題型06A: 線性獨立

06A01 【 成大84資工[4] 】

Consider the vector space of all functions of a variable t. Show that the pair of functions e^t, e^{2t} are linear independent. (10%)

【解】若常數a, b使得$ae^t+be^{2t}=0$ (此爲t之恆等式)

微分得$ae^t+2be^{2t}=0$ (此爲t之恆等式)

前兩式以$t=0$代入得 $\begin{cases} a+b=0, \\ a+2b=0. \end{cases}$

可解得$a=0$, $b=0$.

06A02 【 大同83資工[3] 】

Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Show that functions $f(t)=e^{2t}$, $g(t)=t^2$ and $h(t)=t$ are linearly independent.

【解】設常數a, b, c使 $ae^{2t}+bt^2+ct=0$. (此爲t之恆等式)

以$t=0$代入上式得 $a=0$.

$\therefore\ bt^2+ct=0$

上式微分得 $2bt+c=0$

以$t=0$及$t=1$代入得 $c=0$, $2b+c=0$

$\therefore\ b=0$

故得證 f, g, h爲線性獨立. (綜線CH6定義19)

06A03 【 雲技84電資Y[3](a) 】

(a) Let A be an $m\times n$ matrix. Find the nullspace of A if A has linearly independent column vectors.

【解】將A的行依序記爲C_1, C_2, ... , C_n. 令 $x=[x_1, x_2, ..., x_n]^T$.

$x \in$ [nullspace of A] $\Longleftrightarrow$ $Ax=o$ (綜線CH5定義19)

$$\Longleftrightarrow \begin{bmatrix} C_1 & \dots & C_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = o$$

$\Longleftrightarrow$ $x_1C_1+x_2C_2+\dots+x_nC_n=0$ (綜線CH2定理6)

$\Longleftrightarrow$ $x_1=x_2=\dots=x_n=0$ (綜線CH6定理9)

$\Longleftrightarrow$ $x=o$

０６Ａ**04** 【 台大77資工[5](iv)& 】

True or false, with counterexample if false:

(ⅳ) If a square matrix A has independent columns, so does A^2.

【解】(ⅳ) True. 證明如下:

A has independent columns,

$\therefore$ $Ax=o \Longrightarrow x=o$ (綜線CH6定理15)

$\therefore$ $AAx=o \Longrightarrow Ax=o \Longrightarrow x=o$

$\therefore$ A^2 has independent columns. (綜線CH6定理15)

【加強演練】(本題的推廣)

若矩陣A的各行線性獨立,且矩陣B的各行線性獨立，求證 AB 的各行線性獨立.

[解] $\because$ A的行線性獨立

$\therefore$ $Ax=o \Longrightarrow x=o$ ....(A) (綜線CH6定理15)

$\because$ B的行線性獨立

$\therefore$ $Bx=o \Longrightarrow x=o$ ....(B) (綜線CH6定理15)

若 $ABx=o$

則 $Bx=o$ (套用(A)式)

$\therefore$ $x=o$ (套用(B)式)

此即$ABx=o \Longrightarrow x=o$

$\therefore$ AB的行線性獨立 (綜線CH6定理15)

０６Ａ**05** 【 交大86資工[9] 】

Let $C^t[a,b)$ be the set of all functions that have a continuous tth derivative on $[a,b)$.

Consider n functions $f_k(x)=\cos(\frac{2k\pi}{n}+x)$ in $C^{n-1}[0, 2\pi)$ for $k=0, \ldots, n-1$.

Are these n functions linearly independent ? Justify your answer.

【解】No. 除了$n=1$的特殊情形外, 這些 f_k 都是線性相關. 證明如下:

$$f_k(x)=\cos(\frac{2k\pi}{n})\cos x-\sin(\frac{2k\pi}{n})\sin x \in \text{span}\{\cos x, \sin x\}$$

∵ $\{\cos x, \sin x\}$ 是span$\{\cos x, \sin x\}$ 的生成集, 只含兩向量.

