題型03A: 解線性方程式

０３A**０１** 【 朝陽85工工[6] 】

Solve the given linear system:

$$\begin{aligned} x+y+z-2w&=-4\\ 2y+z+3w&=4\\ 2x+y-z+2w&=5\\ x-y+w&=4 \end{aligned}$$

【解】

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & -4\\ 0 & 2 & 1 & 3 & 4\\ 2 & 1 & -1 & 2 & 5\\ 1 & -1 & 0 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & -4\\ 0 & 2 & 1 & 3 & 4\\ 0 & -1 & -3 & 6 & 13\\ 0 & -2 & -1 & 3 & 8 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & -4\\ 0 & 0 & -5 & 15 & 30\\ 0 & -1 & -3 & 6 & 13\\ 0 & 0 & 5 & -9 & -18 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & -4\\ 0 & 0 & 1 & -3 & -6\\ 0 & -1 & -3 & 6 & 13\\ 0 & 0 & 5 & -9 & -18 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & -4\\ 0 & 0 & 1 & -3 & -6\\ 0 & -1 & -3 & 6 & 13\\ 0 & 0 & 0 & 6 & 12 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & -2 & -4\\ 0 & 0 & 1 & -3 & -6\\ 0 & -1 & -3 & 6 & 13\\ 0 & 0 & 0 & 1 & 2 \end{array}\right]$$

$$\sim\left[\begin{array}{cccc|c}1&1&1&0&0\\0&0&1&0&0\\0&-1&-3&0&1\\0&0&0&1&2\end{array}\right]\sim\left[\begin{array}{cccc|c}1&1&0&0&0\\0&0&1&0&0\\0&-1&0&0&1\\0&0&0&1&2\end{array}\right]\sim\left[\begin{array}{cccc|c}1&0&0&0&1\\0&0&1&0&0\\0&-1&0&0&1\\0&0&0&1&2\end{array}\right]$$

$$\sim\left[\begin{array}{cccc|c}1&0&0&0&1\\0&1&0&0&-1\\0&0&1&0&0\\0&0&0&1&2\end{array}\right],\quad\therefore\ x=1, y=-1, z=0, w=2\quad.$$

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0 A**02** 【 清大86工工[4] 】

Describe the solutions of $Ax=b$, where

(a)

$$A=\begin{bmatrix}3&5&-4\\-3&-2&4\\6&1&-8\end{bmatrix}, b=\begin{bmatrix}7\\-1\\-4\end{bmatrix}$$

(b)

$$A=\begin{bmatrix}1&2&-7\\-2&-3&9\\0&-2&10\end{bmatrix}, b=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

indicate specifically their solutions as the parametric vector form, such as $x=p+tv$. This is to specify what are p, v, and t respectively.

【解】(a)

$$\left[\begin{array}{ccc|c}3&5&-4&7\\-3&-2&4&-1\\6&1&-8&-4\end{array}\right]\sim\underset{\cdots\cdot}{\text{經列運算}}\sim\left[\begin{array}{ccc|c}1&0&-4/3&-1\\0&1&0&2\\0&0&0&0\end{array}\right]$$

$$\therefore\ x=\begin{bmatrix}-1+(4/3)t\\2\\t\end{bmatrix}=\begin{bmatrix}-1\\2\\0\end{bmatrix}+t\begin{bmatrix}4/3\\0\\1\end{bmatrix}$$

(b) $$\left[\begin{array}{ccc|c}1&2&-7&0\\-2&-3&9&0\\0&-2&10&0\end{array}\right]\sim\overset{\text{經列運算}}{\cdots\cdots}\sim\left[\begin{array}{ccc|c}1&0&3&0\\0&1&-5&0\\0&0&0&0\end{array}\right]$$

$$\therefore\ x=\begin{bmatrix}-3t\\5t\\t\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}+t\begin{bmatrix}-3\\5\\1\end{bmatrix}$$

０３A**03** 【 中正83資工[1] 】

Consider the case in which the matrix equation $AX=B$ is the equation

$$\begin{bmatrix}-1&1&-2&0&-2&0\\-2&2&-4&4&-6&-2\\2&-2&4&-2&5&1\\2&-2&4&4&2&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\end{bmatrix}=\begin{bmatrix}-3\\-8\\7\\6\end{bmatrix}$$

(a) Determine the rank of the matrix A. (5%)

(b) Determine the particular solution and the general solution of $AX=B$. (10%)

【解】先對分隔矩陣$[A\mid B]$做列運算:

$$\left[\begin{array}{cccccc|c} -1 & 1 & -2 & 0 & -2 & 0 & -3 \\ -2 & 2 & -4 & 4 & -6 & -2 & -8 \\ 2 & -2 & 4 & -2 & 5 & 1 & 7 \\ 2 & -2 & 4 & 4 & 2 & 0 & 6 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cccccc|c} 1 & -1 & 2 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(a) 由前面列運算左半部得知 $\operatorname{rank}A=3$.　　　　(綜線CH6定理23)

(b) 由前面列運算得知 $[\,3\quad 0\quad 0\quad 0\quad 0\quad 1\,]^T$ 爲一個特解, 通解爲:

$$\begin{cases} x_1=3+r-2s-2t \\ x_2=r \\ x_3=s \\ x_4=t/2 \\ x_5=t \\ x_6=1 \end{cases}\quad ,\ r,s,t\text{爲任意數.}$$

(綜線CH3範例7)

０３A**04** 【 成大81資工甲乙[1] 】

For the following system of three equations in four unknowns:

$$\begin{aligned} 7x_3+14x_4&=-7 \\ 2x_1-8x_2+4x_3+18x_4&=0 \\ 3x_1-12x_2-x_3+13x_4&=7 \end{aligned}$$

(a) Find a row–echelon form　(5%)

(b) Determine the solutions of the original system of equations.　(5%)

【分析】(1) row–echelon form 爲 Noble & Daniel 的用語, 與 Hoffman 所說的 row–reduced echelon form 完全相同, 也就是列簡化梯形。　(綜線CH3定義4)

(2) Noble & Daniel 將 pivot 全是 1 的梯形矩陣稱爲 Gauss–reduced form。

【解】(a)

$$\left[\begin{array}{cccc|c} 0 & 0 & 7 & 14 & -7 \\ 2 & -8 & 4 & 18 & 0 \\ 3 & -12 & -1 & 13 & 7 \end{array}\right]\begin{array}{l} \\ (1/2) \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 0 & 0 & 7 & 14 & -7 \\ 1 & -4 & 2 & 9 & 0 \\ 3 & -12 & -1 & 13 & 7 \end{array}\right]\begin{array}{l} (1/7) \\ (-3) \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 0 & 0 & 1 & 2 & -1 \\ 1 & -4 & 2 & 9 & 0 \\ 0 & 0 & -7 & -14 & 7 \end{array}\right]\begin{array}{l} (-2)\ (7) \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 0 & 0 & 1 & 2 & -1 \\ 1 & -4 & 0 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -4 & 0 & 5 & 2 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(b) 原方程式化為

$$\begin{cases} x_1 - 4x_2 \qquad + 5x_4 = 2 \\ \qquad\qquad\quad x_3 + 2x_4 = -1 \end{cases}$$

∴ 原方程式之解為

$$\begin{cases} x_1 = 2 + 4t - 5u \\ x_2 = t \\ x_3 = -1 - 2u \\ x_4 = u \end{cases} \qquad , t, u \text{ 為任意數.}$$

０３Ａ**05** 【 交大81資工[3](c) 】

Let

$$A = \begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

(c) (2%) For a linear system $Ax = b$ with column vectors x and b, suppose the system is consistent and x_0 is a slolution to the systems, find the set of solutions to the system.

【解】 $\{x \mid Ax=b\}$

$= \{x_0+u \mid u\in \mathrm{NS}(A)\}$ (綜線CH3定理2)

$$=\left\{x_0+t\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}\ \middle|\ t\in\mathbb{R}\right\}$$ (本題由(a)已解出NS(A))

03A06 【 大同82資工[12] 】

Which of the following matrices is in a row echelon form:

(a) $\begin{pmatrix}1&0&1&0&1\\0&1&0&0&2\\0&0&0&1&3\end{pmatrix}$; (b) $\begin{pmatrix}0&1&1&0&1\\1&0&0&0&2\\0&0&0&1&3\end{pmatrix}$;

(c) $\begin{pmatrix}1&0&1&0&1\\0&1&0&1&2\\0&0&0&1&3\end{pmatrix}$; (d) all of the above.

【解】 選 (a) (綜線CH3定義4)

03A07 【 清大84工工[7] 】

Let $T:\mathbb{R}^2\longrightarrow\mathbb{R}^3$ be a linear transformation such that

$T(x_1, x_2)=(x_1+2x_2, -x_1-3x_2, -3x_1-2x_2)$.

Find x such that $T(x)=(-4, 7, 0)$. (10%)

【解】

解方程式 $\begin{cases}x_1+2x_2=-4\\-x_1-3x_2=7\\-3x_1-2x_2=0\end{cases}$

$$\left[\begin{array}{cc|c} 1 & 2 & -4 \\ -1 & -3 & 7 \\ -3 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & -4 \\ 0 & -1 & 3 \\ 0 & 4 & -12 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{array}\right]$$

$\therefore\ x=(2,-3)$

03A08 【交大85工工[1]】

In coding a message, a blank space was represented by 0, an A by 1, a B by 2, a C by 3, and so on. The message was transformed using the matrix T and sent out as

$$M=\begin{bmatrix} 4 \\ 10 \\ 7 \end{bmatrix}, \text{ where } T=\begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 3 \\ 2 & 3 & 2 \end{bmatrix}. \quad \text{What was the message ?}$$

【解】解 $TX=M$,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 5 & 3 & 10 \\ 2 & 3 & 2 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 0 & -1 \end{array}\right] \sim \ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore\ X=[1\ \ 1\ \ 1]^T$, $\therefore$ 原訊息爲 AAA .

