題型05A: 向量空間的基本性質

０５A**01** 【 清大83工工[5] 】

Show that all real $p\times 1$ column matrices with matrix addition and matrix multiplication by a scalar is a vector space. (20%)

【解】設$V=\{v \mid v$爲real $p\times 1$ column matrix $\}$.

由矩陣加法及係數積之定義得知:

$$\forall \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in V, \quad \forall k \in \mathbb{R}:$$

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} \in V, \quad k\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} kv_1 \\ \vdots \\ kv_n \end{bmatrix} \in V.$$

以下驗證各性質:

$$\forall \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}, \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in V, \quad \forall h,k \in \mathbb{R},$$

1°
$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} = \begin{bmatrix} v_1+u_1 \\ \vdots \\ v_n+u_n \end{bmatrix} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

2°
$$\left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right) + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$$

$$= \begin{bmatrix} u_1+v_1+w_1 \\ \vdots \\ u_n+v_n+w_n \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1+w_1 \\ \vdots \\ v_n+w_n \end{bmatrix}.$$

$$= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \left(\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \right)$$

3°

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

4°

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} -u_1 \\ \vdots \\ -u_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

5°

$$(h+k)\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} (h+k)u_1 \\ \vdots \\ (h+k)u_n \end{bmatrix} = \begin{bmatrix} hu_1+ku_1 \\ \vdots \\ hu_n+ku_n \end{bmatrix}.$$

$$= \begin{bmatrix} hu_1 \\ \vdots \\ hu_n \end{bmatrix} + \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix} = h\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + k\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

6°

$$k\left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \right) = k\begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix} = \begin{bmatrix} k(u_1+v_1) \\ \vdots \\ k(u_n+v_n) \end{bmatrix}$$

$$= \begin{bmatrix} ku_1+kv_1 \\ \vdots \\ ku_n+kv_n \end{bmatrix} = \begin{bmatrix} ku_1 \\ \vdots \\ ku_n \end{bmatrix} + \begin{bmatrix} kv_1 \\ \vdots \\ kv_n \end{bmatrix} = k\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + k\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

7°

$$(hk)\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}=\begin{bmatrix}hku_1\\ \vdots\\ hku_n\end{bmatrix}=h\begin{bmatrix}ku_1\\ \vdots\\ ku_n\end{bmatrix}=h(k\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix})$$

8°

$$1\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}=\begin{bmatrix}1u_1\\ \vdots\\ 1u_n\end{bmatrix}=\begin{bmatrix}u_1\\ \vdots\\ u_n\end{bmatrix}$$

05A02 【 中正84資工[1](a) 】

True or False:

(a) The number of vectors in each vector space is infinite.

【解】(a) False.

零空間 $\{o\}$ 只含一個向量.

即使是非零空間, 若有限維, 而所佈的體爲有限體, 則空間中所含的向量也只是有限多個.

05A03 【 台大84資工[1](a) 】

[是非論證題]

a. Let $V=\{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$, define

$$(a_1, a_2)+(b_1, b_2)=(a_1+b_1, a_2+b_2)$$

$$c\cdot(a_1, a_2)=\begin{cases}0 & \text{if } c=0.\\ (a_1, a_2/c) & \text{if } c\neq 0.\end{cases}$$

then $(V,+,\cdot)$ is a vector space.

【解】(a) Disprove.

依所給的運算: $2\cdot(1,0)+3\cdot(1,0)=(1,0)+(1,0)=(2,0)$

$(2+3)\cdot(1,0)=5\cdot(1,0)=(1,0)$

$\therefore\ 2\cdot(1,0)+3\cdot(1,0)\neq(2+3)\cdot(1,0)$

此與向量空間所要求的性質不符. (綜線CH5定義3)

05A04 【 清大81工工[7.1] 】

Which statement is true?

(a) Matrix multiplication is a vector–space operation on the set M of all $n \times n$ matrices..

(b) The set Q of all rational numbers is a real vector space under the usual operations of addition and scalar multiplication.

(c) Every vector space has at least two vectors.

(d) Multiplication of any vector by the zero scalar always yields the zero vector.

(e) Multiplication of a nonzero vector by a nonzero scalar never yields the zero vector.

(f) none of the above.

【解】選(d)(e).

