mathbb{R}^{2\times1}$ ，令 $\begin{bmatrix}x\\y\end{bmatrix}=\alpha\begin{bmatrix}1\\m\end{bmatrix}+\beta\begin{bmatrix}-m\\1\end{bmatrix}$,

可解得 $\alpha=\dfrac{x+my}{1+m^2}$, $\beta=\dfrac{-mx+y}{1+m^2}$

$$T\begin{bmatrix}x\\y\end{bmatrix}=\alpha T\begin{bmatrix}1\\m\end{bmatrix}+\beta T\begin{bmatrix}-m\\1\end{bmatrix}=\alpha\begin{bmatrix}1\\m\end{bmatrix}+\beta\begin{bmatrix}m\\-1\end{bmatrix}$$

$$=\ .....\ =\frac{1}{1+m^2}\begin{bmatrix}1-m^2 & 2m\\2m & m^2-1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

由三角學公式:

$$\frac{1-m^2}{1+m^2}=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\cos(2\theta)\ ,\ \frac{2m}{1+m^2}=\frac{2m}{1+\tan^2\theta}=\sin(2\theta)$$

∴ 上式可改寫爲:

$$T\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\cos(2\theta) & \sin(2\theta)\\\sin(2\theta) & -\cos(2\theta)\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

∴ 所求矩陣爲 $\begin{bmatrix}\cos(2\theta) & \sin(2\theta)\\\sin(2\theta) & -\cos(2\theta)\end{bmatrix}$. (綜線CH7定理15)

０１A**05** 【 中央83資工[4] 】

In the vector space $\mathbb{R}^3$, what is the axis of rotation, and the angle of rotation, of the transformation that takes vector $(x_1, x_2, x_3)^T$ into vector $(x_2, x_3, x_1)^T$? Find the matrix that represents this transformation. (15%)

【解】設此線性映射爲T, 由題得知

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)=\begin{bmatrix} x_2 \\ x_3 \\ x_1 \end{bmatrix}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

令$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$，則$A$爲所求之矩陣.

旋轉軸是A的特徵子空間，並以1爲特徵值.

$$A-1I=\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\sim\overset{\text{列運算}}{\cdots\cdots}\sim\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$，解得特徵向量$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

$\therefore$ 旋轉軸$=\left\{\begin{bmatrix} t \\ t \\ t \end{bmatrix} \middle| \, t\in\mathbb{R}\right\}$

$$A^2=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$A^3=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}=I$$

$\because$ T作用三次得出恆等映射.

$\therefore$ T的旋轉角爲$2\pi/3$，即120°.

０１A**06** 【 清大84資料[1] 】

(a) Find the matrix that represents reflections through the origin in three dimensions.

(b) Find the matrix that represents reflections through the xz plane in three dimensions.

【解說】reflection through the origin 就是 $(x, y, z) \to (-x, -y, -z)$

reflection through the xz–plane 就是 $(x, y, z) \to (x, -y, z)$

【解】(a) 設所求爲T, 由題意得知

$$T([\,x \quad y \quad z\,]^T) = [\,-x \; -y \; -z\,]^T$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore$ 所求爲 $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (綜線CH7定理15)

(b) 設所求爲T, 由題意得知

$$\begin{cases} T([\,1 \quad 0 \quad 0\,])^T = [\,1 \quad 0 \quad 0\,]^T \\ T([\,0 \quad 1 \quad 0\,])^T = [\,0 \quad -1 \quad 0\,]^T \\ T([\,0 \quad 0 \quad 1\,])^T = [\,0 \quad 0 \quad 1\,]^T \end{cases}$$

$$\therefore\ T([\,x \quad y \quad z\,]^T) = x\,[\,1 \quad 0 \quad 0\,]^T + y\,[\,0 \quad -1 \quad 0\,]^T + z\,[\,0 \quad 0 \quad 1\,]^T$$

$$= [\,x \quad -y \quad z\,]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\therefore \text{ 所求爲} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ (綜線CH7定理15)

０１A**07** 【 淡江85資工[4] 】

An $n\times n$ matrix A is called orthogonal if $AA^t = I_n$, where I_n is the $n\times n$ identity matrix. Show that if x is a nonzero vector in $\mathbb{R}^n$, then the $n\times n$ matrix

$$A = I_n - \frac{2}{\|x\|^2} xx^t$$

is both orthogonal and symmetric. (25%)

【解】① 證明symmetric:

$(xx^T)^T = x^{TT}x^T$ (綜線CH2定理23)