03A09 【成大84資工[3]】

A Cryptosystem is one in which a meaningful message block M, called the plaintext, is enciphered (or transformed) into a meaningless message block, called the ciphertext. This transformation is usually specified by a key in such a way that only the authorized users who knows the key can decipher (recover) the ciphertext. Assume that you are an attacker. Knowing that the transformation of a given cryptosystem is a 3*3 matrix transformation and also knowing the following pairs of plaintex–ciphertext:

$(M_1, C_1) = (2, 1, 2), (3, 15, 8)$;

$(M_2, C_2) = (0, 6, -2), (-6, 14, -2)$;

$(M_3, C_3) = (0, 3, -1), (-3, 7, -1)$;

$(M_4, C_4) = (4, 1, 1), (7, 21, 13)$.

(a) Find the enciphering and deciphering keys of the cryptosystem.(10%)

(b) Compute the plaintext of the ciphertext $C = (0, 8, 4)$. (5%)

(c) What is the requirement(s) for a matrix to be a transformation in a cryptosystem. (5%)

【分析】本題含有下列專有名詞:

cryptosystem(密碼系統), plaintext(原始文字), ciphertext(加密後的文字), encipher(加密), decipher(解密), attacker(破解者). 這些對大多數考生可能是生字, 但只須保持鎮靜細心閱讀, 仍可判知題意.

【解】(a) 設所求之 encipher key 為3×3矩陣X. 由題義得知

$$M_i X = C_i \quad , i = 1,2,3,4 .$$

即:

$$(2, 1, 2)\,X = (3, 15, 8) ,$$
$$(0, 6, -2)\,X = (-6, 14, -2) ,$$
$$(0, 3, -1)\,X = (-3, 7, -1) ,$$
$$(4, 1, 1)\,X = (7, 21, 13) .$$

可合併得

$$\begin{bmatrix} 2 & 1 & 2 \\ 0 & 6 & -2 \\ 0 & 3 & -1 \\ 4 & 1 & 1 \end{bmatrix} X = \begin{bmatrix} 3 & 15 & 8 \\ -6 & 14 & -2 \\ -3 & 7 & -1 \\ 7 & 21 & 13 \end{bmatrix}$$ (綜線CH2定理7①)

以列運算解此方陣方程式內的3*3矩陣X: (綜線CH3範例12a之解說)

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 3 & 15 & 8 \\ 0 & 6 & -2 & -6 & 14 & -2 \\ 0 & 3 & -1 & -3 & 7 & -1 \\ 4 & 1 & 1 & 7 & 21 & 13 \end{array}\right] \begin{array}{l} (-2) \\ \leftarrow \\ (-2) \\ \leftarrow \end{array} \sim \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 3 & 15 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & -1 & -3 & 7 & -1 \\ 0 & -1 & -3 & 1 & -9 & -3 \end{array}\right]$$

$$\sim \;.......\; \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 4 & 3 \\ 0 & 1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

解得 $X = \begin{bmatrix} 2 & 4 & 3 \\ -1 & 3 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

再以列運算解deciphering key X^{-1} (綜線CH3範例12b)

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \sim \;....\; \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & 1/2 & -9/4 \\ 0 & 1 & 0 & 1/4 & 1/2 & -3/4 \\ 0 & 0 & 1 & -1/2 & -1 & 5/2 \end{array}\right]$$

解得 $X^{-1} = \begin{bmatrix} 3/4 & 1/2 & -9/4 \\ 1/4 & 1/2 & -3/4 \\ -1/2 & -1 & 5/2 \end{bmatrix}$

(b) 所求爲

$$CX^{-1}=[\,0 \quad 8 \quad 4\,]\begin{bmatrix} 3/4 & 1/2 & -9/4 \\ 1/4 & 1/2 & -3/4 \\ -1/2 & -1 & 5/2 \end{bmatrix} = [\,0 \quad 0 \quad 4\,]$$

(c) 須為可逆矩陣.

０３A**１０**【 交大85資料[8] 】

[填充題]

(1) 用Gauss消去法將一個$n*n$矩陣A作P^TLU分解, 問其所作＋ － × ÷ 這些基本運算之time complexity是多少 ? (2%)

(2) 已將一個$n*n$矩陣A作P^TLU分解後問解線性方程組$AX=b$之 time complexity 是多少 ? 其中A是$n*n$, X是$n*1$, b是$n*1$. (1%)

【解】 (1) $O(n^3)$ (2) $O(n^2)$

【說明】請先參閱綜線CH3演算法4a.

本題問的是complexity, 應該用$O(...)$ 表示, 而不是要算真正的次數.

(1) 將某列遍乘一數需做n次乘法, 複雜度是$O(n)$. 將某列的k倍加入另一列需做n次乘法及n次加法, 共$2n$次運算, 複雜度也是$O(n)$. 將某兩列對調不必做運算, 複雜度是$O(1)$, 但它需做$3n$次寫入動作, 由演算法的立場來看, 複雜度應該是$O(n)$. 準備好一個pivot(列對調)並將它下方都消成零(頂多n次的列加入)需$O(n)+nO(n)=O(n^2)$次運算. 最多有n個pivot, 所以整個過程需要 $nO(n^2)=O(n^3)$次運算.

(2) 由$P^TLUX=b$, 可令 $LUX=b_1$, $UX=b_2$.

1° 解 $P^Tb_1=b$ 的 b_1 只是將b內的n個數重排, 它的複雜度依P的資料結構而定, 需要$O(n)$或$O(n^2)$, 頂多是$O(n^2)$.

2° b_1求得後, 欲解 $Lb_2=b_1$, 因L為下三角矩陣, 可先解出b_2內的第一個數, 再代回解第二個數, 如此依序解出整個b_2在此過程的每一回, 將已求得的數代回頂多要$O(n)$次寫入, 相乘頂多要$O(n)$次, 相加,移項, 並解出一個數頂多要$O(n)$次, 共需$O(n)+O(n)+O(n)=O(n)$次運算. 因需解n次, 所以共需

$nO(n)=O(n^2)$次運算.

3° b_2求得後, 欲解 $UX=b_2$, 因U為上三角矩陣, 可先解出X內的最後一個數, 再代回解倒數第二個數, 如此依序解出X. 這個過程與2°類似, 也是需要 $nO(n)=O(n^2)$次運算.

$\therefore$ 全部共需$O(n^2)+O(n^2)+O(n^2)=O(n^2)$次運算.

題型03B: 方程式解的性質

03B01 【交大86資工[6](a)】

[是非倒扣題]

A consistent linear system with coefficient matrix A has an infinite number of solutions if and only if A can be row–reduced to an echelon matrix that includes some column containing no pivot.

【解】True. (綜線CH3定理10)

03B02 【中央86資工[1](a)(b)】

[是非論證題]

(a) A consistent linear system has infinitely many solutions if and only if at least one column in the coefficient matrix does not contain a pivot position.

(b) The linear system $Ax=o$ always has solution.

【解】(a) True.

對有解(consistence)的方程組來說,

有無限多解 $\Longleftrightarrow$ 有自由變數 $\Longleftrightarrow$ 有某行不含pivot位置. (綜線CH3範例7)

【解】(b) True.

$x=o$已是一個解.

03B03 【清大86工工[1](abefgij)】

[是非論證題]

(a) A homogeneous system always has a solution.

(b) A nonhomogeneous system has a nontrival solution if and only if the numbers of equations of the system equal the numbers of the variables.

(e) Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and let A be the standard matrix of T. Then T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^n$.

(f) If an $m \times n$ matrix A has m pivot columns, then the equation $Ax=b$ has a unique solution for every b in $\mathbb{R}^m$

(g) $A_{n\times n}$ is an invertible matrix iff the equation $Ax=o$ has nontrivial solutions.

(i) Given a $b\in\mathbb{R}^m$, if the equation $Ax=b$ is consistent, then the column space of $A_{m\times n}$ is $\mathbb{R}^m$

(j) Any complex system can be simplified by investigating its corresponding eigensystem.

【解】(a) Yes. 至少零向量是一個解. (綜線CH3定理11)

(b) No. 舉例說明如下:

$$\begin{cases} x+y=1, \\ 2x+2y=3 \end{cases}$$ 無解. (綜線CH3範例8)

(e) No.

矩陣A是$m\times n$矩陣, 它的column都是$m\times 1$矩陣, 通常不在$\mathbb{R}^n$之中.

若將題目改爲 "... the columns of A span $\mathbb{R}^m$" 則答案變成Yes.
詳情請參閱綜線CH6定理7. 本題極可能是命題上的筆誤, 作答時應儘量解釋清楚.

(f) No.

例如2x3矩陣$A=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$有2個pivot column, 但$Ax=b$總是有無限多組解. (綜線CH3範例7)

(g) No. 剛好講反了. (綜線CH3定理18)

A可逆時, $Ax=o \implies x=A^{-1}o=o$, 必無nontrivial解.

(i) No. $Ax=b$有解只代表b落在A的column space中. (綜線CH3定理2)

例如 $A=\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$, $b=\begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

使$Ax=b$有解, 但A的column space並非$\mathbb{R}^2$.

(j) No.

依第(1)(2)小題的用詞, complex system 應是指複數係數的聯立方程式. 也就是$Ax=b$, $A\in\mathbb{C}^{m\times n}$, $x\in\mathbb{C}^{n\times 1}$, $b\in\mathbb{C}^{m\times 1}$.

因必須$m=n$才可能討論A的eigenvalue，所以本題答No.

$$\left(\begin{array}{l}\text{當}A\text{爲方陣時，必有可逆矩陣}P\text{，及三角矩陣}R\text{，使}A=PRP^{-1}\\ \text{令}x=Py\text{，則原方程式可簡化爲}Ry=P^{-1}b \qquad \text{(CH13定理10)}\end{array}\right)$$

03B04 【中央85資工[2](abc)】

[是非論證題]

(a) A linear system with fewer equations than variables cannot have a unique solution.

(b) Two linear systems $Ax=b$ and $Bx=c$ are equivalent if and only if A and B are row equivalent.

(c) If a linear system has no free variables, then it has a unique solution.