分別解說如下:

(a) $M_{n\times n}$當做向量空間時，只考慮“矩陣加法”和“矩陣係數積”兩種運算。 (綜線CH5範例5)

(b) 實數乘有理數未必是有理數，與係數積的要求不符。 (綜線CH5定義3)

(c) 零空間 $\{o\}$ 內只有一個向量。

(d) $0v=(0+0)v=0v+0v$,

兩邊加入 $-0v$ 即得 $o = 0v$ (綜線CH5定理8)

(e) 若純量$k\neq 0$，向量$v\neq o$，則必定 $kv\neq o$

否則，在$kv=o$時，兩邊左乘$1/k$將得出$v=o$，與已知條件矛盾. (綜線CH5定理8)

題型05B: 子空間的判定

05B01 【大同86資工[1]】

Determine whether or not the following sets are subspaces of $\mathbb{R}^3$. Prove your answers. (9%)

(a) $\{(x_1, x_2, x_3)^T \mid x_1+x_3=1\}$

(b) $\{(x_1, x_2, x_3)^T \mid x_1=x_2=x_3\}$

(c) $\{(x_1, x_2, x_3)^T \mid x_3=x_1+x_2\}$

【解】(a) No.

$(1, 0, 0)^T$ 在集合中, 但 $2(1, 0, 0)^T$ 不在集合中.

∴ 此集合不具封閉性.

(b) Yes.

顯然此爲非空集合.

對集合內的$(a,a,a)^T$, $(b,b,b)^T$及任意實數x, y

$x(a,a,a)^T+y(b,b,b)^T=(xa+yb,xa+yb,xa+yb)^T$, 仍在集合之內.

∴ 此集合具封閉性. (綜線CH5定理11)

(c) Yes.

顯然此爲非空集合.

對集合內的$(a,b,a+b)^T$, $(c,d,c+d)^T$及任意實數x, y

$x(a,b,a+b)^T+y(c,d,c+d)^T$

$=(xa+yc,xb+yd,x(a+b)+y(c+d))^T$

$=(xa+yc,xb+yd,(xa+yc)+(xb+yd))^T$

仍在集合之內.

∴ 此集合具封閉性. (綜線CH5定理11)

05B02 【交大79工工[4]】

是非題:

The set of all vectors of the form $(a, b, -a)$ is a subspace of $\mathbb{R}$.

【解】是

【討論】證明如下:

令$V=\{ (a, b, -a) \mid a, b\in \mathbb{R}^3\}$

顯然$V\neq \{o\}$, 以下證明V具有封閉性: (綜線CH5定理11)

$\forall x, y\in \mathbb{R}, \forall\, (p, q, -p), (r, s, -r)\in V,$

$x(p, q, -p)+y(r, s, -r)=(xp+yr, xq+ys, -(xp+yr))\in V$

故得證.

05B**03** 【 中正84資工[1](b) 】

True or False:

(b) $\{ (x_1 ,x_2 ,x_3) \mid x_1+x_2+x_3=1 \}$ is a subspace of $\mathbb{R}^3$.

【解】(b) False.

例如(1,0,0), (0,1,0) 都在此集合內, 但相加後(1,1,0)不在此集合內, 不滿足封閉性.

05B**04** 【 師大84資教[5] 】

(a) Is $S=\{ (x, 1)^t \mid x\in \mathbb{R}\}$ a subspace of $\mathbb{R}^2$?

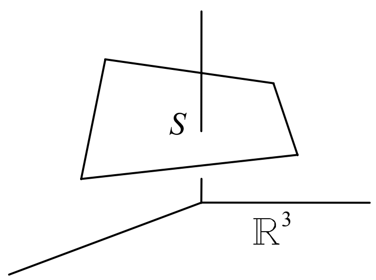
(b) Is $S=\{A\in \mathbb{R}^{2\times2} \mid a_{12}=-a_{21}\}$ a subspace of $\mathbb{R}^{2\times2}$?

(c) Is $S=\{(x_1 , x_2 , x_3)^t \mid x_1=x_2\}$ a subspace of $\mathbb{R}^3$?

【解】(a) No. (b) Yes. (c) Yes. (綜線CH5範例12—14)

05B**05** 【 師大84資教[6] 】

Is S a subspace of $\mathbb{R}^3$?



【解】No. 因爲S不通過o. (綜線CH5定理11.)

０５Ｂ**06** 【 元智85工工甲[1](1) 】

Please provide detailed answers for the following questions.

(1) Show that the set of nonsingular 3×3 matrices is not a vector space. (10%)

【解】(1) 條件“non–singular”並不具備封閉性: (綜線CH5定理11)

$$例如A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}2 & 0\\ 0 & -1\end{bmatrix},$$

由行列式得知A, B都是non–singular.

$$但\ A+B=\begin{bmatrix}3 & 0\\ 0 & 0\end{bmatrix}$$ 卻是singular matrix.

０５Ｂ**07** 【 雲技84工工[9] 】

我們稱W子集(subset)爲向量空間(vector space)V的子空間(subspace), 如果W本身定義於V在加法(addition)與乘法(multiplication)上爲一向量空間。 試證明當$a+d=0$時,

$$A=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

A爲一個子空間。

【分析】本考題寫的是中文, 用的卻是英文文法. 讀者應自己辨明意義.