∴ span$\{\cos x, \sin x\}$ 中的獨立集至多含兩向量. (綜線CH6定理18)

∴ $n\geq 3$ 時 $f_0, \ldots, f_{n-1}$ 爲線性相關.

當$n=2$時, $f_0(x)=\cos x$, $f_1(x)=\cos(\pi+x)=-\cos x$. 仍爲線性相關.

06A06 【 大同82資工[1] 】

Let $\varnothing$ denote an empty set. Which of the following statements is wrong ?

(a) span($\varnothing$)$=\{ o \}$;

(b) $\varnothing$ is a subspace of any vector space;

(c) $\varnothing$ is linearly independent;

(d) none.

【解】 選 (b)

【說明】依定義, $\varnothing$線性獨立, 且生成零空間.

$\varnothing$中缺少零向量, 不能成爲向量空間.

06A07 【 清大86工工[1](cd) 】

[是非論證題]

(c) If a set $S=\{v_1, \ldots, v_p\}$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly independent.

(d) If $v_1, \ldots, v_4$ are linearly independent vectors in $\mathbb{R}^4$, then $\{v_1, v_2, v_3\}$ is also linearly independent.

【解】(c) No. 含零向量的集合必定線性獨立. 證明如下:

因可將零向量配1, 其它向量配0, 而得出不全爲零的係數可將這些向量組成零向量. (綜線CH6定義9)

(d) Yes. 證明如下: (綜線CH6定義8要訣3)

假如$\{v_1, v_2, v_3\}$線性相關，則存在不全爲零的純量 a, b, c使$av_1+bv_2+cv_3=0$,

所以 $av_1+bv_2+cv_3+0v_4=o$

這使得v_1, v_2, v_3, v_4變成線性相關，與已知不合.

０６A**08**【 中央85資工[2](e) 】

[是非論證題]

(e) The subset of dependent vectors is dependent.

【分析】應是“A subset of an independent vectors is independent.”(綜線CH6定義8要訣3)

【解】(e) F,

例如 { (1, 1) , (1, 0), (0, 1) } 是 dependent set,

但子集 { (1, 0), (0, 1) } 不是dependent.

０６A**09**【 中央84資工[1](h) 】

(h) If both $\{v_1, v_2, v_3\}$ and $\{v_2, v_3, v_4\}$ are linearly independent sets, then $\{v_1, v_2, v_3, v_4\}$ is linearly independent, where vectors v_1, v_2, v_3 and v_4 are in $\mathbb{R}^4$.

【解】(h) False. 例如

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}\right\} \text{線性獨立,} \quad \left\{\begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\\0\end{bmatrix}\right\} \text{線性獨立,}$$

$$\text{但} \left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\\0\end{bmatrix}\right\} \text{線性相關.}$$

０６A１０【 交大83資工[2] 】

Let A be a linearly independent subset of a vector space V. Suppose v is a vector in V which is not in the subspace spanned by A. Show that the set $A \cup \{v\}$ is linearly independent. (6%)

【分析】本題背景請參閱綜線CH6定理21.

【解】將由A span 所得的子空間記爲W. 由題意得知$v \notin W$.

對任意正整數k, $A \cup \{v\}$內的相異向量 $y_1, \dots, y_k$，及任意純量 $\alpha_1, \dots, \alpha_k$.

設 $\alpha_1 y_1 + \dots + \alpha_k y_k = o$，我們要指明每個$\alpha_i$都是0: (綜線CH6定義8)

若每個 y_i 都不是v, 由A的獨立性就可知每個α_i都是0.

若有某個i使得$v = y_i$,

假設$\alpha_i \neq 0$, 則可移項解得 $y_i = \sum_{j \neq i} \frac{-\alpha_j}{\alpha_i} y_j$

這導致$v \in W$的矛盾結果.