【分析】(b) "A and B are row eauivalent" 只是必要條件. 充要條件應是 "$[A \mid b]$ and $[B \mid c]$ are row eauivalent" (綜線CH3定理5)

(c) 將矩陣化爲梯形後, pivot所在的行所對應的未知數稱爲basic variable, 而其它的未知數稱爲free variable. (綜線CH3範例7)

【解】(a) T,

將此linear system寫成矩陣形$Ax=b$, (綜線CH3定義1)

設A爲$m\times n$矩陣，則由題意得知$m<n$.

$\therefore$ rank$A\leq m<n$ (綜線CH8定理15)

$\therefore$ A有$n-$rank$A$個free variable. (綜線CH3範例7)

$\therefore$ $Ax=b$ 無解或不止一解. (綜線CH3定理10)

(b) F,

例如 $\begin{bmatrix}1&1\\0&0\end{bmatrix}\begin{bmatrix}p\\q\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}2&2\\0&0\end{bmatrix}\begin{bmatrix}p\\q\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$

並不equivalent. (綜線CH3定義1)

但 $\begin{bmatrix}1&1\\0&0\end{bmatrix}$ 與 $\begin{bmatrix}2&2\\0&0\end{bmatrix}$ row equivalent. (綜線CH3定義3)

(c) F,

例如 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

沒有free variable, 但卻無解. (綜線CH3定理10)

０３B**05** 【 中央84資工[1](abcd) 】

True or False. (一定要有說明、證明或反例. 每小題5分)

(a) Every matrix is row equivalent to a unique matrix in echelon form.

(b) For the linear system $A_{m\times n}X_{n\times 1}=b_{m\times 1}$. A has infinitely many solution if and only if at least one column of A doesn't contain a pivot position.

(c) The linear system $Ax=b$ with more equations than variables cannot have a unique solution.

(d) If the columns of A are linearly independent, then the linear system $Ax=b$ has solution.

【解】(a) False.

例如 $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 彼此列等價, (綜線CH3定義3)

而它們都是echelon form. (綜線CH3定義4)

[討論]

本題若將echelon form改為row–reduced echelon form, 則答案為True.

row–reduced echelon form 的唯一性的證明相當煩瑣.

Hoffman & Kunze: 證在Sec2.5,Theorem 11的Corollary. (篇幅一頁)

Noble & Naniel: 證在Ch4 Theorem4.5 及 Theorem4.11. (篇幅共二頁)

Friedberg, Insel, & Spence : 未證.

而且因各教科書作者對row–reduced echelon form的定義的描述方式不同而有不同的講法.

(b) False.

本題only–if part對，但if part 不對.

例如 $\begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ 第二column不含pivot，但無解.

(c) False.

例如 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$

恰有一組解. 但展開後有二變數，三等式.

(d) False.

例如 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ 無解. 但 $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ 線性獨立.

０３B**06** 【 元智83工工[4] 】

[是非題]

If rank$(A, b)=$rank$(A)+1$, then $Ax=b$ has a solution.

【解】 ×，剛好講反了. (綜線CH3定理10)

０３B**07** 【 元智83工工[6] 】

[是非題]

Let A be a m by n matrix with rank $r\leq m<n$ Then the system $Ax=o$ always has nontrival solutions. (2%)

【解】 ○，此爲定理. (綜線CH3定理11③)

０３B**08** 【 交大79工工[9] 】

是非題:

If A is an $m\times n$ matrix with $m<n$, then the linear system $AX=B$ has a solution for every $m\times 1$ matrix B.

【解】非

【討論】反例如下:

取 $m=2, n=3$,

方程式 $\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}$ 無解

０３B**09** 【 清大82工工[4] 】

Let A be an $m\times n$ matrix and b an $m\times 1$ column vector. If A is a square nonsingular matrix, prove that $Ax=b$ has a solution and that this solution is unique.

【解】 $\because$ A 爲nonsingular, $\therefore$ A^{-1}存在

令 $x=A^{-1}b$, 則 $Ax=A(A^{-1}b)=(AA^{-1})b=Ib=b$

$\therefore$ $Ax=b$ 有解.

若 $Ax_1=b$, 且 $Ax_2=b$, 則

$x_1=A^{-1}(Ax_1)=A^{-1}b=A^{-1}(Ax_2)=x_2$

$\therefore$ $Ax=b$ 的解唯一 .

０３B**10** 【 交大79工工[2] 】

是非題:

If $|A|=0$, then the linear system $AX=B$, $B\neq o$, has no solution

【解】非

【討論】反例如下:

取 $A=\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$, $B=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

則 $\det A=\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix}=0$, 且$B\neq o$,

但方程式$AX=B$有無限多個解: $\begin{bmatrix} 1-t \\ t \end{bmatrix}$, t為任意常數

03B11 【 清大86工工[3] 】

Determine whether the linear systems are consistent. If they are consistent, are their solutions "unique" ? Given the reason for your answer. (10%)

(a) $2x_2+2x_3=0$

$x_1-2x_4=-3$

$x_3+3x_4=-4$

$-2x_1+3x_2+2x_3+x_4=4$

(b) $-3x_2-6x_3+4x_4=9$

$-x_1-2x_2-x_3+3x_4=1$

$-2x_1-3x_2+3x_4=-1$

$x_1+4x_2+5x_3-9x_4=-7$

【解】(a)

$$\left[\begin{array}{cccc|c} 0 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & -2 & -3 \\ 0 & 0 & 1 & 3 & -4 \\ -2 & 3 & 2 & 1 & 4 \end{array}\right] \underset{\sim \;\cdots\cdots\; \sim}{\text{經列運算}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 & -6 \end{array}\right]$$

此方程式無解(Not consistent). (綜線CH3定理10)

(b)
$$\left[\begin{array}{cccc|c} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array}\right] \underset{\cdots\cdots}{\overset{\text{經列運算}}{\sim}} \sim \left[\begin{array}{cccc|c} -1 & -2 & -1 & 3 & 1 \\ 0 & -3 & -6 & 4 & 9 \\ 0 & 0 & 0 & -5/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

此方程式有解(consistent), 解不唯一. (綜線CH3定理10)

０３B**１２** 【 元智81工工[4] 】

設
$$\begin{cases} x_1+x_2+x_3+x_4+x_5+x_6+x_7=0 \\ x_1+x_3=0 \\ 2x_1+x_5+3x_7=0 \\ -x_1+x_3+x_5+x_7=0 \\ \sqrt{3}\,x_1+x_4-x_5-x_7=0 \\ x_1+x_2-x_3+x_4-x_5+x_6-x_7=0 \end{cases}$$

爲一線性聯立方程組。 試問此方程組是否會有nontrivial解? (即不爲零之解)

【參考章節】 綜線CH3定理11④

【解】此方程組必有nontrivial解. 理由如下:

令其係數矩陣爲 A, 則 A 爲 6×7 矩陣. (綜線CH3定義1)

考慮線性映射 $T: \mathbb{R}^7 \longrightarrow \mathbb{R}^6$, $T(x)=Ax$ (綜線CH7定理6)

則 $\dim(\mathbb{R}^7)=\dim \operatorname{Ker}T + \dim \operatorname{Im}T$ (綜線CH8定理8)

$\leq \dim \operatorname{Ker}T + \dim \mathbb{R}^6$ (綜線CH8定義5要訣2, CH6定理22a)

$\therefore\ 7 \leq \dim \operatorname{Ker}T + 6$

$\therefore\ \dim \operatorname{Ker}T \geq 1$

$\therefore\ \operatorname{Ker}T \neq \{o\}$ (綜線CH6定理19)

$\therefore\ \exists\, x\neq o$ 使得 $Tx=0$ (綜線CH8定義5)

$\therefore\ \exists\, x\neq o$ 使得 $Ax=0$

【加強演練】

已知 $[a_1, a_2, a_3, a_4]$, $[b_1, b_2, b_3, b_4]$, $[c_1, c_2, c_3, c_4]$ 兩兩之間

都不成比例, 問下列方程式是否可能有解?

$$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \\ a_3x+b_3y=c_3 \\ a_4x+b_4y=c_4 \end{cases}$$

[解] 可能. 例如

$$\begin{cases} x+2y=3 \\ 4x+5y=6 \\ 5x+7y=9 \\ x+y=1 \end{cases}$$

０３B**13** 【 交大82資工[3](a) 】

Determine true or false for the following statements. For each statements, you can obtain 2 points for a correct answer, –1 point for a wrong answer, and 0 point for a blank answer.

(a) Let $Ax=b$ be a system representating two lines in a plane. These two lines might coincide if $Ax=b$ is consistent.

【解釋名詞】consistent 表示方程式有解, coincide 爲重合

【解】(a) True. 解說如下:

$Ax=b$ 表示平面上的兩線, 意指

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} t \\ u \end{bmatrix},$$

即

$$\begin{cases} px_1+qx_2=t \\ rx_1+sx_2=u \end{cases}$$

此方程式有解時可能代表

(1) 交於一點的兩線 (恰有一解), 或

(2) 重合的兩線 (有無限多組解)。

０３Ｂ**14** 【 中央84資工[2] 】

Give four methods to determine a linear system $A_{m\times n}\, x_{n\times 1} = b_{m\times 1}$ has solution.

【分析】本題涉及CH1, CH3, CH5, CH8, CH9. 本題無標準答案, 各人依經驗作答.

【解】(1) 直接利用列運算(高斯消去法)求解, 就可判定是否有解.

(1') $Ax=b$有解 $\Longleftrightarrow$ $\text{rank}[A \mid b]=\text{rank}A$

(2) $Ax=b$可切成m個n變數的一次方程式. 各代表$\mathbb{R}^n$ 中的一個集合(通常是$n-1$維的超平面),

原方程式有解 $\Longleftrightarrow$ 此m個集合的交集非空.

(3) $Ax=b$有解 $\Longleftrightarrow$ $b\in$ [A的column space]

(3') $Ax=b$有解 $\Longleftrightarrow$ b可表爲A的column的線性組合.

(3") 考慮線性映射T：$\mathbb{R}^n \longrightarrow \mathbb{R}^m$, $T(x)=Ax$.

$Ax=b$ 有解 $\Longleftrightarrow$ $b\in$[T的值域]

(4) $Ax=b$有解 $\Longleftrightarrow$ b對[A的column space]的正投影仍爲b.