$= xx^T$

$\therefore A^T = (I - (2/\|x\|^2)xx^T)^T$

$= I^T - (2/\|x\|^2)(xx^T)^T$ (綜線CH2定理23)

$= I - (2/\|x\|^2)\, xx^T = A$

② 證明orthogonal:

$(xx^T)(xx^T) = x(x^Tx)x^T$ (矩陣乘法結合律)

$= x\,[\,\|x\|^2]\, x^T$ (此式爲三個矩陣相乘)

$= \|x\|^2 xx^T$ (此式爲矩陣的係數積)

$\therefore AA^T = AA$ (由①)

$= (I - (2/\|x\|^2)xx^T)^2$

$= I - (4/\|x\|^2)\, xx^T + (4/\|x\|^4)(xx^T)(xx^T)$ (綜線CH2定理16)

$= I - (4/\|x\|^2)\, xx^T + (4\|x\|^2/\|x\|^4)(xx^T) = I$

０１A**08** 【 中正81資工[5] 】

For nonzero w in $\mathbb{R}^p$, the $p\times p$ Householder matrix H_w is defined as

$$H_w = I_p - \left(\frac{2}{w^T w}\right) w\, w^T$$

where I is the identity matrix, w^T is the transpose of w. Prove that H_w is symmetric and orthogonal.

【解】爲求簡潔, 以下將H_w 記爲H, 將I_p 記爲I, 將實數 $2/(w^T w)$ 記爲α,

$H^T = (I - \alpha ww^T)^T = I - \alpha(ww^T)^T$

$= I - \alpha ww^T = H$ (綜線CH2定理23)

$\therefore$ H 爲 symmetric .

$H^T H = HH = [I - \alpha ww^T]^2$

$= I - 2\alpha ww^T + \alpha^2 ww^T ww^T$ (綜線CH2定理16)

$= I - 2\alpha ww^T + \alpha^2 w(2/\alpha)w^T$

$= I - 2\alpha ww^T + 2\alpha ww^T = I$

$\therefore$ H 爲 orthogonal . (綜線CH2定義25)

０１A**09** 【 清大83資料[3] 】

A Householder matrix can be defined as $H = I - 2uu^t$, where $u \in \mathbb{R}^n$ is a unit column vector. Let $H_1, H_2, \ldots, H_{10}$ be Householder matrices. What is $\det(H_1 H_2 \ldots H_{10})$? (5%)

【分析】 設$W = \{u\}^\perp$, H代表以W爲“鏡面”的鏡射(reflection).

【解】 對 Householder matrix $H = I - 2uu^t$,

令 $W_1 = \{\, ku \mid k \in \mathbb{R} \,\}$, $W_2 = W_1^\perp$.

W_1 爲特徵值–1的特徵子空間, W_2 爲特徵值 1 的特徵子空間.

$\therefore$ $\det H = (-1)(1)^{n-1} = -1$

由此可知所求

$\det(H_1 H_2 \ldots H_{10}) = \det(H_1)\det(H_2)\ldots\det(H_{10}) = (-1)^{10} = 1$

０１A**10** 【 清大83資料[5] 】

Let $R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation defined as

$$R_\theta=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

What are Kernel(R_θ) and Image($\mathbb{R}^3$) under R_θ? (5%)

【分析】矩陣解釋成線性映射通常是用左乘的方式. 此題將線性映射與矩陣混為一談, 略有混淆之嫌. 幾何上, 本題以z軸為轉軸, 將整個$\mathbb{R}^3$ 旋轉θ角.

【解】 1°設

$$\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

乘開得

$$\begin{cases}x\cos\theta-y\sin\theta=0\\ x\sin\theta+y\cos\theta=0\\ z=0\end{cases}$$

前兩式平方相加得 $x^2+y^2=0$, $\therefore\ x=y=0$

$\therefore$ Kernel(R_θ)$=\{\,o\,\}$

2°由 Kernel(R_θ)$=\{\,o\,\}$ 得知R_θ 為一對一映射. (綜線CH8定理7)

而R的定義域與對應域都是3維空間,

$\therefore$ R_θ為映成. (綜線CH8定理11)

$\therefore$ Image($\mathbb{R}^3$) under R_θ 就是$\mathbb{R}^3$.

０１A**１１** 【 元智80工工[3] 】

Let $\vec{n}=\begin{pmatrix}1\\ 0\\ 1\end{pmatrix}\in\mathbb{R}^{3\times 1}$ and let U be the plane through the origin with $\vec{n}$ as its normal vector. Let P be the orthogonal projection of $\mathbb{R}^{3\times 1}$ on U.

