題型04A: 行列式的性質

０４A01 【 中央86資工[1](g) 】

[是非論證題]

(g) $A^3 = O$ if and only if $\det A = 0$.

【解】(g) False.

$$例如A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \det A = 0, \quad 但\ A^3 = \begin{bmatrix} 8 & 0 \\ 0 & 0 \end{bmatrix} \neq O$$

【討論】本題只有 “only if” 成立:

$$A^3 = O \implies \det(A^3) = 0 \iff (\det A)^3 = 0 \iff \det A = 0$$

０４A02 【 交大85資料[11] 】

[填充題]

(1) $n*n$矩陣A, 問$A \cdot \operatorname{adj} A = ?$ (2%)

(2) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, 問$\operatorname{adj} A = ?$ (1%)

【解答】(1) $(\det A) I_n$ (綜線CH4定義16,定理17)

(2) $\begin{bmatrix} 1 & -4 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ (綜線CH4定義16)

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【說明】(2) 依定義算出即可. 但因極易算錯, 應依第一小題驗算.

０４A03【 師大84資教[2] 】

Let U be the row echelon form of matrix A.

(a) If A is singular, what is $\det(U)$?

(b) If A is nonsingular, what is $\det(U)$?

【解】(a) $\det U = 0$, (b) $\det U \neq 0$

【討論】基本列運算不會改變行列式的"等於零"或"不等於零". (綜線CH4定理7)

可逆 $\Longleftrightarrow$ 行列式非零. (綜線CH4定理17)

０４A04【 師大84資教[3] 】

Answer yes/no for the following questions.

(a) $\det(A+B)=\det(A)+\det(B)$?

(b) $\det(AB)=\det(A)\det(B)$?

(c) $\det(A^{-1})=1/\det(A)$?

【解】(a) No. (綜線CH4定理7要訣6)

(b) Yes. (綜線CH4定理6)

(c) Yes. (綜線CH4定理6要訣3)

０４A05【 台大83資工[6] 】

[複選題]

Which of the following statements is (or are) true ?

(1) A matrix M is invertible if and only if $\det(M)=0$,

(2) $\det(A^t)=-\det(A)$,

(3) $\det(AB)=\det(A)\det(B)$ for all $A, B\in M_{n\times n}(F)$,

(4) $\det(B)=-\det(A)$ if B is a matrix obtained from A by interchanging two rows,

(5) If Q is an invertible matrix, then $\det(Q^{-1})=(\det(Q))^{-1}$

【解】 選(3)(4)(5).

【討論】(1) 講反了. 應是 if and only if $\det(M)\neq 0$. (綜線CH4定理17)

(2) 應是 $\det(A^t)=\det(A)$. (綜線CH4定理5)

(3) 此爲定理. (綜線CH4定理6)

(4) 此爲定理. (綜線CH4定理7①)

(5) 此爲定理. (綜線CH4定理6要訣3)

04A06 【 中央83資工[3](ab) 】

True or False (Give a reason if true, and give a counterexample if false).

Let A and B be two different $n\times n$ nonsingular matrices.

(a) $\det A^{T}=\det A$. (3%)

(b) $\det(A+B)=\det A+\det B$ (3%)

【解】 (a) True. 此爲定理. (綜線CH4定理5)

(b) False. 反例如下: (綜線CH4定理7要訣6, 習題7.2)

$$\det\left(\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)=\det\begin{bmatrix}2 & 2\\ 3 & 5\end{bmatrix}=4$$

$$\det\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=-2+1=-1$$

04A07 【 元智83工工[7] 】

[是非題]

Let A and B be two 10 by 10 matrix such that

$$B_{\cdot j}=-A_{\cdot 1}-A_{\cdot 2}-\ldots-A_{\cdot j-1}+A_{\cdot j},\quad j=1,2,\ldots,10,$$

then $\det(A)=\det(B)$. (2%)

【解】○, 解說如下:

題目指明B的第j行由A的第j行減去A的某些行而得, 所以B的行列式與A的行列式相等. (綜線CH4定理7)

04A08 【 元智80工工[2] 】

Let A, B be two $n\times n$ matrices. If n is odd and $AB=-BA$, prove that either A or B is not invertible.

【解】 $\because$ $AB = -BA$ (已知)

$\therefore$ $\det(AB) = \det(-BA)$

$\therefore$ $\det(AB) = -\det(BA)$ (綜線CH4定理7要訣5)

$\therefore$ $\det A \det B = -\det B \det A$ (綜線CH4定理6)

$\therefore$ $2 \det A \det B = 0$ (移項)

$\therefore$ $\det A \det B = 0$ $\therefore$ $\det A = 0$ 或 $\det B = 0$

$\therefore$ A不可逆或B不可逆 (綜線CH4定理17)

04A09 【 大同80資工[3](d) 】

True or False:

(d) If all diagonal entries of matrix A are zero, then A is singular.

【解】(d) False, 反例如下:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \text{則}\det A = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

$\therefore$ A可逆 (non–singular) (綜線CH4定理17)

04A10 【 大同80資工[3](b)& 】

True or False:

(b) If matrix A is invertible and its rows are in reverse order in matrix B, then B is invertible.

【解】(b) True, 證明如下:

設A為$n \times n$矩陣, 則

$$B = \begin{bmatrix} \Box & & & & 1 \\ & & & \cdot & \\ & & \cdot & & \\ & 1 & & & \\ 1 & & & & \Box \end{bmatrix} A$$ (綜線CH3定理5)

$\because$ A可逆, $\therefore$ $\det A \neq 0$ (綜線CH4定理17)

$$\therefore\ \det B = \det\begin{bmatrix} \square & & & & & 1 \\ & & & & \cdot & \\ & & & \cdot & & \\ & & \cdot & & & \\ & 1 & & & & \\ 1 & & & & & \square \end{bmatrix} \det A \neq 0$$

∴ B可逆 (綜線CH4定理17)

[另證] 請參閱第八章

04A 11 【大同80資工[3](j)】

True or False:

(j) $\det A = 0$, then at least one of the cofactors must be zero.

【解】(j) False, 反例如下:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},\ 則\ \det A = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1-1 = 0$$

但 $\mathrm{cof}_{11}A = 1$, $\mathrm{cof}_{12}A = -1$, $\mathrm{cof}_{21}A = -1$, $\mathrm{cof}_{22}A = 1$,(綜線CH4定義10)

全都不是0.