(4') 任取$A^TAx=A^Tb$的一個解u. (綜線CH9定理21a要訣4)

$Ax=b$有解 $\Longleftrightarrow$ $Au=b$ (綜線CH9定理21a要訣1)

０３Ｂ**15** 【 交大84資科[2]& 】

The equation of a strait line in the x–y plane has the form $ax+by=c$. Consider three strait lines, with equations

$$a_i x+b_i y=c \quad \text{for } i=1,2,3.$$

Prove that if the three lines all pass through a common point (x,y) and only (x,y), then $\det[A,B,C]=0$, where A,B,C are 3×1 and $\langle A\rangle_i = a_i$, $\langle B\rangle_i = b_i$, $\langle C\rangle_i = c_i$.

【討論】(1) 題目中 " and only (x, y) " 的條件並不需要.

(2) 這個題目在高中數學就已出現過, 但在此應套用線性代數的定理來解題.

【解】由題意可知: 方程式

$$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \\ a_3x+b_3y=c_3 \end{cases}$$

有解. 所以

$$\operatorname{rank}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \operatorname{rank}\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix}$$ (綜線CH8定理18)

≤ 2 (綜線CH8定理15)

所以 $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ 不可逆. (綜線CH8定理17)

所以 $\det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = 0$ (綜線CH4定理17)

０３B **16** 【 淡江83資工[1] 】

If A is a nonsquare matrix, why does there not exist a matrix that $AX=I$, $XA=I$ even when the two I's are of different sizes? (Hint: If A is $m\times n$ with $m>n$, there is a nonzero Y such that $YA=O$, contrary to the relation $O=YAX=YI=Y$. Hence $AX=I$ cannot hold.)

【分析】綜線CH8定理16a的解法較簡單，也較標準. 但與題目提示的方向不符.

【解】設A爲$m\times n$矩陣.

∵ A是nonsquare, ∴ $m\neq n$.

若$m>n$, 假設$AX=I$成立，欲導出矛盾.

考慮方程式$YA=O$, Y爲$1\times m$矩陣, O爲$1\times n$零矩陣.

此方程式展開後爲有m個未知數, n個等式的聯立方程組.

∵ $m>n$,

∴ 此方程組經列運算求解，化爲row–reduced echelon form後的非零列數小於等於n，也就小於m.

∴ 此方程組有自由變數.

∴ 此方程組有不全爲零的解. (綜線CH3定理11④)

∴ 存在$Y \neq O$, 使得$YA = O$.

而 $Y = YI = YAX = OX = O$, 得出矛盾.

若$m < n$, 假設$XA = I$成立, 欲導出矛盾.

考慮方程式$AY = O$, Y爲$n\times1$矩陣, O爲$m\times1$零矩陣.

此方程式展開後成爲有n個未知數m個等式的聯立方程組.

∵ $m < n$,

∴ 此方程組經列運算求解, 化爲row–reduced echelon form後的非零列數小於等於m, 也就小於n.

∴ 此方程組有自由變數.

∴ 此方程組有不全爲零的解. (綜線CH3定理11④)

∴ 存在$Y \neq O$, 使得$AY = O$.

而 $Y = IY = XAY = XO = O$, 得出矛盾.

題型03C:　文字方程式

０３Ｃ**０１**【 交大81資料[1] 】

The matrix below is the augmented matrix for some system of equations. Find the values of the parameters k for which the system has no, one, and infinitely many solutions; when there are infinitely many, find the general form of the solution in terms of arbitrary parameters.

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & k & 6 & 6 \\ -1 & 3 & k-3 & 0 \end{array}\right]$$

【解】對增廣矩陣執行列運算:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & k & 6 & 6 \\ -1 & 3 & k-3 & 0 \end{array}\right] \begin{array}{c} (-2)\quad(1) \\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & k+4 & 0 & 4 \\ 0 & 1 & k & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & k & 1 \\ 0 & k+4 & 0 & 4 \end{array}\right] (-(k+4)) \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & k & 1 \\ 0 & 0 & -k(k+4) & -k \end{array}\right]$$

以下分三種情況分別求解:

1°若$k=-4$, 則得

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right],$$

此時無解.

2°若$k=0$, 則得

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

(2)

$$\therefore \text{ 解爲} \begin{cases} x_1 = 3-3t \\ x_2 = 1 \\ x_3 = t \end{cases}, \ t\text{爲任意參數}$$

3°若$k\neq-4$ 且$k\neq0$, 則第三列再除以 $\dfrac{-1}{k(k+4)}$:

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & k & 1 \\ 0 & 0 & 1 & 1/(k+4) \end{array}\right] \begin{array}{c} \\ \\ (-k) \quad (-3) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 0 & (k+1)/(k+4) \\ 0 & 1 & 0 & 4/(k+4) \\ 0 & 0 & 1 & 1/(k+4) \end{array}\right] \begin{array}{c} \\ (2) \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & (k+9)/(k+4) \\ 0 & 1 & 0 & 4/(k+4) \\ 0 & 0 & 1 & 1/(k+4) \end{array}\right]$$

$$\therefore \text{ 恰有一解: } \begin{cases} x = (k+9)/(k+4) \\ y = 4/(k+4) \\ z = 1/(k+4) \end{cases}$$

０３Ｃ**02** 【 交大82資料[4] 】

For what value of a does the system have one solution, no solution, infinitely many solutions

$$\begin{aligned} x+2y \quad\quad -3z &= 4 \\ 3x-\ y \quad\quad +5z &= 2 \\ 4x+\ y+(a^2-14)z &= a+2 \end{aligned}$$

【解】

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{array}\right] \begin{array}{c}(-3)\ (-4)\\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{array}\right] \begin{array}{c} \\ (-1) \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{array}\right]$$

$\therefore$ 當$a^2\neq 16$時恰有一解，當$a=-4$時無解，當$a=4$時有無限多解.

０３Ｃ**03** 【交大84資料[3]】

Find the values of k for which the following equations possess no, one, and infinitely many solutions.

$$x-2y+3y=1$$
$$2x+ky+6z=6$$
$$-x+3y+(k-3)z=0$$

【解】以列運算解此方程式:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 2 & k & 6 & 6 \\ -1 & 3 & k-3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & k+4 & 0 & 4 \\ 0 & 1 & k & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & k & 1 \\ 0 & k+4 & 0 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 1 \\ 0 & 1 & k & 1 \\ 0 & 0 & -k(k+4) & -k \end{array}\right]$$

$k=-4$時無解. $k=0$ 時有無限多解. $k\in\mathbb{R}\setminus\{0,-4\}$時恰有一解.

03C**04** 【清大78資料[1]】

Let $A=\begin{bmatrix} 1 & 9 & 8 & 9 \\ 0 & 5 & 1 & 4 \\ 1 & -1 & 6 & a \end{bmatrix}$

Find the null space of A for every $a\in\mathbb{R}$.

【解】 1. 設N為A的null space, 則 $x\in N \Longleftrightarrow Ax=0$.

令$x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, 則 $\begin{cases} x_1+9x_2+8x_3+9x_4=0 \\ \quad\;\; 5x_2+\ x_3+4x_4=0 \\ x_1-\ x_2+6x_3+ax_4=0 \end{cases}$

對A作列運算

$$\begin{bmatrix} 1 & 9 & 8 & 9 \\ 0 & 5 & 1 & 4 \\ 1 & -1 & 6 & a \end{bmatrix} \sim \begin{bmatrix} 1 & 9 & 8 & 9 \\ 0 & 5 & 1 & 4 \\ 0 & -10 & -2 & a-9 \end{bmatrix} \sim \begin{bmatrix} 1 & 9 & 8 & 9 \\ 0 & 5 & 1 & 4 \\ 0 & 0 & 0 & a-1 \end{bmatrix}$$

對a加以討論,

(ⅰ) 若$a\neq 1$, 則繼續計算如下:

$$\sim \begin{bmatrix} 1 & 9 & 8 & 9 \\ 0 & 5 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 9 & 8 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} (-8)$$

$$\sim\begin{bmatrix} 1 & -31 & 0 & 0 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (\text{註}1)$$

$$\therefore \begin{cases} x_1 - 31x_2 \qquad\quad = 0 \\ \qquad 5x_2 + x_3 \quad = 0 \\ \qquad\qquad\quad x_4 = 0 \end{cases}$$

$$\therefore \begin{cases} x_1 = -31t_2 \\ x_2 = t_2 \\ x_3 = -5t_2 \\ x_4 = 0 \end{cases} \qquad , t_2 \text{ 爲任意常數}$$

$$\therefore\ a \neq 1\text{時}A\text{的null space爲}\left\{ t\begin{bmatrix} -31 \\ 1 \\ -5 \\ 0 \end{bmatrix} \middle|\ t \in \mathbb{R} \right\}$$

(ⅱ) 若$a=1$, 則繼續計算如下:

$$\begin{bmatrix} 1 & 9 & 8 & 9 \\ 0 & 5 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} (-8) \sim \begin{bmatrix} 1 & -31 & 0 & -23 \\ 0 & 5 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (\text{註}2)$$

$$\text{即}\begin{cases} x_1 - 31x_2 \qquad - 23x_4 = 0 \\ \qquad 5x_2 + x_3 + \ \ 4x_4 = 0 \end{cases}$$

$$\therefore \begin{cases} x_1 = 31t_2 + 23t_4 \\ x_2 = t_2 \\ x_3 = -5t_2 - 4t_4 \\ x_4 = t_4 \end{cases}, t_2, t_4 \text{ 為任意常數}$$

$\therefore$ $a = 1$時A的null space為

$$\left\{ t_2 \begin{bmatrix} 31 \\ 1 \\ -5 \\ 0 \end{bmatrix} + t_4 \begin{bmatrix} 23 \\ 0 \\ -4 \\ 1 \end{bmatrix} \middle| \; t_2, t_4 \in \mathbb{R} \right\}$$

註１． 照“正規”的高斯消去法至此應將第二列除以5, 並作如下之運算.

$$\sim \begin{bmatrix} 1 & 9 & 8 & 0 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} (-9) \sim \begin{bmatrix} 1 & 0 & 31/5 & 0 \\ 0 & 1 & 1/5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

但此法引入分數, 不利於筆算.