Compute $P\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and then find the 3×3 matrix A, such that

$$P\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \forall \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^{3\times 1}$$

【參考章節】CH1習題13.1

【解】

令 $v=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$，並令$u$為$v$對$n$方向的正投影.

$$u=\frac{v\cdot n}{n\cdot n}n$$ (綜線CH1定義8④)

$$=\frac{x+z}{2}n=\frac{x+z}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\therefore\ Pv=v-u=\begin{pmatrix} x \\ y \\ z \end{pmatrix}-\frac{x+z}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}=\frac{1}{2}\begin{pmatrix} x-z \\ 2y \\ -x+z \end{pmatrix}$$

$$=\begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\therefore\ A=\begin{pmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{pmatrix}$$

題型01B: 幾何問題 (原題型17D)

本章的題型有某些題目需用到第八章, 第十三章, 及附錄D. 最好是由第二章開始研讀.

０１B**01** 【 成大81資工甲乙[4] 】

Given a straight line in 2-space that passes through the two points $P(-2, -3)$ and $Q(5, 1)$

(a) Find a vector equation for the straight line. (5%)

(b) Determine a direction vector for the straight line. (5%)

【解】(a) $\overrightarrow{PQ} = (5, 1) - (-2, -3) = (7, 4)$

設此線上之動點為$X(x, y)$, 則 $\overrightarrow{PX} = t\,\overrightarrow{PQ}$.

∴ 此線之向量方程式為 $\overrightarrow{OX} = \overrightarrow{OP} + t\,\overrightarrow{PQ}$, t為參數. (綜線CH1定義20)

(b) (a)小題中的 $\overrightarrow{PQ}$ 就是此線的derection vector之一.

【加強演練】

考慮空間中的兩點 $P(-2, -3, 2)$, $Q(5, 1, 7)$. 試求通過P點, 並垂直於 $\overrightarrow{PQ}$ 的平面方程式.

Ans: $7(x+2)+4(y+3)+5(z-2)=0$

０１B**02** 【 中央85資工[3] 】

Give two geometric meanings for that the linear system $Ax=b$ is consistent.

【解】設A為$m\times n$矩陣.

① 設A的各列依序為$R_1, R_2, \ldots, R_m$, 而 $b=[\,b_1, b_2, \ldots, b_m\,]^T$

$Ax=b \iff \forall i=1,2,\ldots,m,\quad R_i\cdot x=b_i$ (綜線CH3定義1)

各個 $R_i\cdot x=b_i$ 都代表 n維空間上的超平面(hyperplane).

$Ax=b$ 有解就是這m個超平面的交集非空.

② 設A的各行依序為$C_1, C_2, \ldots, C_n$.

$Ax=b \iff \exists x_1,\ldots,x_n$ 使得$b=x_1 C_1 + x_2 C_2 + \ldots + x_n C_n$ (綜線CH3定理2)

$Ax=b$ 有解就是b掉在 span$\{C_1, C_2, \ldots, C_n\}$ 之內.

０１B**03** 【 中央82資電[2] 】

For a system of linear equations,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

explain when the system is singular from row picture

$$\text{(i.e., } \quad \begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1, \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2, \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3, \end{array} \quad)$$

and column picture

$$\text{(i.e., } \quad x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ a_3 \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad).$$

【解】由列的觀點, 此聯立方程式是由三個平面構成,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1,$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2,$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3,$$

係數矩陣的三個列分別代表此三個平面的法向量(normal vector). 解方程式就是要求出這三個平面的交集.

聯立方程式爲singular $\iff$ 係數行列式爲零 $\iff$ 此三平面的法向量共面

這時三平面的交集可能爲一平面, 一直線, 或$\emptyset$.

由行的觀點，此聯立方程式是試圖將 $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ 表成係數矩陣的三個行的線性組合，

聯立方程式為singular $\Longleftrightarrow$ 係數行列式為零 $\Longleftrightarrow$ 係數矩陣的三個行共面
這時三個行向量展出(span)的行空間(column space)可能是通過原點的一平面，一直線，或 $\{ o \}$.

方程式有解就是 $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ 掉在行空間之上. 在有解時，線性組合的係數並不唯一.

０１B**04** 【 交大79工工[8] 】
是非題:
If X and Y are vectors in $\mathbb{R}^n$, then $\| X-Y \|^2 = \| X \|^2 - \| Y \|^2$.

【參考章節】CH1定義3
【解】非
【討論】本題毫無道理可言，即使在$n=1$也不對，反例隨便找就可以了.