A是矩陣, 當然不能是子空間. 這題原意是要求證

$$K^{2\times2}\ 的子集\ \left\{\begin{bmatrix}a & b\\ c & d\end{bmatrix}\ \middle|\ a+d=0\right\}$$ 是它的子空間.

【解】

若 $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A'=\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$, 滿足 $a+d=0$, $a'+d'=0$.

對任意純量 k, k',

$$kA+k'A'=k\begin{bmatrix} a & b \\ c & d \end{bmatrix}+k'\begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}=\begin{bmatrix} ka+k'a' & kb+k'b' \\ kc+k'c' & kd+k'd' \end{bmatrix}$$

$(ka+k'a')+(kd+k'd')=k(a+d)+k'(a'+d')=k\cdot 0+k'\cdot 0=0$

∴ 此種 A 滿足封閉性, ∴ 是子空間. (綜線CH5定理11)

０５Ｂ**08**【 交大85資料[2] 】

[複選題]

In the following statements, which ones are wrong ?

(a) Using natural operations, two–dimensional geometrical vectors whose "heads" lie in the first quadrant are a real vector space.

(b) Using natural operations, ratios $P_m(x)/Q_n(x)$ —— with Q_n not be the zero polynomial —— of polynomials P_m of degree at most m and Q_n of degree at most n, where m and n are allowed to range over all nonnegative integers, are a real vector space.

(c) The set of x satisfying $Ax=b$, where A is a real $p\times q$ matrix and b is a real $p\times 1$ matrix with $b\neq 0$ is a subspace of $\mathbb{R}_q$.

(d) Let V be a vector space, o be the zero vector of V, then $\{ o \}$ is not a subspace.

(e) The vectors $V_1=[2 \quad -1 \quad 1]^T$, $V_2=[1 \quad 1 \quad 2]^T$, and $V_3=[10 \quad -8 \quad 2]^T$ form a basis for $\mathbb{R}^3$.

【解】選(a)(b)(c)(d)(e).

【說明】注意: 本題是要選出錯的.

(a) 例如 $[1 \quad 1]^T$ 合條件, 但 $(-1)[1 \quad 1]^T$ 不合條件, 違反封閉性.

(b) 例如 $1/(x^n)$, 與 $1/(x^n+1)$ 都合條件, 但相加後卻不合條件.

(c) $\mathbb{R}^q$ 中的零向量不合條件.

(d) 零空間是子空間.

(e) (請參閱題型06C)

$$\begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 2 \\ 10 & -8 & 2 \end{bmatrix} \sim \begin{bmatrix} 0 & -3 & -3 \\ 1 & 1 & 2 \\ 0 & -18 & -18 \end{bmatrix} \sim \begin{bmatrix} 0 & -3 & -3 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ [2 –1 1], [1 1 2], [10 –8 2] 線性相關.(綜線CH6定理23③)

05B**09** 【 台大79資工[4](iv) 】

(Yes or No question and explain the reason:)

Let V be the (real) vector space of all function f from $\mathbb{R}$ to $\mathbb{R}$.

$W=\{f\in V \mid f(x^2)=f(x)^2\}$ Then W is a subspace of V.

【解】 NO

考慮函數 $f(x)=x^3$, (取 $f(x)=1$也可以)

因 $f(x^2)=(x^2)^3=(x^3)^2=f(x)^2$

$\therefore f(x)\in W$

但對$g(x)=2f(x)=2x^3$

$g(x^2)=2x^6\neq(2x^3)^2=(g(x))^2$ (至少在$x=1$時不等)

$\therefore 2f(x)\notin W$

$\therefore$ W不具有封閉性, 不爲V的子空間 (綜線CH5定義10)

【加強演練】

(1) 已知$f:\mathbb{R}^+\longrightarrow\mathbb{R}$，滿足 $f(x^c)=(f(x))^c$， $x>0, c\in\mathbb{R}$. 試求f的一般型.

(2) 問所有此種函數所成的集合是否形成向量空間?

[解] (1) $f(x)=f(e^{\ln(x)})=f(e)^{\ln(x)}$

令 $a=\ln f(e)$, 則 $f(e)=e^a$ (對數的定義)

$\therefore f(x)=e^{a\ln(x)}=x^a$ (對數的基本性質)

(2) 對 $f(x)=x^2$, $g(x)=x^3$,

$(f+g)(x)=x^2+x^3$, 不再是乘冪函數

$\therefore$ 此集合對加法缺乏封閉性, 不能形成向量空間.