註２． 同上, 若在此先做第二列除以5, 則最後答案的表示法會稍有不同, 但這只相當於取了另一個基底, 並不是做錯.

０３C**05** 【 清大75資科[1] 】

Compute the solution of the system

$ax + y + z = 1$

$x + ay + z = 1$

$x + y + az = 1$

for all possible values of a.

【解】請參閱綜線CH3範例9.

０３C**06** 【 清大85工工[3] 】

Discuss the solution set of the following system of equations with α

an arbitrary parameter.

$$\begin{aligned} x-3y&=-2\\ 3x-2y&=\alpha\\ 2x+y&=3 \end{aligned}\qquad (10\%)$$

【解】對原方程式的分隔矩陣做列運算:

$$\left[\begin{array}{cc|c} 1 & -3 & -2\\ 3 & -2 & \alpha\\ 2 & 1 & 3 \end{array}\right]\sim\left[\begin{array}{cc|c} 1 & -3 & -2\\ 0 & 7 & \alpha+6\\ 0 & 7 & 7 \end{array}\right]\sim\ \ldots\ \sim\left[\begin{array}{cc|c} 1 & 0 & 1\\ 0 & 1 & 1\\ 0 & 0 & \alpha-1 \end{array}\right]$$

當 $\alpha\neq 1$時, 解集合為空集合. (綜線CH3範例7,8,9)

當 $\alpha=1$時, 解集合為 $\{[1\quad 1]^T\}$

０３C**０７** 【 交大83工工[6] 】

Find all right–hand sides b for which $Ax=b$ has solutions, and find all solutions, where

$$A=\begin{bmatrix} 4 & -1 & 2 & 6\\ -1 & 5 & -1 & -3\\ 3 & 4 & 1 & 3 \end{bmatrix}$$

【解】設$b=[p\quad q\quad r]^T$ ，試解$Ax=b$.

$$\left[\begin{array}{cccc|c} 4 & -1 & 2 & 6 & p\\ -1 & 5 & -1 & -3 & q\\ 3 & 4 & 1 & 3 & r \end{array}\right]\sim\left[\begin{array}{cccc|c} 0 & 19 & -2 & -6 & p+4q\\ -1 & 5 & -1 & -3 & q\\ 0 & 19 & -2 & -6 & 3q+r \end{array}\right]$$

$$\sim\left[\begin{array}{cccc|c} 0 & 19 & -2 & -6 & p+4q\\ -1 & 5 & -1 & -3 & q\\ 0 & 0 & 0 & 0 & -p-q+r \end{array}\right]$$

$\therefore$ 有解 $\Longleftrightarrow -p-q+r=0 \Longleftrightarrow r=p+q$

$$\Longleftrightarrow b=\begin{bmatrix} p \\ q \\ p+q \end{bmatrix}, p,q\text{爲任意數}.$$

有解時繼續求解如下:

$$\sim\left[\begin{array}{cccc|c} 1 & -5 & 1 & 3 & -q \\ 0 & 1 & -2/19 & -6/19 & p/19+4q/19 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim\left[\begin{array}{cccc|c} 1 & 0 & 9/19 & 27/19 & 5p/19+q/19 \\ 0 & 1 & -2/19 & -6/19 & p/19+4q/19 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore x=\begin{bmatrix} -(9/19)s-(27/19)t+5p/19+q/19 \\ (2/19)s+(6/19)t+p/19+4q/19 \\ s \\ t \end{bmatrix}, \quad s,t\text{爲任意數}.$$

#

03C08 【 中正85資工[2] 】

Let $A=\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 1 \end{bmatrix}$ and $b=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(a) Under what conditions on b so that $Ax=b$ has a solution.

(b) Find a basis for the nullspace of A.

(c) Find the general solution to $Ax=b$, where a solution exists.

【解】先對分隔矩陣做列運算:

$$\left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 2 & 4 & 0 & 1 & b_3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 3 & b_1 \\ 0 & 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & -5 & -2b_1+b_3 \end{array}\right]$$

$$\sim \;.... \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -b_1/5+3b_3/5 \\ 0 & 0 & 0 & 1 & 2b_1/5-b_3/5 \\ 0 & 0 & 0 & 0 & b_2 \end{array}\right]$$

(a) 有解的充要條件爲 $b_2=0$. (綜線CH3定理10)

(b) $x\in$ null space of $A \iff$ (綜線CH3範例7)

$$x=\begin{bmatrix} -2s \\ s \\ t \\ 0 \end{bmatrix}=s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}+t\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad s,t\text{爲任意純量.}$$

$$\therefore \text{ 基底可取爲 } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(c) $Ax=b$的通解爲 (綜線CH3範例7)

$$x = \begin{bmatrix} -2s - b_1/5 + 3b_3/5 \\ s \\ t \\ 2b_1/5 - b_3/5 \end{bmatrix}, \quad s, t \text{為任意純量}.$$

題型03D: 求算逆矩陣

03D01 【清大83資料[1]】

Let $L=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 2 & 1 & 1\end{bmatrix}$, Find L^{-1}. (5%)

【解】做列運算:

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ -2 & 1 & 0 & 0 & 1 & 0\\ 2 & 1 & 1 & 0 & 0 & 1\end{array}\right]\sim\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 2 & 1 & 0\\ 0 & 1 & 1 & -2 & 0 & 1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 2 & 1 & 0\\ 0 & 0 & 1 & -4 & -1 & 1\end{array}\right] \qquad \therefore\ L^{-1}=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ -4 & -1 & 1\end{bmatrix}$$

03D02 【交大80工工[8]】

已知$A=\begin{bmatrix}3 & -2 & 1\\ 5 & 6 & 2\\ 1 & 0 & -3\end{bmatrix}$，求反矩陣$A^{-1}=?$

【解】

$$\left[\begin{array}{ccc|ccc} 3 & -2 & 1 & 1 & 0 & 0 \\ 5 & 6 & 2 & 0 & 1 & 0 \\ 1 & 0 & -3 & 0 & 0 & 1 \end{array}\right] (-5)(-3) \sim \left[\begin{array}{ccc|ccc} 0 & -2 & 10 & 1 & 0 & -3 \\ 0 & 6 & 17 & 0 & 1 & -5 \\ 1 & 0 & -3 & 0 & 0 & 1 \end{array}\right] (3)$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & -2 & 10 & 1 & 0 & -3 \\ 0 & 0 & 47 & 3 & 1 & -14 \\ 1 & 0 & -3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & 0 & 1 \\ 0 & -2 & 10 & 1 & 0 & -3 \\ 0 & 0 & 47 & 3 & 1 & -14 \end{array}\right] (1/47)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -3 & 0 & 0 & 1 \\ 0 & -2 & 10 & 1 & 0 & -3 \\ 0 & 0 & 1 & 3/47 & 1/47 & -14/47 \end{array}\right] (-10)\ (3)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/47 & 3/47 & 5/47 \\ 0 & -2 & 0 & 17/47 & -10/47 & -1/47 \\ 0 & 0 & 1 & 3/47 & 1/47 & -14/47 \end{array}\right] (1/-2)$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9/47 & 3/47 & 5/47 \\ 0 & 1 & 0 & -17/94 & 5/47 & 1/94 \\ 0 & 0 & 1 & 3/47 & 1/47 & -14/47 \end{array}\right]$$

$$\therefore\ A^{-1} = \begin{bmatrix} 9/47 & 3/47 & 5/47 \\ -17/94 & 5/47 & 1/94 \\ 3/47 & 1/47 & -14/47 \end{bmatrix}$$

０３D**03** 【 清大79資科[4](1) 】

(1) Compare A^{-1} where $A=\begin{bmatrix}1&1&0&1\\0&1&1&0\\0&0&1&1\\0&0&0&1\end{bmatrix}$

【勘誤】本題原題有筆誤，第一小題的Compare應改爲Compute.

【解】(1)

$$\left[\begin{array}{cccc|cccc}1&1&0&1&1&0&0&0\\0&1&1&0&0&1&0&0\\0&0&1&1&0&0&1&0\\0&0&0&1&0&0&0&1\end{array}\right]\begin{array}{c}\\ \\ \\(-1)\end{array}\sim\left[\begin{array}{cccc|cccc}1&1&0&0&1&0&0&-1\\0&1&1&0&0&1&0&0\\0&0&1&0&0&0&1&-1\\0&0&0&1&0&0&0&1\end{array}\right]\begin{array}{c}\\ \\(-1)\\ \\ \end{array}$$

$$\sim\left[\begin{array}{cccc|cccc}1&1&0&0&1&0&0&-1\\0&1&0&0&0&1&-1&1\\0&0&1&0&0&0&1&-1\\0&0&0&1&0&0&0&1\end{array}\right]\begin{array}{c}\\(-1)\\ \\ \\ \end{array}\sim\left[\begin{array}{cccc|cccc}1&0&0&0&1&-1&1&-2\\0&1&0&0&0&1&-1&1\\0&0&1&0&0&0&1&-1\\0&0&0&1&0&0&0&1\end{array}\right]$$

$$\therefore\ A^{-1}=\begin{bmatrix}1&-1&1&-2\\0&1&-1&1\\0&0&1&-1\\0&0&0&1\end{bmatrix}$$

０３D**04** 【 成大81資工丙[3](ab) 】

(a) Please find the inverse matrix of

$$A=\begin{pmatrix}1 & 1 & 1\\ 0 & 2 & 3\\ 5 & 5 & 1\end{pmatrix}$$

if possible. (5%)

(b) If $B=AX$,

please get the solution of X, given $B=\begin{pmatrix}2\\ 3\\ 4\end{pmatrix}$ (5%)