０１B**05** 【 交大84資工[5] 】
Let $\Delta = \{ x \in \mathbb{R}^n \mid x \geq o,\ \sum x_j = 1 \}$ be the $(n-1)$-dimensional standard simplex. Its center $a_0 = \dfrac{1}{n}(1, 1, \ldots, 1)$. Let $B(a_0, r)$ be the largest $(n-1)$-dimensional ball centerd at a_0 in Δ with radius r, and let $B(a_0, R)$ be the smallest $(n-1)$-dimensional ball centered at a_0 with radius R and containing Δ. Calculate r and R.

【分析】(1) 由 a_0 的寫法可判定題目的$\mathbb{R}^n$ 其實是$\mathbb{R}^{1\times n}$.　　(與上題不一致!)
(2) 對 $x=(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,
$x \geq o$ 表示 “$x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0$”.

但這並不是通行的寫法.

(4) 以$n=1$ (這是退化的情形) 來看,

$\Delta_1=\{ x_1 \mid x_1\geq 0,\quad x_1= 1 \} = \{ 1 \}$

中心 $a_0= 1$.

(5) 以$n=2$來看,

$\Delta_2=\{ (x_1 , x_2) \mid x_1\geq 0, x_2\geq 0, x_1+x_2= 1 \}$

這是$\mathbb{R}^2$上由(1, 0)連到(0, 1)的線段.

中心 $a_0=(1/2 , 1/2)$.

(6) 以$n=3$來看,

$\Delta_3=\{(x_1 , x_2 , x_3) \mid x_1\geq 0, x_2\geq 0, x_3\geq 0, x_1+x_2+x_3=1 \}$

這是$\mathbb{R}^3$上的正三角形, 以(1,0,0), (0,1,0), (0,0,1)爲頂點.

中心 $a_0=(1/3, 1/3, 1/3)$.

Δ_3 與$x_1 x_2$平面的交集爲 $\{ [x_1 , x_2 ,0] \mid x_1\geq 0, x_2\geq 0, x_1+x_2=1 \}$

可視爲Δ_2.

(7) $n=4$時, Δ_4 爲$\mathbb{R}^4$ 上的正三角錐(正四面體), 它的頂點爲(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1). 中心$a_0=(1/4,1/4,1/4,1/4)$.

Δ_4 與 "$x_1 x_2 x_3$超平面" 的交集爲

$\{(x_1, x_2, x_3,0) \mid x_1\geq 0, x_2\geq 0, x_3\geq 0, x_1+x_2+x_3=1 \}$

可視爲Δ_3.

(8)對一般的n, Δ_n 是$\mathbb{R}^n$ 中以(1,0,0,...0), (0,1,0,...,0), ... (0,0,0,...,1)爲頂點的最小凸集合(convex set).

中心 $a_0= (1/n, 1/n, \ldots , 1/n)$.

Δ_n 與 "$x_1 x_2 \ldots x_{n-1}$超平面"的交集可視爲Δ_{n-1} .

(9) 對$\mathbb{R}^n$ 中的n+1個點$v_0 , v_1 , \ldots , v_p$,

若 $v_1-v_0 , v_2-v_0 , \ldots , v_p-v_0$ 線性獨立,

則稱包含這p個點的最小凸集合爲n-simplex. simplex 是數學上homology theory 的起步.

【解】已知中心點$a_0=(1/n, 1/n, \ldots , 1/n)$,

令 $V=(0, 0, \ldots , 1)$, V爲Δ_n 的頂點.

外接球半徑$R = \| \overrightarrow{a V_0} \| = \| (-1/n, -1/n, ..., -1/n, (n-1)/n) \|$

$= (1/n) \| (-1, -1, ..., -1, n-1) \| = (1/n)\sqrt{(n-1)+(n-1)^2} = \sqrt{(n-1)/n}$

令$M = (1/(n-1), 1/(n-1), ..., 1/(n-1), 0)$,

M為“Δ_n 的一面”的中心, 也是內切球的切點.

內切球半徑$r = \| \overrightarrow{a_0 M} \| = \| (1/(n(n-1)))\,(1, 1, ..., 1, 1-n) \|$

$= (1/(n(n-1)))\ \| (1, 1, ..., 1, 1-n) \|$

$= (1/(n(n-1)))\sqrt{(n-1)+(n-1)^2} = 1 / \sqrt{n(n-1)}$

０１B06 【 清大85資料[5] 】

Let $x, b \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Define

$$f(x) = x^t A^t Ax - 2b^t x + b^t b.$$

Find $\nabla f(x)$, the gradient of f.