０５Ｂ**１０** 【 交大83資工[3] 】

Let V be the vector space of all functions f from $\mathbb{R}$ to $\mathbb{R}$. Which of the following sets of functions are subspaces of V? Note that for given f and g in V and the scalar c, we define $(f+g)(x)=f(x)+g(x)$ and $(cf)(x)=cf(x)$ (NO JUSTIFICATION IS REQUIRED)

(a) all f such that $f(x^2)=f(x)^2$

(b) all f such that $f(0)=f(1)$

(c) all f such that $f(3)=1+f(-5)$

(d) all f such that $f(-1)=0$

(e) all f which are continuous (5%)

【解】選(b)(d)(e)

【討論】(a) 設 f 滿足要求條件. (例如 $f(x)=x^3$)

對$g=2f$. $g(x^2)=(2f)(x^2)=2f(x^2)=2(f(x))^2$,

$g(x)^2=(2f(x))^2=4(f(x))^2$.

∴ 通常 $g(x^2)\neq g(x)^2$.

(b) 證明封閉性:

設 f, g滿足要求, a,b爲常數.

$(af+bg)(0)=(af)(0)+(bg)(0)=af(0)+bg(0)$

$=af(1)+bg(1)=(af)(1)+(bg)(1)=(af+bg)(1)$.

∴ $af+bg$合乎條件.

(c) 設 f合乎要求. (例如 $f(x)=(x+5)/8$).

令$g=2f$, 則g必不合乎要求.

(d) 仿(b)可證封閉性成立.

(e) 連續函數的線性組合仍爲連續函數. (微積分的定理)

０５Ｂ**１１** 【 中央86資工[1](h) 】

[是非論證題]

(h) Let $\mathbb{R}^n$ be the set of all vectors with n entries. $\mathbb{R}^n$ is a subspace of $\mathbb{R}^{n+1}$

【解】(h) False.

$\mathbb{R}^n\cap\mathbb{R}^{n+1}=\varnothing$.

$\mathbb{R}^n$ 並不是 $\mathbb{R}^{n+1}$ 的子集合, 所以不能是子空間.

【討論】有些數學系的書可將 $\mathbb{R}^n$ 視爲(identify) $\{ (x_1, x_2, ..., x_n, 0) \mid x_1, x_2, ..., x_n \in \mathbb{R} \}$.
如此本題就變成True. 但一般書通常都不這樣處理.

05B12 【 交大86資工[10] 】

Let $\mathbb{R}$ be the set of real numbers. Let B be a nonzero matrix in $\mathbb{R}$ Define $H = \{2 \cdot M + B \mid M \in \mathbb{R}^{m \times n}\}$, where the operators $+$ and $\cdot$ denote the matrix addition and the scalar multiplication, respectively. Is $(H, +, \cdot)$ a subspace of $(\mathbb{R}^{m \times n}, +, \cdot)$? Justify your answer.

【解】Yes.

$\forall Z \in \mathbb{R}^{m \times n}$, 令 $M = (1/2)(Z - B)$, 則 $Z = 2 \cdot M + B$.

$\therefore H = \mathbb{R}^{m \times n}$, 當然是 $\mathbb{R}^{m \times n}$ 的子空間.

05B13 【 台大77資工[5](ii) 】

(True or false, with counterexample if false:)

The intersection of two subspaces of a vector space cannot be empty

【解】 true.

因每個子空間都含零向量. 所以交集至少含零向量.

05B14 【 台大80資工[1] 】

[True or False Problem]

If V is a vector space, and W_1 , W_2 are its subspaces, then $W_1 \cap W_2$ and $W_1 \cup W_2$ are also subspaces of V.

【參考章節】CH5定理22, 範例22a, 定理24

【要訣】只要 p 錯或 q 錯, (p 且 q)就錯.

【解】 False; $W_1 \cup W_2$ 未必是 V 的子空間. 反例如下:

取 $V = \mathbb{R}^2 = \{(x, y) \mid x, y \in \mathbb{R}\}$,

$W_1 = \{(x, 0) \mid x \in \mathbb{R}\}$, $W_2 = \{(0, y) \mid y \in \mathbb{R}\}$.

則 W_1, W_2 爲 V 的subspace,

但 $W_1 \cup W_2$ 並非 V 的subspace.

０５Ｂ**１５**【 元智83工工[1] 】

[是非題]

If V and W are two subspaces of $\mathbb{R}^n$, then both $V \cup W$ and $V \cap W$ are subspaces of $\mathbb{R}^n$. (2%)

【解】× , (本題爲上題之特例)

０５Ｂ**１６**【 台大83資工[1] 】

[複選題]

If V is a vector space, and W_1 , W_2 are its subspaces, then which of the following is (or are) also subspace(s) of V :

(1) $W_1 \cap W_2$ (2) $W_1 \cup W_2$

(3) $W_1 + W_2$ (4) $(W_1 \cap W_2) + (W_1 \cup W_2)$.

【解】 選(1)(3).