【解】

(a)

$$\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 2 & 3 & 0 & 1 & 0\\ 5 & 5 & 1 & 0 & 0 & 1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 2 & 3 & 0 & 1 & 0\\ 0 & 0 & -4 & -5 & 0 & 1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 2 & 3 & 0 & 1 & 0\\ 0 & 0 & 1 & 5/4 & 0 & -1/4\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1 & 1 & 0 & -1/4 & 0 & 1/4\\ 0 & 2 & 0 & -15/4 & 1 & 3/4\\ 0 & 0 & 1 & 5/4 & 0 & -1/4\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc}1 & 1 & 0 & -1/4 & 0 & 1/4\\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8\\ 0 & 0 & 1 & 5/4 & 0 & -1/4\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1 & 0 & 0 & 13/8 & -1/2 & -1/8\\ 0 & 1 & 0 & -15/8 & 1/2 & 3/8\\ 0 & 0 & 1 & 5/4 & 0 & -1/4\end{array}\right)$$

$$\therefore\ A^{-1}=\begin{pmatrix}13/8 & -1/2 & -1/8\\ -15/8 & 1/2 & 3/8\\ 5/4 & 0 & -1/4\end{pmatrix}=(1/8)\begin{pmatrix}13 & -4 & -1\\ -15 & 4 & 3\\ 10 & 0 & -2\end{pmatrix}$$

(b) $X=A^{-1}B$

$$=(1/8)\begin{pmatrix}13 & -4 & -1\\ -15 & 4 & 3\\ 10 & 0 & -2\end{pmatrix}\begin{pmatrix}2\\ 3\\ 4\end{pmatrix}=(1/8)\begin{pmatrix}10\\ -6\\ 12\end{pmatrix}=\begin{pmatrix}5/4\\ -3/4\\ 3/2\end{pmatrix}$$

０３Ｄ**05** 【 清大80資料[4]& 】

Let

$$A = \begin{bmatrix} 1 & -1 & \cdots & \cdots & \cdots & -1 \\ & 1 & -1 & \cdots & \cdots & \cdot \\ & & 1 & \cdots & \cdots & \cdot \\ & 0 & & & \cdots & \cdot \\ & & & & \cdot & -1 \\ & & & & & 1 \end{bmatrix} \in \mathbb{R}^{10\times 10}$$

find the inverse matrix A^{-1}

【解】

$$\left[\begin{array}{cccccccccc|cccccccccc}
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

$$\sim \left[\begin{array}{cccccccccc|cccccccccc}
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]$$

$$
\sim\left[\begin{array}{cccccccccc|cccccccccc}
1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\\
0 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\\
0 & 0 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2\\
0 & 0 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2\\
0 & 0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2\\
0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\sim\left[\begin{array}{cccccccccc|cccccccccc}
1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4\\
0 & 1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4\\
0 & 0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 4\\
0 & 0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 4\\
0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 4\\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 4\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

$$
\sim\left[\begin{array}{cccccccccc|cccccccccc}
1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8\\
0 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8\\
0 & 0 & 1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 4 & 8\\
0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 4 & 8\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 4 & 8\\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4 & 8\\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 4\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right]
$$

....

$$
\sim\left[\begin{array}{cccccccccc|cccccccccc}
1&0&0&0&0&0&0&0&0&0&1&1&2&4&8&16&32&64&128&256\\
0&1&0&0&0&0&0&0&0&0&0&1&1&2&4&8&16&32&64&128\\
0&0&1&0&0&0&0&0&0&0&0&0&1&1&2&4&8&16&32&64\\
0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&2&4&8&16&32\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&2&4&8&16\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&2&4&8\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&2&4\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1&2\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&1\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&1
\end{array}\right]
$$

分隔線右邊即爲所求之A^{-1}.

【另解】請參閱第二章

０３D**06**【 元智83工工[11] 】

Suppose that you want to compute the inverse of a 100 by 100 nonsingular matrix. What method should you use ? Why ?

【分析】本題的要求並不很清晰.

【解】將此矩陣(稱爲A)造出100×200的矩陣$[A,I]$, 用列運算將$[A,I]$化爲$[I,B]$, 所得到的B就是A的inverse. 如果左邊不能化成I, A就是不可逆. 由於此過程所耗費時間過大(需做$O(n^3)$＝一百萬次等級的乘算). 應該寫一個程式讓電腦來做這件事. 如果有現成的軟體可用就直接用. 如果朋友有現成的軟體就請他(她)幫忙. 如果......)

題型03E: 列矩陣與LU分解

03E01 【 清大81資料[16] 】

Let the matrices P, A, and Q be defined as follows.

$$P=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix},\ A=\begin{bmatrix}2&4&-2\\1&-6&7\\1&0&2\end{bmatrix},\ Q=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

What is PAQ ?

(a) $\begin{bmatrix}1&0&2\\1&-6&7\\2&4&-2\end{bmatrix}$ (b) $\begin{bmatrix}-2&4&2\\7&-6&1\\2&0&1\end{bmatrix}$ (c) $\begin{bmatrix}7&-6&1\\-2&4&2\\2&0&1\end{bmatrix}$

(d) $\begin{bmatrix}-6&1&7\\4&2&-2\\0&1&2\end{bmatrix}$ (e) $\begin{bmatrix}-6&7&1\\4&-2&2\\0&2&1\end{bmatrix}$

【參考章節】 綜線CH3定理15,範例13a 及 CH3定理24,範例23a.

【解】選(c). 計算如下: (直接乘或利用綜線CH3定理15,定理24):

$$PAQ=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&4&-2\\1&-6&7\\1&0&2\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

$$=\begin{bmatrix}1&-6&7\\2&4&-2\\1&0&2\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}=\begin{bmatrix}7&-6&1\\-2&4&2\\2&0&1\end{bmatrix}$$

03E**02** 【 交大85資料[9] 】

[填充題]

下列兩個矩陣是否能分解成為 elementary matrix 之乘積? 若能, 分解之. 若不能,說出理由.

(1) $A=\begin{bmatrix}0&1\\1&2\end{bmatrix}$ (2%)　　(2) $B=\begin{bmatrix}1&0\\-2&3\end{bmatrix}\cdot\begin{bmatrix}0&1\\0&1\end{bmatrix}$ (2%)

【解】 (1)

能, $A=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}$

(2) 不能, 因B不可逆.

【說明】(本題請先參閱綜線CH3定理16)

(1)

$$A=\begin{bmatrix}0&1\\1&2\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&2\\0&1\end{bmatrix}$$ (綜線CH3範例8)

(2) 基本矩陣都是可逆矩陣, (綜線CH3定義13要訣)

它們的乘積也必須是可逆矩陣. (綜線CH2定理12)

$$\text{但}\ \det B=\det\left(\begin{bmatrix}1 & 0\\ -2 & 3\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right)$$

$$=\det\begin{bmatrix}1 & 0\\ -2 & 3\end{bmatrix}\det\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\qquad \text{(綜線CH4定理6)}$$

$$=\det\begin{bmatrix}1 & 0\\ -2 & 3\end{bmatrix}\cdot 0 = 0$$

$\therefore$ B不可逆. (綜線CH4定理17)

０３Ｅ03 【 清大86資料[9] 】

Find the LU decomposition of the matrix A defined next. (No pivoting is necessary).

$$A=\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 3 & 7\end{bmatrix}$$

【解】

$$A=\begin{bmatrix}1 & 1 & 2\\ 2 & 1 & 3\\ 3 & 3 & 7\end{bmatrix}\begin{matrix}(-2) & (-3)\\ \leftarrow & \\ \leftarrow & \end{matrix}\sim\begin{bmatrix}1 & 1 & 2\\ 0 & -1 & 1\\ 0 & 0 & 1\end{bmatrix}$$

$$\therefore\ A=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 1 & 2\\ 0 & -1 & 1\\ 0 & 0 & 1\end{bmatrix}\qquad \text{(綜線CH3定理27)}$$

$$=\begin{bmatrix}1&0&0\\2&1&0\\3&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&2\\0&1&1\\0&0&1\end{bmatrix}$$ (綜線CH3範例14b)

$$=\begin{bmatrix}1&0&0\\2&-1&0\\3&0&1\end{bmatrix}\begin{bmatrix}1&1&2\\0&1&1\\0&0&1\end{bmatrix}=LU$$ (綜線CH3範例23a)

03E**04** 【 交大85資工[6] 】

Let

$$A=\begin{bmatrix}2&3&1\\4&1&4\\3&4&6\end{bmatrix}$$

Find the LU–decomposition of matrix A, where matrices L and U are lower triangular matrix and upper triangular matrix, respectively.

【分析】依Nobel & Daniel 的用詞: LU–decomposition 是以U爲單位上三角, L_0U_0–decomposition 是以L_0爲單位上三角. 有列對調時寫$A=P^TLU$ 及 $A=P^TL_0U_0$.
依Strang的用詞: LU–facterization (又稱triangular facterization) 是以L爲單位上三角. 有列對調時稱$PA=LU$.

【解】

$$A=\begin{bmatrix}2&3&1\\4&1&4\\3&4&6\end{bmatrix}\begin{matrix}(1/2)\\ \\ \\ \end{matrix}\sim\begin{bmatrix}1&3/2&1/2\\4&1&4\\3&4&6\end{bmatrix}\begin{matrix}(-4)\ (-3)\\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & -5 & 2 \\ 0 & -1/2 & 9/2 \end{bmatrix} \begin{matrix} \\ (-1/5) \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & -2/5 \\ 0 & -1/2 & 9/2 \end{bmatrix} \begin{matrix} \\ (1/2) \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & -2/5 \\ 0 & 0 & 43/10 \end{bmatrix} \begin{matrix} \\ \\ (10/43) \end{matrix} \sim \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ $A=LU$, 其中

$$L=\begin{bmatrix} 2 & 0 & 0 \\ 4 & -5 & 0 \\ 3 & -1/2 & 43/10 \end{bmatrix}, U=\begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & -2/5 \\ 0 & 0 & 1 \end{bmatrix}.$$

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０３E**05** 【 清大80資科[8] 】

Let

$$A=\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 11 \end{bmatrix},$$

find an LU decomposition of matrix A, where L is a lower triangular matrix with all diagonal elements 1, and U is an upper triangular matrix.