04A 12 【交大79資料[7]】

Prove that there is no n by n matrix X over real number where n is an odd number such that $X^2 + I_n = O_n$ (I is the n by n identity matrix).

【解】 ∵ $X^2 = -I$

∴ $\det(X^2) = \det(-I) = -1$ (綜線CH4定理7要訣5)

∴ $(\det X)^2 = -1$ (綜線CH4定理6)

此在實數系內造成矛盾.

04A 13 【清大79資料[1](2)】

For each of the following statements, explain whether it is true or false.

(2) There is a 5×5 real matrix A with $A^4 = -I$.

【解】(2) False. 以矛盾證法證明如下:

若存在5×5實數矩陣A使得$A^4=-I$, 兩邊取行列式, 得:

$\det(A^4)=\det(-I)$,

$\therefore\ (\det A)^4=-1$ (綜線CH4定理6)

但實數det$A$的4次方應該是正數, 因而產生矛盾.

０４Ａ**１４** 【 清大81資料[19] 】

Let $Q\in\mathbb{R}^{n\times n}$ be an orthogonal matrix. What is the determinant of Q?

(a) 1 (b) –1 (c) 1 or –1 (d) n (e) n^2

【解】選(c). 解說如下:

$\because\ Q^TQ=I,\quad \therefore\ \det(Q^TQ)=\det(I)$

$\therefore\ \det(Q^T)\det Q=1\quad \therefore\ (\det Q)^2=1\quad \therefore\ \det Q=\pm 1$

０４Ａ**１５** 【 交大80資工[1](bcde) 】

True (T) or False (F): (1 for each)

(b) $\det(A+B)=\det(A)+\det(B)$.

(c) $\det(AB)=\det(A)\det(B)$.

(d) If $A\overset{R}{\sim}I$, then $\det(A)=1$.

(e) If A is $n\times n$, then $\det(\mathrm{Adj}(A))=\det(A)^n$.

【參考章節】(b) CH4定理7 (c) CH4定理6 (d) CH3定義3 (e) CH4定理17

【解】(b) False. 反例如下:

$$\text{取}A=\begin{bmatrix}1&0\\0&1\end{bmatrix},\ B=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\text{則}\ \det(A+B)=\det\begin{bmatrix}2&0\\0&2\end{bmatrix}=4$$

但$\det A+\det B=1+1=2$

(c) True

此爲定理 (綜線CH4定理6)

本定理依各書之公理體系不同而有各種完全不同之證法. 有興趣的讀者可參考各種線性代數之書.

(d) False. 反例如下:

取$A=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$, 則$A\overset{R}{\sim}I$

但 $\det A=2$

(e) [要訣] $\det(\operatorname{adj}A)=(\det A)^{n-1}$ (綜線CH4定理17要訣(2))

False. 反例如下: (只須使 $\det A\neq 1$ 即可)

取$A=\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}$, $n=2$.

則 $\operatorname{adj}A=\begin{bmatrix}\operatorname{cof}_{11}A & \operatorname{cof}_{21}A\\ \operatorname{cof}_{12}A & \operatorname{cof}_{22}A\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}$

$(\det A)^{n}=\left(\begin{vmatrix}1 & 0\\ 0 & 2\end{vmatrix}\right)^{2}=4\neq\det(\operatorname{adj}A)$

０４Ａ１６ 【 大同82資工[7] 】

Let $D: F^{n\times n}\longrightarrow F$ be a determinant function defined by $D(M)=|M|$, where M is any $n\times n$ matrix over F, i.e., $M\in F$ Then, which of the following statements is true:

(a) D is a linear transformation;

(b) $D(M)\neq 0$ if any two rows in M are nonzero and distinct;

(c) D is a linear function of each row of an $n\times n$ matrix when the other $n-1$ rows are held fixed;

(d) none.

【解】 選 c

【說明】(a) 錯, 例如

$$D\left(\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}+\begin{bmatrix}0 & 0\\3 & 4\end{bmatrix}\right)\neq D\begin{bmatrix}1 & 2\\0 & 0\end{bmatrix}+D\begin{bmatrix}0 & 0\\3 & 4\end{bmatrix}$$

(b) 錯, 例如

$(1 \quad 2)$, $(3 \quad 6)$ 非零且相異, 但

$$D\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}\neq 0$$

(c) 此爲定理。

０４A１７【 大同82資工[二1] 】

If A is a 2×1 matrix, and B is a 1×2 matrix. Prove that AB is not invertible.

【解】

令 $A=\begin{bmatrix}a\\b\end{bmatrix}$, $B=\begin{bmatrix}c & d\end{bmatrix}$,

則 $AB=\begin{bmatrix}ac & ad\\bc & bd\end{bmatrix}$, $\det(AB)=\begin{vmatrix}ac & ad\\bc & bd\end{vmatrix}=0$

∴ AB 不可逆 (綜線CH4定理17)

(本題是【 大同82資工[二2] 】的特例, 當然也可採用它的證法。)

題型04B: 求算行列式

04B01 【 清大86資料[10](a) 】

(a) Evaluate the determinant of A^{-1} where $A=BCD$, and

$$B=\begin{bmatrix}1&2&1\\0&2&3\\0&0&1\end{bmatrix},\quad C=\begin{bmatrix}1&0&1\\-1&1&0\\0&3&2\end{bmatrix},\ D=\begin{bmatrix}1&0&0\\-2&1&0\\5&2&5\end{bmatrix}.\quad (5\%)$$

【解】(a) $\det B=1\cdot 2\cdot 1=2$,

$$\det C=\begin{vmatrix}1&0&1\\-1&1&0\\0&3&2\end{vmatrix}=\begin{vmatrix}1&0&1\\0&1&1\\0&3&2\end{vmatrix}=\begin{vmatrix}1&1\\3&2\end{vmatrix}=-1\ .$$

$\det D=1\cdot 1\cdot 5=5$

$\det(A^{-1})=(\det A)^{-1}=(\ (\det B)(\det C)(\det D)\ )^{-1}$ (綜線CH4定理6)

$=(\ 2\cdot(-1)\cdot 5\ )^{-1}=-1/10$.

04B02 【 師大84資教[4] 】

Let A and B be 3 by 3 matrices with $\det(A)=4$ and $\det(B)=5$.