【討論】$W_1 \cap W_2 \subseteq W_1 \cup W_2$, 所以

$(W_1 \cap W_2) + (W_1 \cup W_2) = W_1 \cup W_2$. (註一)

子空間的聯集通常不再是子空間. (綜線CH5定理22,定理24,定理27)

[註一] $(W_1 \cap W_2) + (W_1 \cup W_2) = W_1 \cup W_2$ 證明如下:

[⊇] $\forall\, v \in W_1 \cup W_2$, $v = o + v \in (W_1 \cap W_2) + (W_1 \cup W_2)$.

[⊆] $\forall\, u \in W_1 \cap W_2$, $\forall v \in W_1 \cup W_2$, (此時 $v \in W_1$ 或 $v \in W_2$)

若$v \in W_1$, 則 $u + v \in (W_1 \cap W_2) + W_1 = W_1$

若$v \in W_2$, 則 $u + v \in (W_1 \cap W_2) + W_2 = W_2$

$\therefore\ u + v \in W_1 \cup W_2$.

題型05C: 子空間的計算

05C01 【 交大80資工[1](a)(ⅰ,ⅱ,ⅳ) 】

True (T) or False (F): (1 for each)

(a) Suppose A is row equivalent to B, i.e., $A \overset{R}{\sim} B$.

ⅰ. NS(A)＝NS(B)

ⅱ. CS(A)＝CS(B)

ⅳ. RS(A)＝RS(B)

【解】(a) (ⅰ) True. 請參閱CH5定理20

(a) (ⅱ) False. 反例如下:

$$取A=\begin{bmatrix}1\\2\end{bmatrix},\ B=\begin{bmatrix}1\\0\end{bmatrix}$$

則 $A \overset{R}{\sim} B$

$$但\ \mathrm{CS}(A)=\left\{ t\begin{bmatrix}1\\2\end{bmatrix} \middle| \ t為純量 \right\}$$

$$\mathrm{CS}(B)=\left\{ t\begin{bmatrix}1\\0\end{bmatrix} \middle| \ t為純量 \right\}$$ (綜線CH5定義16①)

可知 CS(A)≠CS(B)

(a) (ⅳ) True. 請參閱CH5定理17

05C02 【 中央84資工[1](f) 】

(f) If matrices A and B are row equivalent then their column spaces are the same, but their row spaces may be different.

【解】(f) False.

例如: $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ 列等價於 $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

前者的column space 爲 $\left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| t \in \mathbb{R} \right\}$,

後者的column space 爲 $\left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} \middle| t \in \mathbb{R} \right\}$,

並不相等.

本題正好講反了. 應該是

... then their row spaces are the same, but their column spaces may be different.

05C**03**【 交大84資科[5] 】

Show that the row space of matrix AB is a subspace of the row space of matrix B .

【參考章節】綜線CH5定理21a.

【解】 $x \in \mathrm{RSP}(AB)$

$\Longrightarrow \exists u$ 使得 $x = u(AB)$ (CH5定理17)

$\Longrightarrow \exists u$ 使得 $x = (uA)B$

$\Longrightarrow x \in \mathrm{RSP}B$ (CH5定理17)

[另證]

對任意的k,

"AB的第k列" 就是 "(A的第k列)B", (CH2定理7①)

它是B的列的線性組合. (CH2定理7②)

所以"AB的第k列" 在RSPB之內.

所以拿AB的列做出的線性組合也都在RSPB之內. (RSPB的封閉性)

所以 $\mathrm{RSP}(AB) \subseteq \mathrm{RSP}(B)$.

0 5 C**04** 【 中央82資電[4] 】

Prove that

(a) The column space of matrix AB is contained in the column space of matrix A (5%)

(b) The left nullspace of matrix AB contains the left null space of matrix A. (5%)

【解】(a) $x\in$[column space of AB]

$\Longrightarrow \exists u$ 使得$x=(AB)u$ (綜線CH5定理17)

$\Longrightarrow \exists u$ 使得$x=A(Bu)$

$\Longrightarrow \exists w$ 使得$x=Aw$

$\Longrightarrow x\in$ [column space of A] (綜線CH5定理17)

(b) $x \in$ [left null space of A]

$\Longrightarrow xA=o$ (綜線CH5定義19)

$\Longrightarrow (xA)B=o$

$\Longrightarrow x(AB)=o$

$\Longrightarrow x\in$[left null space of AB] (綜線CH5定義19)

0 5 C**05** 【 交大81資工[3](b) 】

Let

$$A=\begin{bmatrix} 1 & -2 & 1 & 1 \\ -1 & 3 & 0 & 2 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

(b) (3%) Suppose space V is spanned by the first two columns of A, and space W is spanned by the last two columns of A. Find a basis for their intersection $V\cap W$.