$\therefore\ \alpha_i = 0$.

刪除$\alpha_i y_i$ 之後, 這線性組合變成A內的線性組合, 由A的獨立性可知各純量係數都必須是0.

０６A１１【 中正80資工[4] 】

(a) Let P_n be the set of polynomials with real coefficients of degree less than or equal to n. Show that in P_n the polynomials $1, x, x^2, x^3, \dots, x^n$ are independent. (5%)

(b) Suppose that u, v, and w are linearly independent. Prove or disprove $u+v$, $u+w$, and $v+w$ are linearly independent. (5%)

【相關考題】請參閱台大80資工[2].

【解】(a) 若$c_0, c_1, \dots, c_n \in \mathbb{R}$, 使 $c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n = 0$ (零多項式)

比較係數可得 $c_0 = c_1 = c_2 = \dots = c_n = 0$.

$\therefore\ 1, x, x^2, \dots, x^n$ 爲linear independent. (綜線CH6定義9)

(b) [Prove]

若 $a(u+v) + b(u+w) + c(v+w) = o$

則 $(a+b)u + (a+c)v + (b+c)w = o$

$\because\ u, v, w$ 線性獨立

$$\therefore \begin{cases} a+b \quad\quad =0 \\ a \quad\quad +c=0 \\ \quad\quad b+c=0 \end{cases}$$ (綜線CH6定義)

係數行列式 $\begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = -2 \neq 0$

由Cramar's rule可得$a=0, b=0, c=0$. (綜線CH4定理18)

$\therefore$ $u+v, u+w, v+w$ 線性獨立. (綜線CH6定義9)

【加強演練】

prove or Disprove:

If a, b, c, d are linearly independent, then $a+b$, $b+c$, $c+d$, $d+a$ are linearly independent.

[解] Disprove.

例如: $a=(1,0,0,0)$, $b=(0,1,0,0)$, $c=(0,0,1,0)$, $d=(0,0,0,1)$.

$a+b=(1,1,0,0)$, $b+c=(0,1,1,0)$, $c+d=(0,0,1,1)$, $d+a=(1,0,0,1)$.

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$ (由第一行展開)

$$=1-1=0$$

$\therefore$ (1 1 0 0), (0 1 1 0), (0 0 1 1), (1 0 0 1) 線性獨立. (綜線CH6定理14)

０６A**１２**【 清大78資料[3] 】

Let A be a 3 by 4 matrix. Suppose

$$A=P\begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

for some invertible matrix P. Show that the first and third columns of A are linearly independent.

【證】 設A的各行依次爲A_1, A_2, A_3, A_4.

若 $xA_1+yA_3=0$ (欲證$x=y=0$)

則$xA_1+0A_2+yA_3+0A_4=\begin{bmatrix}0\\0\\0\end{bmatrix}$ $\therefore$ $\begin{bmatrix}A_1, A_2, A_3, A_4\end{bmatrix}\begin{bmatrix}x\\0\\y\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

即 $P\begin{bmatrix}0&0&1&1\\1&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}x\\0\\y\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

等式兩邊左乘P^{-1}, 得

$$\begin{bmatrix}0&0&1&1\\1&0&0&0\\1&0&0&0\end{bmatrix}\begin{bmatrix}x\\0\\y\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

即$x\begin{bmatrix}0\\1\\1\end{bmatrix}+y\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$\therefore$ $x=0, y=0$

０６Ａ❶❸【 交大85資料[3] 】

[複選題]

Determine which of the following matrices cannot be written as a linear combination of the others.

$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 6 \end{bmatrix}$$

(a) $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$, (b) $\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$, (c) $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$, (d) $\begin{bmatrix} 1 & 1 \\ 1 & 6 \end{bmatrix}$

【解】選(b).

【要訣】 對$v_1, v_2, \ldots, v_r$.