【參考章節】CH3定理27,範例28

【解】 對A進行列運算:

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 11 \end{bmatrix} \begin{matrix} (-2) \ (-3) \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & -6 & -10 \end{bmatrix} \begin{matrix} \\ (-2) \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 2 \end{bmatrix}$$

$\therefore\ A = LU$,

其中 $U = \begin{bmatrix} 1 & 4 & 7 \\ 0 & -3 & -6 \\ 0 & 0 & 2 \end{bmatrix}$, $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$

03E**06** 【交大83資料[4]】

Let $A = \begin{bmatrix} 2 & 2 \\ 4 & -1 \end{bmatrix}$.Find a unit–lower–triangular matrix L_1 , a nonsingular diagonal matrix D_0 , and a unit-upper-triangular matrix U_1 for which $A = L_1 D_0 U_1$. (5%)

【解】 $\begin{bmatrix} 2 & 2 \\ 4 & -1 \end{bmatrix} \begin{matrix} (-2) \\ \leftarrow \end{matrix} \sim \begin{bmatrix} 2 & 2 \\ 0 & -5 \end{bmatrix} \begin{matrix} (1/2) \\ (-1/5) \end{matrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$\therefore\ A = \begin{bmatrix} 2 & 2 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$L_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, D_0 = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}, U_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

03E**07** 【交大79工工[16]】

Solve the following linear system using the LU–factorization, given below, of its coefficient matrix.

$$\begin{aligned} 6x_1 - 2x_2 - 4x_3 + 4x_4 &= 2 \\ 3x_1 - 3x_2 - 6x_3 + x_4 &= -4 \\ -12x_1 + 8x_2 + 21x_3 - 8x_4 &= 8 \\ -6x_1 \qquad - 10x_3 + 7x_4 &= -43 \end{aligned}$$

【解】1°對係數矩陣A執行高斯消去法:

$$\begin{array}{c}\begin{bmatrix} 6 & -2 & -4 & 4 \\ 3 & -3 & -6 & 1 \\ -12 & 8 & 21 & -8 \\ -6 & 0 & -10 & 7 \end{bmatrix}\end{array}\begin{array}{l}(-1/2)\ (2)\ (1) \\ \\ \\ \\ \end{array} \sim \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 4 & 13 & 0 \\ 0 & -2 & -14 & 11 \end{bmatrix}\begin{array}{l} \\ (2)\ (-1) \\ \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & -10 & 12 \end{bmatrix}\begin{array}{l} \\ \\ (2) \\ \\ \end{array} \sim \begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$\therefore\ A=LU$, 其中

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 \\ -2 & -2 & 1 & 0 \\ -1 & 1 & -2 & 1 \end{bmatrix},\quad U=\begin{bmatrix} 6 & -2 & -4 & 4 \\ 0 & -2 & -4 & -1 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

2°解 $Ax=b=[2\ -4\quad 8\quad -43]^T$:

即 $LUx=b$

令 $y=Ux$, 先解 $Ly=b$:

$$\begin{cases} y_1 = 2 \\ (1/2)y_1+y_2 = -4 \\ -2y_1-2y_2+y_3 = 8 \\ -y_1+y_2-2y_3+y_4 = -43 \end{cases}$$

依序解得

$y_1=\ 2$, $y_2=-5$, $y_3=\ 2$, $y_4=-32$

再解 $Ux=y$:

$$\begin{cases} 6x_1-2x_2-4x_3+4x_4= 2 \\ \quad -2x_2-4x_3- \quad x_4= -5 \\ \qquad\qquad 5x_3-2x_4 = 2 \\ \qquad\qquad\qquad 8x_4 =-32 \end{cases}$$

依序解得

$x_4= -4$， $x_3= -6/5$， $x_2=69/10$， $x_1=9/2$.

０３E**08** 【 交大79資料[8] 】

Find the LU-decomposition $A=P^{\mathrm{T}}LU$ by performing Gauss elimination with interchanges where

$$A=\begin{bmatrix} 2 & 4 & 0 & -2 \\ -4 & -8 & 0 & 3 \\ 3 & 7 & 2 & 4 \\ 5 & 6 & 1 & -8 \end{bmatrix}$$

【解說】LU分解有兩種說法(見CH3定理29)，以交大資料的考題而言，本題應以解2作答.

【解1】(延除型)

$$\begin{bmatrix} 2 & 4 & 0 & -2 \\ -4 & -8 & 0 & 3 \\ 3 & 7 & 2 & 4 \\ 5 & 6 & 1 & -8 \end{bmatrix} \begin{matrix} (2)\ (-3/2)\ (-5/2) \\ \\ \\ \\ \end{matrix} \quad \begin{matrix} \longleftarrow l_1 \\ \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 4 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 7 \\ 0 & -4 & 1 & -3 \end{bmatrix} \begin{matrix} \\ \\ (4) \\ \\ \end{matrix} \quad \begin{matrix} \longleftarrow l_1 \\ \\ \longleftarrow l_2 \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 2 & 4 & 0 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 9 & 25 \end{bmatrix} \begin{array}{l} \longleftarrow l_1 \\ \longleftarrow l_4 \\ \longleftarrow l_2 \\ \longleftarrow l_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 \\ l_4 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix}$$

令$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$,

則 $PA = LU$, 其中

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3/2 & 1 & 0 & 0 \\ 5/2 & -4 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 2 & 4 & 0 & -2 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 9 & 25 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

【解2】(先除型)

$$\begin{array}{c} \begin{bmatrix} 2 & 4 & 0 & -2 \\ -4 & -8 & 0 & 3 \\ 3 & 7 & 2 & 4 \\ 5 & 6 & 1 & -8 \end{bmatrix} \end{array} \begin{array}{l} (1/2) \\ \\ \\ \\ \end{array} \qquad \begin{array}{l} \longleftarrow l_1 \\ \\ \\ \\ \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & -1 \\ -4 & -8 & 0 & 3 \\ 3 & 7 & 2 & 4 \\ 5 & 6 & 1 & -8 \end{bmatrix} \begin{array}{l} (4)\ (-3)\ (-5) \\ \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array} \qquad \begin{array}{l} \longleftarrow l_1 \\ \\ \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 7 \\ 0 & -4 & 1 & -3 \end{bmatrix} \begin{array}{l} \\ \\ (4) \\ \longleftarrow \end{array} \qquad \begin{array}{l} \longleftarrow l_1 \\ \\ \longleftarrow l_2 \\ \\ \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 9 & 25 \end{bmatrix} \begin{array}{l} \\ (1/-1) \\ \\ (1/9) \end{array} \qquad \begin{array}{l} \longleftarrow l_1 \\ \longleftarrow l_4 \\ \longleftarrow l_2 \\ \longleftarrow l_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 25/9 \end{bmatrix} \qquad \begin{array}{l} \longleftarrow l_1 \\ \longleftarrow l_4 \\ \longleftarrow l_2 \\ \longleftarrow l_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} l_1 \\ l_4 \\ l_2 \\ l_3 \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix}$$

令$P=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$,

則 $PA=LU$, 其中

$$L=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ 5 & -4 & 9 & 0 \\ -4 & 0 & 0 & -1 \end{bmatrix}, \quad U=\begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 2 & 7 \\ 0 & 0 & 1 & 25/9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

03E**09** 【 交大80資料[1](A) 】

$$A=\begin{bmatrix} 0 & 0 & 4 & 2 \\ 2 & 4 & 6 & 2 \\ -4 & -8 & -10 & -2 \end{bmatrix}$$

(A) Find a L_0U_0-decomposition $A=P\,{}^tL_0U_0$.

【解】(A)

$$A=\begin{bmatrix}0&0&4&2\\2&4&6&2\\-4&-8&-10&-2\end{bmatrix}\begin{array}{l}\\(2)\\ \\ \end{array}\overset{r}{\sim}\begin{bmatrix}0&0&4&2\\2&4&6&2\\0&0&2&2\end{bmatrix}\begin{array}{l}\\ \\(-2)\end{array}$$

$$\overset{r}{\sim}\begin{bmatrix}0&0&0&-2\\2&4&6&2\\0&0&2&2\end{bmatrix}\begin{array}{l}\leftarrow l_3\\\leftarrow l_1\\\leftarrow l_2\end{array}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}\text{——}\ l_3\ \text{——}\\\text{——}\ l_1\ \text{——}\\\text{——}\ l_2\ \text{——}\end{bmatrix}=\begin{bmatrix}\text{——}\ l_1\ \text{——}\\\text{——}\ l_2\ \text{——}\\\text{——}\ l_3\ \text{——}\end{bmatrix}$$

令$P=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$,

則 $PA=L_0U_0$,

其中 $U_0=\begin{bmatrix}2&4&6&2\\0&0&2&2\\0&0&0&-2\end{bmatrix}, L_0=\begin{bmatrix}1&0&0\\-2&1&0\\0&2&1\end{bmatrix}$.

$\therefore\ A=P^{-1}L_0U_0=P^{t}L_0U_0$

０３Ｅ１０【 台大76資工[3] 】

$$A=\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}$$

(a) Derive three matrices P (a permutation matrix), L (a lower triangular matrix) and U (an upper triangular matrix) such that $PA=LU$.

(b) Use the result of (a) to compute det$A$.