【參考章節】 綜合線性代數CH5範例23

【解】(b) 由題意:

$$V=\left\{ x\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \,\middle|\, x, y \in \mathbb{R} \right\},$$

$$W=\left\{p\begin{bmatrix}1\\0\\1\end{bmatrix}+q\begin{bmatrix}1\\2\\2\end{bmatrix}\,\middle|\, p,q\in\mathbb{R}\right\}.$$

考慮W內的向量 $w=p\begin{bmatrix}1\\0\\1\end{bmatrix}+q\begin{bmatrix}1\\2\\2\end{bmatrix}$,

$w\in V$

$\iff$ $\exists x,y$ 使得 $x\begin{bmatrix}1\\-1\\0\end{bmatrix}+y\begin{bmatrix}-2\\3\\1\end{bmatrix}=w$

$\iff$ x,y的方程式 $\begin{cases}x-2y=p+q\\-x+3y=2q\\y=p+2q\end{cases}$ 有解

以列運算試解前述方程式:

$$\left[\begin{array}{cc|c}1&-2&p+q\\-1&3&2q\\0&1&p+2q\end{array}\right]\sim\left[\begin{array}{cc|c}1&-2&p+q\\0&1&p+3q\\0&1&p+2q\end{array}\right]\sim\left[\begin{array}{cc|c}1&-2&p+q\\0&1&p+3q\\0&0&-q\end{array}\right]$$

$\therefore$ $w\in V \iff q=0$ (綜線CH3定理10)

$$\iff w=p\begin{bmatrix}1\\0\\1\end{bmatrix}$$

$$\therefore\ V\cap W=\left\{ p\begin{bmatrix}1\\0\\1\end{bmatrix} \middle|\ p\in\mathbb{R} \right\}$$

$$\therefore\ 取\ \left\{ \begin{bmatrix}1\\0\\1\end{bmatrix} \right\}$$ 爲$V\cap W$的基底。

05C**06** 【 元智80工工[5] 】

Let $U=L(\,(1, 2, 3, 6), (4, -1, 3, 6), (5, 1, 6, 13)\,)$, $V=L(\,(1, -1, 1, 1), (2, -1, 4, 5)\,)$

(Here, $L(x_1, x_2, \dots , x_r)$ stands for the linear–span by vectors $x_1, x_2, \dots , x_r$)

Find a basis for $U\cap V$.

【解】考慮 $v=p(1,-1,1,1)+q(2,-1,4,5)\in V$:

$v\in U$

$\Longleftrightarrow$ $\exists x, y, z$使得$x(1,2,3,6)+y(4,-1,3,6)+z(5,1,6,13)= p(1,-1,1,1)+q(2,-1,4,5)$

$\Longleftrightarrow$ x, y, z的方程組

$$\begin{cases} x+4y+5z=p+2q \\ 2x-y+z=-p-q \\ 3x+3y+6z=p+4q \\ 6x+6y+13z=p+5q \end{cases}$$

有解 (p, q視爲已知數) .

以列運算試解前述方程式:

$$\left[\begin{array}{ccc|c}1&4&5&p+2q\\2&-1&1&-p-q\\3&3&6&p+4q\\6&6&13&p+5q\end{array}\right]\xrightarrow{(-2)(-3)(-6)}\sim\left[\begin{array}{ccc|c}1&4&5&p+2q\\0&-9&-9&-3p-5q\\0&-9&-9&-2p-2q\\0&-18&-17&-5p-7q\end{array}\right]\xrightarrow{(-1)(-2)}$$

$$\sim\left[\begin{array}{ccc|c}1&4&5&p+2q\\0&-9&-9&-3p-5q\\0&0&0&p+3q\\0&0&1&p+3q\end{array}\right]\sim\left[\begin{array}{ccc|c}1&4&5&p+2q\\0&-9&-9&-3p-5q\\0&0&1&p+3q\\0&0&0&p+3q\end{array}\right]$$

∴ 方程式有解 $\Longleftrightarrow p+3q=0$ (綜線CH3定理10)

$\Longleftrightarrow p=-3q$

∴ $U\cap V=\{v\in V \mid v\in U\}=\{p(1,-1,1,1)+q(2,-1,4,5) \mid p=-3q\}$

$=\{-3q(1,-1,1,1)+q(2,-1,4,5) \mid q$為任意純量 $\}$

$=\{q(-1,2,1,2) \mid q$為任意純量 $\}$

【另解】考慮 $u=x(1,2,3,6)+y(4,-1,3,6)+z(5,1,6,13)\in U$

$u\in V$

$\Longleftrightarrow \exists\, p,q$ 使得$p(1,-1,1,1)+q(2,-1,4,5)=x(1,2,3,6)+y(4,-1,3,6)+z(5,1,6,13)$

$\Longleftrightarrow p,q$的方程組

$$\begin{cases}p+2q=x+4y+5z\\-p-q=2x-y+z\\p+4q=3x+3y+6z\\p+5q=6x+6y+13z\end{cases}$$

有解 (x, y, z視為已知數) .