【說明】(1) 對f：$\mathbb{R}^n \to \mathbb{R}$, $\nabla f = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_n} \right]$. (此為微積分的定義)

(2) 由題目中$f(x)$的公式可判知本題的$\mathbb{R}^n$代表$\mathbb{R}^{n\times 1}$.

【解】本題需用到一些定理:

定理A: 對$a, b \in \mathbb{R}$, 及可微函數 f, g：$\mathbb{R}^n \to \mathbb{R}$, $\nabla(af+bg)(x) = a\nabla f(x) + b\nabla g(x)$.

定理B: 設$c \in \mathbb{R}$, 若g：$\mathbb{R}^n \to \mathbb{R}$, 定義為 $g(x) = c$, 則 $\nabla g(x) = [0, 0, ..., 0]$.

定理C: 設$a \in \mathbb{R}^n$, 若g：$\mathbb{R}^n \longrightarrow \mathbb{R}$, 定義為 $g(x) = a^t x$, 則 $\nabla g(x) = a^t$.

定理D: 設$C \in \mathbb{R}^{n\times n}$, 若$g$：$\mathbb{R}^n \to \mathbb{R}$, 定義為 $g(x) = x^t Cx$, 則$\nabla g(x) = x^t(C + C^t)$.

若C為對稱矩陣, 則公式簡化為 $\nabla g(x) = 2x^t C$.

本題計算如下:

$\nabla f(x) = \nabla(x^t A^t Ax - 2b^t x + b^t b) = \nabla(x^t A^t Ax) - \nabla(2b^t x) + \nabla(b^t b)$ (定理A)

$= 2x^t A^t A - 2b^t$ (定理B,C,D)

定理A,B,C讀者自證.

[定理D的證明]

令 $C = [c_{ij}]$, 則

$f(x)=x^tCx = \sum_i \sum_j c_{ij}\, x_i\, x_j$ (綜線CH10範例2a)

對 $k=1, 2, \ldots, n$,

$$\frac{\partial x_i x_j}{\partial x_k} = \begin{cases} 2x_k\,, & \text{if } i=j=k \\ x_i\,, & \text{if } i\neq j=k \\ x_j\,, & \text{if } j\neq i=k \\ 0\,, & \text{else} \end{cases}$$

$$\frac{\partial f}{\partial x_k} = \sum_i \sum_j c_{ij}\frac{\partial x_i x_j}{\partial x_k} = 2c_{kk}x_k + \sum_{i\neq k} c_{ik}x_i + \sum_{j\neq k} c_{kj}\, x_j = \sum_i c_{ik}x_{ik} + \sum_j c_{kj}\, x_j$$

$= x^t$(C的第k行)+(C的第k列)$x = x^t$(C的第k行)$+x^t$ (C^t 的第k行)

$= x^t$($(C+C^t)$的第k行)

$\therefore\ \nabla f(x)=x^t\,(C+C^t)$ (綜線CH2定理6①)

０１Ｂ**０７** 【 中原86工工[5] 】

Consider a moving body B whose position at time t is given by $\mathrm{R}(t)=t^3\,\mathrm{i}+2t^2\,\mathrm{j}+3t\,\mathrm{k}$

($\mathrm{V}(t)=d\mathrm{R}(t)/dt$ denotes the velocity of B; $\mathrm{A}(t)=d\mathrm{V}(t)/dt$ denotes the acceleration of B)

(a) Find the position of B when $t=1$.

(b) Find the velocity (v) of B when $t=1$.

(c) Find the speed(s) of B (or the normalized v) when $t=1$.

(d) Find the acceleration (a) of B when $t=1$.

【分析】speed是velocity的長度.

【解】(a) $\mathrm{R}(1)=\mathrm{i}+2\,\mathrm{j}+3\,\mathrm{k}$

(b) $\mathrm{V}(t)=d\mathrm{R}(t)/dt=3t^2\,\mathrm{i}+4t\,\mathrm{j}+3\,\mathrm{k}$, $\mathrm{V}(1)=3\,\mathrm{i}+4\,\mathrm{j}+3\,\mathrm{k}$

(c) $\|\mathrm{V}(1)\| = \sqrt{9+16+9}=34$

(d) $\mathrm{A}(t)=d\mathrm{V}(t)/dt=6t\,\mathrm{i}+4\,\mathrm{j}$, $\mathrm{A}(1)=6\,\mathrm{i}+4\,\mathrm{j}$