Find the values of (a) $\det(3A)$, and (b) $\det(2AB)$.

【解】(a) $\det(3A)=27\det A=108$ (綜線CH4定理7)

(b) $\det(2AB)=8\det(AB)=8\det A\det B=160$ (綜線CH4定理6,7)

04B03 【 台大78資工[1] 】

Let

$$A=\begin{bmatrix}1&0&0&0&0\\2&2&0&0&0\\3&1&3&0&0\\1&2&1&4&0\\2&1&1&1&1\end{bmatrix}\quad B=\begin{bmatrix}1&2&1&2&1\\0&2&3&3&2\\0&0&3&1&2\\0&0&0&1&3\\0&0&0&0&4\end{bmatrix}$$

evaluate det(AB).

【解】

$$\det A=\begin{vmatrix}1&0&0&0&0\\2&2&0&0&0\\3&1&3&0&0\\1&2&1&4&0\\2&1&1&1&1\end{vmatrix}=1\cdot\begin{vmatrix}2&0&0&0\\1&3&0&0\\2&1&4&0\\1&1&1&1\end{vmatrix}$$ (對第一列降階) (綜線CH4定理1)

$$=1\cdot 2\cdot\begin{vmatrix}3&0&0\\1&4&0\\1&1&1\end{vmatrix}$$ (再對上式第一列降階)

$$=1\cdot 2\cdot 3\cdot 4=24$$

同法, 對det$B$逐步對第一行降階可得

$$\det B=\begin{vmatrix}1&2&1&2&1\\0&2&3&3&2\\0&0&3&1&2\\0&0&0&1&3\\0&0&0&0&4\end{vmatrix}=1\cdot 2\cdot 3\cdot 1\cdot 4=24$$

$\therefore\ \det(AB)=(\det A)(\det B)$ (綜線CH4定理6)

$=24\cdot 24=576$

０４B**04** 【 交大84資料[2]& 】

The equation of a strait line in the x-y plane has the form $ax+by=c$. Consider three strait lines, with equations

$a_i x + b_i y = c_i$ for i=1,2,3.

Prove that if the three lines all pass through a common point (x,y) and only (x,y), then $\det[A,B,C]=0$, where A,B,C are 3×1 and $\langle A\rangle_i = a_i$, $\langle B\rangle_i = b_i$, $\langle C\rangle_i = c_i$.

【討論】(1) 題目中 " and only (x, y) " 的條件並不需要.

(2) 這個題目在高中數學就已出現過, 但在此應套用線性代數的定理來解題.

【解】$\det[A,B,C]$

$$=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & a_1x+b_1y \\ a_2 & b_2 & a_2x+b_2y \\ a_3 & b_3 & a_3x+b_3y \end{vmatrix}$$

$$=x\begin{vmatrix} a_1 & b_1 & a_1 \\ a_2 & b_2 & a_2 \\ a_3 & b_3 & a_3 \end{vmatrix} + y\begin{vmatrix} a_1 & b_1 & b_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & b_3 \end{vmatrix}$$ (綜線CH4定理7)

$$=x\cdot 0+y\cdot 0=0$$ (兩行相等的行列式爲零)

０４Ｂ**05** 【 清大85資料[3] 】

Let $D_n \in \mathbb{R}^{n\times n}$ be defined as

$$D_n(i,j)=\begin{cases} 2 & \text{if } i=j, \\ -1 & \text{if } |i-j|=1, \\ 0 & \text{elsewhere.} \end{cases}$$

(a) Write down D_4?

(b) Show that $|D_n|=2|D_{n-1}|-|D_{n-2}|$ for $n\geq 3$.

(c) Find the determinant of D_{100}

【分析】本題(a)是基本閱讀測驗.

本題(c)須使用離散數學的"遞迴關係"求解. 不是純線性代數的問題.

【解】(a)

$$D_4 = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

(b) 對$n \geq 3$,

$|D_n|$ (依第一列降階展開)

$$= 2 \cdot |D_{n-1}| - (-1) \cdot \begin{vmatrix} -1 & -1 & 0 & .. & 0 & 0 \\ 0 & 2 & -1 & .. & 0 & 0 \\ 0 & -1 & 2 & .. & 0 & 0 \\ & & \cdots\cdots & & & \\ 0 & 0 & 0 & & 2 & -1 \\ 0 & 0 & 0 & .. & -1 & 2 \end{vmatrix}$$

(再依第一行降階展開)

$= 2 \cdot |D_{n-1}| - (-1) \cdot (-1) \cdot |D_{n-2}|$

$= 2 \cdot |D_{n-1}| - |D_{n-2}|$

(c) 令 $a_n = |D_n|$

由(b)得 $a_n - 2a_{n-1} + a_{n-2} = 0$

另外, $a_1 = \det \begin{bmatrix} 2 \end{bmatrix} = 2$, $a_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$.

以下解此帶初值的 recursive relation :

令 $r^2 - 2r + 1 = 0$, 解得 $r = 1, 1$

$\therefore a_n = p \cdot 1^n + q \cdot n \cdot 1^n$

即 $a_n = p + qn$ ....(甲)

以 a_1, a_2 代入甲式得 $2 = p + q$, $3 = p + 2q$

解得 $p = 1$, $q = 1$.

$\therefore\ a_n = 1 + n$ #

(讀者可自行驗證$a_3=4$, $a_4=5$)

０４Ｂ**06** 【 元智84工工X[1] 】

令$A_{n\times n}=\begin{bmatrix} 2 & 1 & 0 & 0 & 0 \ldots\ldots 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \ldots\ldots 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \ldots\ldots 0 & 0 & 0 \\ . & . & . & . & . \ldots\ldots . & . & \\ 0 & 0 & 0 & 0 & 0 \ldots\ldots 0 & 1 & 2 \end{bmatrix}_{n\times n}$

求證$A_{n\times n}$的行列式值(determinant)是$n+1$.

【解】以數學歸納法證明:

$n=1$時, $\det A_{n\times n}=\det[\,2\,] = 2$.