以列運算試解前述方程式:

$$\left[\begin{array}{cc|c} 1 & 2 & x+4y+5z \\ -1 & -1 & 2x-y+z \\ 1 & 4 & 3x+3y+6z \\ 1 & 5 & 6x+6y+13z \end{array}\right] \begin{array}{c}(1)(-1)(-1)\end{array} \sim \left[\begin{array}{cc|c} 1 & 2 & x+4y+5z \\ 0 & 1 & 3x+3y+6z \\ 0 & 2 & 2x-y+z \\ 0 & 3 & 5x+2y+8z \end{array}\right] \begin{array}{c}(-2)(-3)\end{array}$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & x+4y+5z \\ 0 & 1 & 3x+3y+6z \\ 0 & 0 & -4x-7y-11z \\ 0 & 0 & -4x-7y-10z \end{array}\right] \begin{array}{c}(-1)\end{array} \sim \left[\begin{array}{cc|c} 1 & 2 & x+4y+5z \\ 0 & 1 & 3x+3y+6z \\ 0 & 0 & -4x-7y-11z \\ 0 & 0 & z \end{array}\right]$$

$\therefore$ 方程式有解 $\Longleftrightarrow \begin{cases} -4x-7y-11z=0 \\ z=0 \end{cases}$ (綜線CH3定理0)

$\Longleftrightarrow \begin{cases} -4x-7y=0 \\ z=0 \end{cases} \Longleftrightarrow \begin{cases} 4x+7y=0 \\ z=0 \end{cases}$

$\therefore$ $U\cap V=\{\, u\in U \mid u\in V \,\}$

$=\{\, x(1,2,3,6)+y(4,-1,3,6)+z(5,1,6,13) \mid 4x+7y=0,\ z=0 \}$

$=\{\, x(1,2,3,6)+y(4,-1,3,6)+z(5,1,6,13) \mid x=7t,\ y=-4t,\ z=0,\ t$爲任意純量$\}$

$=\{\, 7t(1,2,3,6)-4t(4,-1,3,6) \mid t$爲任意純量 $\}$

$=\{\, t(-9,18,9,18) \mid t$爲任意純量 $\}=\{\, s(-1,2,1,2) \mid s$爲任意純量 $\}$

05C**07**【元智84工工X[2]】

考慮多項式所成的向量空間. 令$V_3(\mathbb{R})=\langle 1, x, x \rangle$所生成的向量空間.

① 若$v_1=1+x^2$; $v_2=x^2-x$, $v_3=3-2x$; 試問$\langle v_1, v_2, v_3 \rangle$是否能生成$V_3(\mathbb{R})$

(即: $\langle v_1, v_2, v_3 \rangle \overset{?}{=} V_3(\mathbb{R})$) [證明之或給反例] (10%)

② 考慮$U=\langle 1+2x+x^3, 1-x-x^2 \rangle$, $V=\langle x+x^2-3x^3, 2+2x-2x^3 \rangle$

試求空間$U+V$的一組基底. (12%)

又問$\dim(U+V)$是否與$\dim(V_3(\mathbb{R}))$相同. (2%)

③ 求$U\cap V$的一組基底(15%) 與$\dim(U\cap V)$. (1%)

④ 於本題中, $\dim(U+V)+\dim(U\cap V)$是否等於$\dim(U)+\dim(V)$? (5%)

【解】① (請參閱題型06C)

利用isomorphic 的觀念,

將 $a+bx+cx^2$ 視爲 $[\,a \quad b \quad c\,]$, 將$V_3(\mathbb{R})$視爲$\mathbb{R}^{1\times 3}$.

v_1, v_2, v_3 分別視爲 $[\,1 \quad 0 \quad 1\,], [\,0 \quad -1 \quad 1\,], [\,3 \quad -2 \quad 0\,]$.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -2 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & -5 \end{bmatrix}$$

$\therefore\ \dim\langle v_1, v_2, v_3\rangle = 3$ (綜線CH6定理23)

$\therefore\ \langle v_1, v_2, v_3\rangle = V(\mathbb{R})$ (綜線CH6定理22a)

② 將 $a+bx+cx^2+dx^4$ 視爲$[a \quad b \quad c \quad d\,]$, 將$V_4(\mathbb{R})$視爲$\mathbb{R}^{1\times 4}$.

$1+2x+x^3$ 視爲$[1 \quad 2 \quad 0 \quad 1]$, $1-x-x^2$視爲$[1 \quad -1 \quad -1 \quad 0]$.