【解】(a)

$$\begin{bmatrix}0&1&1&1\\1&0&1&1\\1&1&0&1\\1&1&1&0\end{bmatrix}\begin{array}{l} \\ (-1)\quad(-1)\quad\longleftarrow l_1 \\ \\ \\ \end{array}$$

$$\sim\begin{bmatrix}0&1&1&1\\1&0&1&1\\0&1&-1&0\\0&1&0&-1\end{bmatrix}\begin{array}{l} (-1)\quad(-1)\quad\longleftarrow l_2 \\ \qquad\qquad\qquad\longleftarrow l_1 \\ \\ \\ \end{array}$$

$$\sim\begin{bmatrix}0&1&1&1\\1&0&1&1\\0&0&-2&-1\\0&0&-1&-2\end{bmatrix}\begin{array}{l} \qquad\longleftarrow l_2 \\ \qquad\longleftarrow l_1 \\ \\ (-2)\quad\longleftarrow l_3 \end{array}$$

$$\sim \begin{array}{cl} \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & -2 \end{bmatrix} & \begin{array}{l} \longleftarrow l_2 \\ \longleftarrow l_1 \\ \longleftarrow l_4 \\ \leftarrow l_3 \end{array} \end{array}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} l_2 \\ l_1 \\ l_4 \\ l_3 \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix}$$

令 $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

則 $PA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 3 \end{bmatrix} = LU$

(b) $\det P = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \ldots = 1$

$\therefore\ \det A = (\det P)(\det A) = \det(PA)$

$= \det(LU) = (\det L)\ (\det U)$

$$=\begin{vmatrix}1&0&0&0\\0&1&0&0\\1&1&1&0\\1&1&2&1\end{vmatrix}\begin{vmatrix}1&0&1&1\\0&1&1&1\\0&0&-1&-2\\0&0&0&3\end{vmatrix}$$

$=1\cdot(-3)=-3$

03E11 【 交大81資料[4](a) 】

$$A=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-1&0&1&0\\3&0&0&1\end{bmatrix}\begin{bmatrix}3&2&3&4\\0&2&1&1\\0&4&2&5\\0&6&3&1\end{bmatrix}$$

(a) Find a $P^{T}L_0U_0$ decomposition for matrix A.

【解】(a)

$$A=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-1&0&1&0\\3&0&0&1\end{bmatrix}\begin{bmatrix}3&2&3&4\\0&2&1&1\\0&4&2&5\\0&6&3&1\end{bmatrix}\quad(-2)\ (-3)$$

$$=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-1&0&1&0\\3&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&3&0&1\end{bmatrix}\begin{bmatrix}3&2&3&4\\0&2&1&1\\0&0&0&3\\0&0&0&-2\end{bmatrix}\quad(2/3)$$

$$=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-1&0&1&0\\3&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&2&1&0\\0&3&0&1\end{bmatrix}\cdot$$

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-2/3&1\end{bmatrix}\begin{bmatrix}3&2&3&4\\0&2&1&1\\0&0&0&3\\0&0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-1&2&1&0\\3&3&-2/3&1\end{bmatrix}\begin{bmatrix}3&2&3&4\\0&2&1&1\\0&0&0&3\\0&0&0&0\end{bmatrix}$$

$$P=\begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{bmatrix},\ L_0=\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-1&2&1&0\\3&3&-2/3&1\end{bmatrix},\ U_0=\begin{bmatrix}3&2&3&4\\0&2&1&1\\0&0&0&3\\0&0&0&0\end{bmatrix}$$

０３Ｅ**12**【 交大85資科[7] 】

[填充題]

$$A=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\cdot\begin{bmatrix}-2&0&0\\-3&3&0\\5&0&1\end{bmatrix}\cdot\begin{bmatrix}1&-1&2\\0&1&3\\0&-3&-1\end{bmatrix}$$

(1) 求A的一個$P^{\mathrm{T}}LU$分解. (3%)

(2) 寫出rank$A$. (1%)

(3) det$A$＝? (1%)

(4) detA^{-1}＝? (1%)

(5) 前述矩陣A, 我們欲解方程組 $AX=b$, 若用Gauss 消去法先將A做成一個upper triangular matrix. 問若需要在A之左方乘以哪些elementary matrix (以矩陣乘法方式一步一步依序寫出). (3%)

(6) 寫出最後之upper trangular matrix. (1%)

(7) 若$b=\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$解$AX=b$, 求$X=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$＝? (2%)

【解】(1)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} -2 & 0 & 0 \\ -3 & 3 & 0 \\ 5 & -3 & 8 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

(2) 3

(3) –48

(4) –1/48

(5) 依序左乘下列基本矩陣即可將A化成上三角:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix},\quad \begin{bmatrix}1&0&0\\0&1/3&0\\0&0&1\end{bmatrix},\quad \begin{bmatrix}1&0&0\\0&1&0\\0&3&1\end{bmatrix},\quad \begin{bmatrix}1&0&0\\0&1&0\\0&0&1/8\end{bmatrix}$$

(6) $\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}$

(7) $AX=b$ 即 $P^{T}LUX=b$. 令 $Y=UX$, $Z=LY=LUX$.

由 $P^{T}Z=b$ 得 $Z=Pb=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\begin{bmatrix}2\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\2\end{bmatrix}$

由 $LY=Z$, 即

$$\begin{bmatrix}-2&0&0\\-3&3&0\\5&-3&8\end{bmatrix}\begin{bmatrix}Y_1\\Y_2\\Y_3\end{bmatrix}=\begin{bmatrix}0\\0\\2\end{bmatrix}$$

乘開得 $-2Y_1=0$, $-3Y_1+3Y_2=0$, $5Y_1-3Y_2+8Y_3=2$

依序得出 $Y_1=0$, $Y_2=0$, $Y_3=1/4$.

由 $UX=Y$, 即

$$\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}\begin{bmatrix}X_1\\X_2\\X_3\end{bmatrix}=\begin{bmatrix}0\\0\\1/4\end{bmatrix}$$

乘開得 $X_1-X_2+2X_3=0$, $X_2+3X_3=0$, $X_3=1/4$.

依序得出 $X_3=1/4$, $X_2=-3/4$, $X_1=-5/4$.

$\therefore$

$$X = \begin{bmatrix} -5/4 \\ -3/4 \\ 1/4 \end{bmatrix}$$

【說明】

(1) 因A已做了一部份的$P^{T}LU$分解，不該將A乘出而重頭算起.

(本題考得較活，需完全了解綜線CH3範例28的[解說]才會作答.)

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ -3 & 3 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & -3 & -1 \end{bmatrix} \quad (3)$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ -3 & 3 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 8 \end{bmatrix}$$

$$=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}-2&0&0\\-3&1&0\\5&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&3&0\\0&-3&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&0&8\end{bmatrix}\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix}\begin{bmatrix}-2&0&0\\-3&3&0\\5&-3&8\end{bmatrix}\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}^{T}\begin{bmatrix}-2&0&0\\-3&3&0\\5&-3&8\end{bmatrix}\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}$$

(2)

$$\mathrm{rank}A=\mathrm{rank}\left(\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}^{T}\begin{bmatrix}-2&0&0\\-3&3&0\\5&-3&8\end{bmatrix}\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}\right)$$

$$=\mathrm{rank}\left(\begin{bmatrix}-2&0&0\\-3&3&0\\5&-3&8\end{bmatrix}\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}\right)$$ (綜線CH8定理16)

$$=\mathrm{rank}\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}$$ (綜線CH8定理16)

$$=3$$ (綜線CH6定理23)

(3)

$$\det A=\det\left(\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}^{T}\begin{bmatrix}-2&0&0\\-3&3&0\\5&-3&8\end{bmatrix}\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}\right)$$

$$\det A=\det\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}\det\begin{bmatrix}-2&0&0\\-3&3&0\\5&-3&8\end{bmatrix}\det\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}$$

(綜線CH4定理5,6)

$$=1\cdot(-48)=-48$$

(4)

$\det(A^{-1})=(\det A)^{-1}=-1/48$. (綜線CH4定理6)

(5)&(6)

由(1), $A=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}^{-1}\begin{bmatrix}-2&0&0\\-3&3&0\\5&-3&8\end{bmatrix}\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}$ (正交矩陣)

$\therefore \begin{bmatrix}-2&0&0\\-3&3&0\\5&-3&8\end{bmatrix}^{-1}\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}A=\begin{bmatrix}1&-1&2\\0&1&3\\0&0&1\end{bmatrix}$ (移項)

其中

$$\begin{bmatrix}-2&0&0\\-3&3&0\\5&-3&8\end{bmatrix}^{-1}$$

$$= \left(\begin{bmatrix} -2 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \right)^{-1} \quad \text{(綜線CH4定理21)}$$

$$= \left(\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \cdot \right.$$

$$\left. \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix} \right)^{-1} \quad \text{(綜線CH3定理21)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \cdot$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \quad \text{(綜線CH2定理12)}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/8 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot$$

$$\begin{bmatrix}1&0&0\\3&1&0\\-5&0&1\end{bmatrix}\begin{bmatrix}-1/2&0&0\\0&1&0\\0&0&1\end{bmatrix}$$ (綜線CH3範例4a)

$$=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1/8\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&3&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1/3&0\\0&0&1\end{bmatrix}\cdot$$

$$\begin{bmatrix}1&0&0\\3&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-5&0&1\end{bmatrix}\begin{bmatrix}-1/2&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

∴ 得出前面的解

(7) 前面已有P^TLU分解, 所以題目的用意是不要直接解$AX=b$.

０３E**１３**【 交大85資料[10] 】

[填充題]

$$A=\begin{bmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1/2&0&1\end{bmatrix},\quad 求A^{-1}=?$$

【解】

$$A^{-1}=\begin{bmatrix}0&0&0&1\\0&1&0&0\\1/2&0&0&0\\0&-1/2&1&0\end{bmatrix}$$

#

【說明】本題A已分解成列矩陣，利用基本矩陣的性質求解較快.

$$A^{-1}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&1/2&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix}^{-1}\begin{bmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0\end{bmatrix}^{-1}$$ (綜線CH2定理2)

$$=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&-1/2&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1/2&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{bmatrix}$$ (綜線CH3定義13要訣)
(綜線CH3定義24要訣)

$$=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1/2&0\\0&-1/2&0&1\end{bmatrix}\begin{bmatrix}0&0&0&1\\0&1&0&0\\1&0&0&0\\0&0&1&0\end{bmatrix}$$ (綜線CH3定理4)

$$=\begin{bmatrix}0&0&0&1\\0&1&0&0\\1/2&0&0&0\\0&-1/2&1&0\end{bmatrix}$$

０３E**14**【 師大84資教[14] 】

Let P be an n by n permutation matrix and A be an n by m matrix. AP reorders the columns of A. How about $A^{-1}P$ does?

【解】$A^{-1}P$ 重排A^{-1}的columns

【討論】本題極不合理. 因在$m \neq n$時不能有AP. 而且A也會不可逆.