$n=2$時, $\det A_{n\times n}=\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}=4-1=3$

假設此公式在$<n$時已成立, 則

$$\det A_{n\times n}=\begin{vmatrix} 2 & 1 & 0 & 0 & 0 \ldots 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \ldots 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \ldots 0 & 0 & 0 \\ . & . & . & . & . \ldots . & . & . \\ 0 & 0 & 0 & 0 & 0 \ldots 0 & 1 & 2 \end{vmatrix}_{n\times n}$$

(由第一行降階展開)

$$=2\cdot\det A_{(n-1)\times(n-1)}-\begin{vmatrix} 1 & 0 & 0 & 0 \ldots 0 & 0 & 0 \\ 1 & 2 & 1 & 0 \ldots 0 & 0 & 0 \\ & & & \ldots\ldots\ldots\ldots & & & \\ 0 & 0 & 0 & 0 \ldots 0 & 1 & 2 \end{vmatrix}_{(n-1)\times(n-1)}$$

$$=2\cdot\det A_{(n-1)\times(n-1)}-\begin{vmatrix}2&1&0\ldots0&0&0\\&&\ldots\ldots\ldots&&\\&&\ldots\ldots\ldots&&\\0&0&0\ldots0&1&2\end{vmatrix}_{(n-2)\times(n-2)}$$

$=2\cdot\det A_{(n-1)\times(n-1)}-\det A_{(n-2)\times(n-2)}$

$=2(n-1+1)-(n-2+1)=n+1$

故得證.

04B**07** 【 大同83資工[5] 】

Let A, B be two $n\times n$ matrices defined by:

$$A=\begin{bmatrix}1+\lambda&\lambda&\lambda&\ldots\ldots&\lambda\\\lambda&1+\lambda&\lambda&\ldots\ldots&\lambda\\\lambda&\lambda&1+\lambda&&\lambda\\.&.&.&.&\\.&.&.&.&\\\lambda&\lambda&\lambda&\ldots\ldots&1+\lambda\end{bmatrix},$$

$$B=\begin{bmatrix}\lambda&\lambda&\lambda&\ldots\ldots&\lambda\\\lambda&1+\lambda&\lambda&\ldots\ldots&\lambda\\\lambda&\lambda&1+\lambda&&\lambda\\.&.&.&.&\\.&.&.&.&\\\lambda&\lambda&\lambda&\ldots\ldots&1+\lambda\end{bmatrix},$$

(a) Find det(A). (5%)

(b) Find det(B). (3%)

(c) Find $\lim_{\lambda\to\infty} \dfrac{\det(B)}{\det(A)}$. (2%)

【解】 (a)

$$\det A = \begin{vmatrix} 1+\lambda & \lambda & \lambda & \cdots\cdots & \lambda \\ \lambda & 1+\lambda & \lambda & \cdots\cdots & \lambda \\ \lambda & \lambda & 1+\lambda & & \lambda \\ . & . & . & & \\ \lambda & \lambda & \lambda & \cdots\cdots & 1+\lambda \end{vmatrix}$$

(將各列都加入第一列:)

$$= \begin{vmatrix} 1+n\lambda & 1+n\lambda & 1+n\lambda & \cdots\cdots & 1+n\lambda \\ \lambda & 1+\lambda & \lambda & \cdots\cdots & \lambda \\ \lambda & \lambda & 1+\lambda & & \lambda \\ . & . & . & & . \\ \lambda & \lambda & \lambda & \cdots\cdots & 1+\lambda \end{vmatrix}$$

$$=(1+n\lambda) \begin{vmatrix} 1 & 1 & 1 & \cdots\cdots & 1 \\ \lambda & 1+\lambda & \lambda & \cdots\cdots & \lambda \\ \lambda & \lambda & 1+\lambda & & \lambda \\ . & . & . & & . \\ \lambda & \lambda & \lambda & \cdots\cdots & 1+\lambda \end{vmatrix}$$

(將第一列的$-\lambda$倍加入各列:)

$$=(1+n\lambda)\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$$

$$=1+n\lambda$$

(b)

$$\det B=\begin{vmatrix} \lambda & \lambda & \lambda & \cdots & \lambda \\ \lambda & 1+\lambda & \lambda & \cdots & \lambda \\ \lambda & \lambda & 1+\lambda & & \lambda \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & & \\ \lambda & \lambda & \lambda & \cdots & 1+\lambda \end{vmatrix}$$

$$=\lambda\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ \lambda & 1+\lambda & \lambda & \cdots & \lambda \\ \lambda & \lambda & 1+\lambda & & \lambda \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \\ \lambda & \lambda & \lambda & \cdots & 1+\lambda \end{vmatrix}=\lambda\cdot 1 \qquad \text{(同前)}$$

$$=\lambda$$

(c) $\lim\limits_{\lambda\to\infty}\dfrac{\det(B)}{\det(A)}=\lim\limits_{\lambda\to\infty}\dfrac{\lambda}{1+n\lambda}=\lim\limits_{\lambda\to\infty}\dfrac{1/\lambda}{1/\lambda+n}=\dfrac{1}{n}$

０４B**08**【 清大86資料[7] 】

Find $\det(A_n)$ if $A=(a_{ij})$, where

$$a_{ij}=\begin{cases}1 & \text{if } i=j \text{ or } i=j+1,\\ -1 & \text{if } i=j-1,\\ 0 & \text{otherwise.}\end{cases}$$

【解】此矩陣對第一列降階展開可得知(細節略，請參閱綜線CH4範例13)：

$\det(A_n)=\det(A_{n-1})+\det(A_{n-2})$.

另外， $\det(A_1)=1$， $\det(A_2)=2$.

解此recursive relation可得

$$\det(A_n)=(1/\sqrt{5})\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)$$

【討論】未學過離散數學的同學可依下法求解:

$$\begin{bmatrix} z_n \\ z_{n-1}\end{bmatrix}=\begin{bmatrix} z_{n-1}+z_{n-2} \\ z_{n-1}\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix} z_{n-1} \\ z_{n-2}\end{bmatrix}$$

$$\therefore\ \begin{bmatrix} z_{n+1} \\ z_n\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^n\begin{bmatrix} z_1 \\ z_0\end{bmatrix}$$

我們可依關係式反推，令$z_0=1$. 再依綜線CH16範例5或範例19的方法求解.

０４B**09**【 淡江80資工[4] 】

令$A=(a_{ij})_{i,j=1,\cdots,n}$， $a_{ij}\in\mathbb{R}$. 在求$\det(A)$時，需要幾次的乘法運算?

試說明其 algorithm.