$x+x^2-3x^3$ 視爲$[0 \quad 1 \quad 1 \quad -3]$, $2+2x-2x^3$ 視爲$[2 \quad 2 \quad 0 \quad -2]$.

$$\begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & -3 \\ 2 & 2 & 0 & -2 \end{bmatrix} \underset{\cdots\cdots}{\overset{\text{列運算}}{\sim \qquad \sim}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

#

$\therefore\ U+V$的基底可取爲 $\{1-3x^3, x+2x^3, x^2-5x^3\}$

$\dim(U+V)=3=\dim V_3(\mathbb{R})$.

③ 仍將各多項式視爲列矩陣,

考慮V中的一般向量$v=p(0, 1, 1, -3)+q(2, 2, 0, -2)$:

$v\in U \iff \exists\, x, y$ 使$x(1, 2, 0, 1)+y(1, -1, -1, 0)=v$.

試解x,y的方程式 $\begin{cases} x+y=2q \\ 2x-y=p+2q \\ -y=p \\ x=-3p-2q \end{cases}$

$$\left[\begin{array}{cc|c} 1 & 1 & 2q \\ 2 & -1 & p+2q \\ 0 & -1 & p \\ 1 & 0 & -3p-2q \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & p+2q \\ 2 & 0 & 2q \\ 0 & -1 & p \\ 1 & 0 & -3p-2q \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & p+2q \\ 0 & 0 & -2p-2q \\ 0 & -1 & p \\ 0 & 0 & -4p-4q \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & p+2q \\ 0 & -1 & p \\ 0 & 0 & p+q \\ 0 & 0 & p+q \end{array}\right]$$

$v\in U \Longleftrightarrow$ 前述x, y的方程式有解 $\Longleftrightarrow p+q=0 \Longleftrightarrow q=-p$

$\therefore\ U\cap V=\{\, p(0, 1, 1, -3)-p(2, 2, 0, -2) \mid p\in\mathbb{R}\}=\{\, p(-2, -1, 1, -1) \mid p\in\mathbb{R}\}$

$\therefore\ U\cap V$的基底可取爲 $\{-2-x+x^2-x^3\}$,

$\dim(U\cap V)=1$.

④ $\because\ 1+2x+x^3, 1-x-x^2$ 線性獨立, $\therefore\ \dim U=2$,

$\because\ x+x^2-3x^3, 2+2x-2x^3$ 線性獨立, $\therefore\ \dim V=2$.

$\therefore\ \dim(U+V)+\dim(U\cap V)=4=\dim U+\dim V$.

05C**08**【 台大82資工[6] 】

Let $V=M_{2\times 2}(F)$,

$$W_1=\left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix}\in V \ :\ a, b, c\in F \right\},$$

and

$$W_2 = \left\{ \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} \in V \ : \ a, b \in F \right\},$$

(a) Prove that W_1 and W_2 are subspaces of V. (10%)

(b) Find the dimensions of W_1 , W_2 , $W_1 + W_2$, $W_1 \cap W_2$. (10%)

【解】(a) 對W_1 證明封閉性: (綜線CH5定理11)

$$\forall\, p, q \in F, \quad \forall \begin{bmatrix} a & b \\ c & a \end{bmatrix}, \begin{bmatrix} d & e \\ f & d \end{bmatrix} \in W_1 ,$$

$$p\begin{bmatrix} a & b \\ c & a \end{bmatrix} + q\begin{bmatrix} d & e \\ f & d \end{bmatrix} = \begin{bmatrix} pa+qd & pb+qe \\ pc+qf & pa+qd \end{bmatrix} \in W_1$$

對W_2 證明封閉性:

$$\forall\, p, q \in F, \quad \forall \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix}, \begin{bmatrix} 0 & c \\ -c & d \end{bmatrix} \in W_2 ,$$

$$p\begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} + q\begin{bmatrix} 0 & c \\ -c & d \end{bmatrix} = \begin{bmatrix} 0 & pa+qc \\ -(pa+qc) & pb+qd \end{bmatrix} \in W_2$$

(b)

$$\begin{bmatrix} a & b \\ c & a \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\therefore \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\} \text{ 生成 } W_1$$

而此集合顯然是線性獨立, 所以形成W_1 的基底.

∴ $\dim W_1 = 3$

$$\begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} = a\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

∴ $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ 生成 W_2

而此集合顯然是線性獨立, 所以形成W_2 的基底.

∴ $\dim W_2 = 2$

$| \; W_1 \cap W_2 = \{ M \in W_2 \mid M \in W_1 \}$

$$= \left\{ \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} \in V: \; b = 0 \right\}, = \left\{ \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \in V: \; a \in F \right\},$$

顯然 $\left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$ 是 $W_1 \cap W_2$ 的基底.

∴ $\dim(W_1 \cap W_2) = 1$

$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$ (綜線CH6定理25)

$= 3 + 2 - 1 = 4$