【解】 先利用Gauss消去法化為梯形， (綜線CH3演算法4a)

再將主對角線上的數乘起來就得到行列式值。

當pivot在倒數第i列($i=n, n-1, \ldots\ldots, 2$)時，

定出倍數需1次乘算，整列向下消再需$(i-1)$次乘算，共需i次乘算.

而下方有$i-1$列需消出0，所以需$i(i-1)=i^2-i$次乘算.

化為梯形矩陣總共需要

$$(n^2-n)+.....+(k^2-k)+.....+(2^2-2)$$

$$=\sum_{i=1}^{n} i^2-\sum_{i=1}^{n} i$$

$$=\frac{n(n+1)(2n+1)}{6}-\frac{n(n+1)}{2}=\frac{n^3-n}{3}$$ 次乘算

將主對角線上的數乘起來需要 $n-1$ 次乘算

$\therefore$ 共需 $\frac{n^3-n}{3}+n-1=\frac{(n-1)(n^2+n+3)}{3}\approx\frac{n^3}{3}$

０４B**１０** 【 交大81資工[2](ab) 】

Decide the correctness of the following inductions. You should concisely explain the reasons for the correct ones and the incorrect ones.

(a) (2%) From the formula

det$A$ = ±(product of the all pivots in the Gaussian elimination)

we can conclude that it is useless to escape a very small pivot by exchanging rows, where det$A$ denotes the determinant of matrix A.

(b) (2%) Matrices A, B, and C are $p\times p$. Suppose that det(AB) = 0, we can conclude that all the inverses of matrices A, B, and AB do not exist.

【參考章節】(b) 綜合線性代數CH4定理17

【分析】(a)本題來自於“G. Strang: Linear Algebra and Its Applications, Academic press”.

茲節錄該書Chapter 4: DETERMINANT 第一節內的三句話供參考:

（ 為尊重著作權，若同學需進一步資料，請自行參閱該書。 ）

“...., from the formula det$A$ = ±(product of the povots), it follows that regardless of the order of elimination, the product of the pivots remains the same apart from sign. Years ago, this led to the belief that it was useless to escape a very small pivot by exchanging rows, since eventually the small pivot would catch up with us. But when usually happens in practice, if an abnormally small pivot is not avoided, is that it is very soon followed by an abnormally large one; this brings the product back to normal but leaves the numerical solution in ruins.”

(b) 可逆 $\Longleftrightarrow$ 行列式不爲零

$\det(AB)=0 \Longleftrightarrow \det(A)\det(B)=0 \Longleftrightarrow \det(A)=0$或$\det(B)=0$

$\Longleftrightarrow$ A不可逆或B不可逆

【解】(a) Incorrect. 解說如下:

在Gauss elimination過程中, 若執行額外的“列對調”將產生不同的pivot.
本題的公式顯示: “pivot乘積的絕對值不因列對調而不同”. 但在解方程式時避開過小的pivot仍有助於避免不必要的大誤差。
若在進行Gauss elimination時不避開過小的pivot, 通常很快就會遇到過大的pivot這將容易造成數值解的大誤差。

(b) Incorrect. 反例如下:

令 $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}, B=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$,

則 $AB=\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$, $\det(AB)=0$.

A, B 合乎要求的條件, 但A卻可有inverse.

【特別解說】

考慮下列方程式:

$$\begin{cases}\varepsilon x+ay=c\\ x+by=d,\end{cases}$$

其中a,b,c,d是絕對值正常的數值, 而 ε 是絕對值(相對於其它數)非常小的數值。我們比較下列二種演算法即可得知利用列對調避開小pivot的重要

[演算法一: 利用列對調避開 ε]

$$\left[\begin{array}{cc|c}\varepsilon & a & c\\ 1 & b & d\end{array}\right]\sim\left[\begin{array}{cc|c}1 & b & d\\ \varepsilon & a & c\end{array}\right]\begin{array}{c}(-\varepsilon)\\ \hookleftarrow\end{array}\sim\left[\begin{array}{cc|c}1 & b & d\\ 0 & a-\varepsilon b & c-\varepsilon d\end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & b & d \\ 0 & 1 & (c-\varepsilon d)/(a-\varepsilon b) \end{array}\right] (-b) \qquad \text{...........①}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & d-b(c-\varepsilon d)/(a-\varepsilon b) \\ 0 & 1 & (c-\varepsilon d)/(a-\varepsilon b) \end{array}\right]$$

精確解爲 $\begin{cases} x=d-b(c-\varepsilon d)/(a-\varepsilon b) \\ y=(c-\varepsilon d)/(a-\varepsilon b) \end{cases}$

因εd 極小於c, εb極小於a, 所以近似解爲

$$x=d-bc/a\,, \quad y=c/a \qquad \text{.......②}$$

本演算法在數値計算的場合下, 由①式起始變成

$$\sim \left[\begin{array}{cc|c} 1 & b & d \\ 0 & 1 & c/a+\delta \end{array}\right] (-b)$$

(式中的 δ是小量的誤差)

$$\sim \left[\begin{array}{cc|c} 1 & 0 & d-bc/a-b\delta \\ 0 & 1 & c/a+\delta \end{array}\right]$$

因 $-b\delta$仍爲小誤差, 可得出②式的合理結果.

[演算法二: 不避開ε]

$$\left[\begin{array}{cc|c} \varepsilon & a & c \\ 1 & b & d \end{array}\right] (-1/\varepsilon) \sim \left[\begin{array}{cc|c} \varepsilon & a & c \\ 0 & b-a/\varepsilon & d-c/\varepsilon \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} \varepsilon & a & c \\ 0 & 1 & (d-c/\varepsilon)/(b-a/\varepsilon) \end{array}\right]$$

因a/ε極大於b, c/ε極大於d, 所以上式變成

$$\left[\begin{array}{cc|c} \varepsilon & a & c \\ 0 & 1 & c/a+\delta \end{array}\right] \begin{matrix} \leftarrow \\ (-a) \end{matrix} \qquad (\delta\text{是小量的誤差})$$

$$\sim \left[\begin{array}{cc|c} \varepsilon & 0 & -a\delta \\ 0 & 1 & c/a+\delta \end{array}\right] \begin{matrix} (1/\varepsilon) \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & -a\delta/\varepsilon \\ 0 & 1 & c/a+\delta \end{array}\right]$$

此種演算法解出

$$x=-a(\delta/\varepsilon), \qquad y=c/a+\delta$$

y仍得出合理的近似值, 但x的值來自於兩個小量相除, 將因δ的變化產生劇烈的誤差. (這類似於微積分中不定型0/0的情況)

題型04C: Vandermonde行列式

04C01 【中正79資工[3]】

Let $A=\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$

Compute det(A) and derive your result.

【解】請參閱綜合線性代數第4章定理14.

04C02 【台大76資工[2]】

Use algebraic concepts to prove the uniqueness of polynomial interpolation :

If $x_1, \dots, x_n, y_1, \dots, y_n$ are given real numbers, x_i 's are all distinct, then there is at most one polynomial f with real coefficients and of degree $n-1$ such that

$$f(x_i)=y_i,\ 1\le i\le n.$$

【解】 設 $f(x)=a_0+a_1x+\dots+a_{n-2}x^{n-2}+a_{n-1}x^{n-1}$ 滿足 $f(x_i)=y_i$, $i=1,2,\dots,n$

則
$$\begin{cases} a_0+x_1a_1+\dots+x_1^{n-2}a_{n-2}+x_1^{n-1}a_{n-1}=y_1 \\ a_0+x_2a_1+\dots+x_2^{n-2}a_{n-2}+x_2^{n-1}a_{n-1}=y_2 \\ \cdots\cdots\cdots \\ a_0+x_na_1+\dots+x_n^{n-2}a_{n-2}+x_n^{n-1}a_{n-1}=y_n \end{cases}$$

此爲以a_0, a_1,..., a_{n-1} 爲未知數的聯立方程式, 其係數行列式

$$\begin{vmatrix} 1 & x_1 & x_1^2 \ldots\ldots & x_1^{n-1} \\ 1 & x_2 & x_2^2 \ldots\ldots & x_2^{n-1} \\ & \cdots\cdots\cdots & \cdots\cdots\cdots & \\ 1 & x_n & x_n^2 \ldots\ldots & x_n^{n-1} \end{vmatrix}$$

$= \prod_{i<j}(x_j - x_i)$ (綜線CH4定理14)

$\neq 0$ (因$x_1, \ldots, x_n$相異)

$\therefore\ a_{n-1}, a_{n-1}, \ldots, a_0$ 有唯一解. (綜線CH4定理18)

【另解】若$f(x)=a_{n-1}x^{n-1}+a_{n-2}x^{n-2}+\ \ldots\ +a_1x+a_0$,

$g(x)=b_{n-1}x^{n-1}+b_{n-2}x^{n-2}+\ \ldots\ +b_1x+b_0$

滿足 $f(x_i)=y_i$ 且 $g(x_i)=y_i$, $i=1, 2, \ldots, n$.

令 $D(x)=f(x)-g(x)$.

則 $\forall\ i=1,2,\ldots, n$, $D(x_i)=f(x_i)-g(x_i)=0$

由因式定理可得 $x-x_i \mid D(x)$

因諸 x_i 相異, 所以 $(x-x_1)(x-x_2)\ldots(x-x_n) \mid D(x)$

但 $\deg D(x) \leq n-1$, $\therefore\ D(x)=0$,

$\therefore\ f(x)=g(x)$

【加強演練】(Lagrange Interpolation formula)

求一個$n-1$次多項式 $f(x)$ 滿足 $f(a_i)=b_i$, $i=1, 2, \ldots, n$.

【解】

對$i=1, 2, \ldots, n$, 令 $P_i(x) = \prod_{j\neq i} \dfrac{x-a_j}{a_i-a_j}$.

則 $P_i(a_i)=\begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$

令 $f(x)=\sum_{i=1}^{n} b_iP_i(x)$, 則 $f(x)$爲$n-1$次多項式, 且 $f(a_k)=\sum_{i=1}^{n} b_iP_i(a_k)=b_k$

04C**03** 【 清大76資料[7](c) 】

Prove or disprove the following statements.

(c) Let $(x_1, y_1)\ldots(x_{n+1}, y_{n+1})$ be $n+1$ points in the plane with $x_i\neq x_j$. Then there may have more than one polynomial $P(x)$ of degree less then or equal to n such that

$P(x_i)=y_i \quad (i=1, \ldots, n+1)$ (5%)

【解】(c) (disprove) 請參閱上題.

０４Ｃ**04** 【 中正81資工[2] 】

Let $b_1, b_2, \ldots, b_r$ and $c_1, c_2, \ldots, c_r$ be real numbers, and $t_1, t_2, \ldots, t_r$ be distinct real numbers. Prove that there exists precisely one polynomial P of degree less than or equal to $2r-1$ such that $P(t_i)=b_i$, $P'(t_i)=c_i$, for $1\le i\le r$, where P' denotes the derivative of P.

【解】[唯一性] 設$P(x)$, $Q(x)$都合乎要求, 欲證 $P(x)=Q(x)$.

令 $D(x)=P(x)-Q(x)$, 則

$$\begin{cases} \deg D(x) \le 2r-1 \text{ , 且} \\ D(t_i)=0 \text{ , } \quad D'(t_i)=0 \quad ; \quad i=1, 2, \ldots, r \end{cases}$$

對任意$i=1, 2, \ldots, r$

考慮 $D(x)$ 在 t_i 的 Tayler 展開式,

由 $D(t_i)=0$, $D'(t_i)=0$ 可知 $(x-t_i)^2$ 整除 $D(x)$

$\therefore$ $(x-t_1)^2(x-t_2)^2\ldots\ldots(x-t_r)^2$ 整除 $D(x)$

但 $\deg D(x) \le 2r-1$

$\therefore$ $D(x)=0$

即 $P(x)=Q(x)$.

[存在性] 對r進行數學歸納法:

1°當$r=1$時:

令 $P(x)=b_1+c_1(x-t_1)$

則 $P(x)$ 滿足 $\begin{cases} \deg P(x) \le 2r-1, \text{ 且} \\ P(t_1)=b_1 \text{ , } P'(t_1)=c_1, \end{cases}$

2°假設當$r=n$時此種$P(x)$存在.

3°當$r=n+1$時: (欲證此種$P(x)$存在)

考慮多項式 $f(x)$, 並令 $P(x)=b_{n+1}+c_{n+1}(x-t_{n+1})+(x-t_{n+1})^2 f(x)$

則 $P'(x)=c_{n+1}+2(x-t_{n+1}) f(x)+(x-t_{n+1})^2 f'(x)$.

對 $i=1,2,\ldots,n$,

令 $\beta_i=\dfrac{b_i-b_{n+1}-c_{n+1}(t_i-t_{n+1})}{(t_i-t_{n+1})^2}$ ， $\gamma_i=\dfrac{c_i-c_{n+1}-2(t_i-t_{n+1})\beta_i}{(t_i-t_{n+1})^2}$

由歸納法的假設得知存在多項式 $f(x)$ 滿足

$$\begin{cases} \deg f(x) \le 2n-1\text{ , 且} \\ f(t_i)=\beta_i\text{ , } \quad f'(t_i)=\gamma_i\text{ ; } \quad i=1, 2, ..., n \end{cases}$$

於是 $P(x)$ 滿足

$$\begin{cases} \deg P(x) \le (2n-1)+2 = 2(n+1)-1 \quad , \\ P(t_i)=b_i\text{ , } \quad P'(t_i)=c_i \quad , i=1,...,n \end{cases}$$

又由$P(x)$的定義得知 $P(t_{n+1})=b_{n+1}$, $P'(t_{n+1})=c_{n+1}$.

∴ $P(x)$合乎要求.

０４Ｃ**05**【 交大83資料[5] 】

Let t_1 , t_2 ,...,t_r be distinct real numbers; let $b_1, b_2, ... , b_r$ and $b_1', b_2', ... , b_r'$ be real numbers. Prove that there exits precisely one polynomial P of degree less than or equal to $2r-1$ such that $P(t_i)=b_i$ and $P'(t_i)=b_i'$ for $1\le i\le r$, where P' denotes the derivative of P. (7%)

【解】此題與上題(中正81資工[2])相同. 以下提供另一種解法:

令 $P(x)=a_0+a_1x+a_2x^2+a_3x^3+.....+a_{2r-1}x^{2r-1}$

則$P'(x)= a_1+2a_2x+3a_3x^2+4a_4x^3+.....+(2r-1)a_{2r-1}x^{2r-2}$

以所給條件代入上二式:

$$\begin{cases} b_1=a_0+a_1t_1+a_2t_1^2+a_3t_1^3+.....+a_{2r-1}t_1^{2r-1} \\ b_1'= \quad a_1+2a_2t_1+3a_3t_1^2+.....+(2r-1)a_{2r-1}t_1^{2r-2} \\ b_2=a_0+a_1t_2+a_2t_2^2+a_3t_2^3+.....+a_{2r-1}t_2^{2r-1} \\ b_2'= \quad a_1+2a_2t_2+3a_3t_2^2+.....+(2r-1)a_{2r-1}t_2^{2r-2} \\ \quad .... \\ \quad .... \\ b_r=a_0+a_1t_r+a_2t_r^2+a_3t_r^3+.....+a_{2r-1}t_r^{2r-1} \\ b_r'= \quad a_1+2a_2t_r+3a_3t_r^2+.....+(2r-1)a_{2r-1}t_r^{2r-2} \end{cases}$$

試解此含2r個等式,2r個係數a_0 , a_1 , ... , a_{2r-1} 的聯立方程組:

此方程組的係數行列式

$$\begin{vmatrix} 1 & t_1 & t_1^2 & t_1^3 & \cdots\cdots & t_1^{2r-1} \\ 0 & 1 & 2t_1 & 3t_1^2 & \cdots\cdots & (2r-1)t_1^{2r-2} \\ 1 & t_2 & t_2^2 & t_2^3 & \cdots\cdots & t_2^{2r-1} \\ 0 & 1 & 2t_2 & 3t_2^2 & \cdots\cdots & (2r-1)t_2^{2r-2} \\ & & \cdots\cdots & \cdots\cdots & & \\ & & \cdots\cdots & \cdots\cdots & & \\ 1 & t_r & t_r^2 & t_r^3 & \cdots\cdots & t_r^{2r-1} \\ 0 & 1 & 2t_r & 3t_r^2 & \cdots\cdots & (2r-1)t_r^{2r-2} \end{vmatrix}$$

$= \prod_{i<j}(t_j-t_i)^4$ (綜線CH4定理14b)

∵ $t_1, t_2, \ldots, t_r$ 兩兩相異, ∴ 係數行列式≠0 .

∴ 由Cramer's rule得知此方程組恰有一組解.

∴ 滿足所給條件的2r–1次多項式存在且唯一.

題型04D: Cramer法則

０４D**０１**【 中央83資工[1] 】

Given a linear system $Ax=b$,

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Let $A_{cof}=\begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$,

where A_{ij} is the cofactor of a_{ij}

(a) What is the cofactor A_{ij} of a_{ij} ? (5%)

(b) Show that $A^{-1}=(1/\det A)A_{cof}$ (5%)

(c) Show that $x_j=\det B_j/\det A$, where B_j is the matrix obtained from A by replacing the j-th column with the vector b (Cramer's rule). (5%)

【分析】 本題(b)(c)考定理證明. 請參閱綜線CH4定理17, CH4定理18.

【解】(a) 將原矩陣刪除第i列及第j行, 將所得的(n–1)×(n–2)矩陣取行列式, 再乘上$(-1)^{i+j}$, 就得到a_{ij} 的cofactor. (綜線CH4定義10)

(b) 令 $A\cdot A_{cof}=[m_{ij}]$, 由矩陣乘法得知

$$m_{ij}=\sum_{t=1}^{n} a_{it}A_{tj}=\begin{cases} \det A, & i=j \\ 0, & i\neq j \end{cases}$$ (綜線CH4定理11)

$\therefore\ A\cdot(\dfrac{1}{\det A}A_{cof})=I$

同法可證 $(\dfrac{1}{\det A}A_{cof})\cdot A=I$.

$\therefore\ A$可逆且 $A^{-1}=\dfrac{1}{\det A}A_{cof}$.

(c) 令 $A_{cof}=[\alpha_{jk}]$, 則 $\alpha_{jk}=A_{kj}$

$x=A^{-1}b=\dfrac{1}{\det A}(A_{cof})b=\dfrac{1}{\det A}[\alpha_{jk}][b_k]$

$\therefore\ x_j=\dfrac{1}{\det A}\sum_{k=1}^{n}\alpha_{jk}b_k=$ $\det B_j$ 的第j行展開式. (綜線CH4定理11)

04D**02** 【中央86資工[3]】

Let A be an $n\times n$ invertible martix. Show that $Ax=b$ has solution

$$x_i=\frac{\det A_i(b)}{\det A},\quad i=1,2,\ldots,n$$

where $A_i(b)$ is the matrix obtained from A by replacing the ith column with vector b.

【解】設b內的第t個數記為b_t 將 $\det A_i(b)$ 依i行展開, 得

$\det A_i(b)=\sum_{t=1}^{n}b_t\mathrm{cof}_{ti}A$ (綜線CH4定理11)

由 $Ax=b$, 得 $x=A^{-1}b=(\det A)^{-1}(\mathrm{adj}A)b$ (綜線CH定理17)

$\therefore\ x_i=(\det A)^{-1}\sum_{t=1}^{n}b_t\,\mathrm{cof}_{ti}A$ (綜線CH4定義16)

$=(\det A)^{-1}\det A_i(b)$

04D**03** 【清大86工工[1](h)】

[是非論證題]

(h) We always can apply Cramer's Rule to find a solution of $Ax=b$.

【解】(h) No. 只有在未知數個數與等式個數相等時才有可能套用Cramer's rule.
(事實上還需要det$A \neq 0$才能套用) (綜線CH4定理18)

0 4 D**04**【 中原86工工[2] 】

Solve the following equations using determinants:

$$3y+2x=z+1$$
$$3x+2z=8-5y$$
$$3z-1=x-2y$$

【解】原方程式整理得

$$\begin{cases} 2x+3y-z=1 \\ 3x+5y+2z=8 \\ x-2y-3z=-1 \end{cases}$$

係數行列式為 $\begin{vmatrix} 2 & 3 & -1 \\ 3 & 5 & 2 \\ 1 & -2 & -3 \end{vmatrix} = \ldots = 22$

$$\begin{vmatrix} 1 & 3 & -1 \\ 8 & 5 & 2 \\ -1 & -2 & -3 \end{vmatrix} = \ldots = 66, \qquad x = 66/22 = 3$$

$$\begin{vmatrix} 2 & 1 & -1 \\ 3 & 8 & 2 \\ 1 & -1 & -3 \end{vmatrix} = \ldots = -22, \qquad y = -22/22 = -1$$

$$\begin{vmatrix} 2 & 3 & 1 \\ 3 & 5 & 8 \\ 1 & -2 & -1 \end{vmatrix} = \ldots = 44, \qquad z = 44/22 = 2$$

0 4 D**05** 【 清大81工工[6] 】

Solve the following system by Cramer's rule.

$$2x_1 + x_2 = 6$$

$5x_1 - 2x_2 = 4$

【解】

$$D = \begin{vmatrix} 2 & 1 \\ 5 & -2 \end{vmatrix} = -4-5 = -9$$

$$x_1 = \frac{1}{D}\begin{vmatrix} 6 & 1 \\ 4 & -2 \end{vmatrix} = \frac{-12-4}{-9} = \frac{-16}{-9} = 16/9$$

$$x_2 = \frac{1}{D}\begin{vmatrix} 2 & 6 \\ 5 & 4 \end{vmatrix} = \frac{8-30}{-9} = \frac{-22}{-9} = 22/9$$

(綜線CH4定理18)

題型04E: 方塊行列式

04E01 【 台大85資工[4] 】

[複選題]

Which of the following are true.

(1) The product of two matrices always has rank equal to the lesser of the ranks of the two matrices.

(2) Let A, B be square matrices. Then, $\det(A \cdot B) = \det(A) \cdot \det(B)$.

(3) Let A, B, and C be square matrices. Then,

$$\det\begin{pmatrix} A & B \\ O & C \end{pmatrix} = \det(A) \cdot \det(C).$$

(4) Let E_1 and E_2 be two elementary matrices, which symbolizes a row-interchange and column-interchange operations, respectively. Then, $\det(E_1 \cdot A \cdot E_2) = \det(A)$.

【解】選(2)(3)(4)

【討論】 (1) False. 反例如下:

$$\text{rank}\begin{bmatrix} 0 & 1 \end{bmatrix} = 1, \quad \text{rank}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1,$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}. \quad \text{rank}\begin{bmatrix} 0 \end{bmatrix} = 0.$$

本題若將equal to 改爲less than or equal to 就成立. (綜線CH8定理16)

(2) True. 此爲定理. (綜線CH4定理6)

(3) True. 此爲定理. (綜線CH4定理20)

(4) True.

$\det(E_1AE_2) = \det E_1 \det A \det E_2$ (綜線CH4定理6)

$= (-1) \cdot \det A \cdot (-1) = \det A$ (綜線CH3定義13, CH3定義22,CH4定理7)

04E02【清大76資料[5](a)】

Assume A and B are two $n\times n$ square matrices, show that

(a) $\det\begin{bmatrix} A & B \\ B & A \end{bmatrix} = \det(A+B)\det(A-B)$ (10%)

【解】請參閱綜線CH4範例23.

04E03【中正84資工[2]】

Let A, B, C, D be $n\times n$ matrices and I be $n\times n$ identity matrix with A invertible.

(a) Find matrices X and Y to produce the block LU factorization

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & O \\ X & I \end{bmatrix}\cdot\begin{bmatrix} A & B \\ O & Y \end{bmatrix}$$ and then show that

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\cdot\det(D-CA^{-1}B).$$

(b) Show that if $AC=CA$, then $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD-CB)$

【分析】本小題的背景請參閱綜線CH4定理21.

【解】(a) 由塊狀乘法, (綜線CH2定理8)

$C=XA+IO=XA$, $D=XB+IY=XB+Y$.

$\therefore X=CA^{-1}$, $Y=D-XB=D-CA^{-1}B$

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det\begin{bmatrix} I & O \\ X & I \end{bmatrix}\det\begin{bmatrix} A & B \\ O & Y \end{bmatrix}$$ (綜線CH4定理6)

$=\det I\det I\det A\det Y$ (綜線CH4定理20)

(b)

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \cdot \det(D - CA^{-1}B)$$ (由(a))

$$= \det(AD - ACA^{-1}B)$$ (綜線CH4定理6)

$$= \det(AD - CAA^{-1}B)$$ (已知)

$$= \det(AD - CB)\ .$$