(1) v_i可寫為其它向量的線性組合

$\Longleftrightarrow$ $\exists c_1, \ldots, c_r$ 使得 $c_1v_1+\ldots+c_rv_r=o$, 且$c_i\neq 0$.

(2) v_i不可寫為其它向量的線性組合

$\Longleftrightarrow$ 若 $c_1 v_1+\ldots+c_rv_r=o$, 則$c_i=0$.

請參閱綜線CH6定理10的證明.

【解說】試解方程式

$$a\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}+b\begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}+c\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}+d\begin{bmatrix} 1 & 1 \\ 1 & 6 \end{bmatrix}=O$$

即

$$\begin{cases} a-b+2c+d=0 \\ -a+2b-3c+d=0 \\ -a+3b-3c+d=0 \\ 2a+b+2c+6d=0 \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 2 & -3 & 1 \\ -1 & 3 & -3 & 1 \\ 2 & 1 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 2 & -1 & 2 \\ 0 & 3 & -2 & 4 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ $a=-5t$, $b=0$, $c=2t$, $d=t$

∴ 第二個矩陣不可寫爲其它向量的線性組合

０６Ａ**１４**【 元智84工工Y[1] 】

For the following matrix A,

(a) determine its rank.

(b) represent the dependent rows as a linear combination of the independent rows.

(c) Find a basis for the nullspace of A. The nullspace of A is formed by the set of vectors x satisfying $Ax=o$.

$$A=\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & -2 \\ 2 & 4 & -2 & 2 & 0 & -4 \\ 0 & 0 & 3 & 1 & 2 & 2 \\ 2 & 4 & 1 & 3 & 2 & -2 \\ 1 & -1 & -1 & 1 & -2 & 1 \end{bmatrix}$$

【解】(a) (請參閱題型08B)

對A做列運算, (爲使第二小題方便, 儘量不做列對調)

$$\begin{bmatrix} 1 & 2 & -1 & 1 & 0 & -2 \\ 2 & 4 & -2 & 2 & 0 & -4 \\ 0 & 0 & 3 & 1 & 2 & 2 \\ 2 & 4 & 1 & 3 & 2 & -2 \\ 1 & -1 & -1 & 1 & -2 & 1 \end{bmatrix} \begin{matrix} (-2)(-1) \\ \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 2 & 2 \\ 0 & 0 & 3 & 1 & 2 & 2 \\ 0 & -3 & 0 & 0 & -2 & 3 \end{bmatrix} (-1)$$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 & -2 & 3 \end{bmatrix}$$ 可調成有三個非零列的梯形矩陣.

∴ 得知rank$A=3$. (綜線CH6定理23)

(b) 將原矩陣各列依序記爲r_1, r_2, r_3, r_4, r_5 .

由(a)可知r_2 , r_4 爲dependent rows.

∵ $r_2-2r_1=o,\quad r_4-2r_1-r_3=o$

∴ $r_2=2r_1,\quad r_4=2r_1+r_3$

(c) (請參閱題型06C)

接續(a), 經列運算將A化爲列簡化梯形矩陣,

$$A \sim \ldots\sim \begin{bmatrix} 1 & 0 & 0 & 4/3 & -2/3 & 2/3 \\ 0 & 1 & 0 & 0 & 2/3 & -1 \\ 0 & 0 & 1 & 1/3 & 2/3 & 2/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ A的nullspace爲

$$\left\{\begin{bmatrix} -4r/3+2s/3-2t/3 \\ -2s/3+t \\ -r/3-2s/3-2t/3 \\ r \\ s \\ t \end{bmatrix} \middle| \ r,s,t\text{爲純量}\right\}$$

$$=\left\{\frac{r}{3}\begin{bmatrix} -4 \\ 0 \\ -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}+\frac{s}{3}\begin{bmatrix} 2 \\ -2 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}+\frac{t}{3}\begin{bmatrix} -2 \\ 3 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix} \middle| \ r,s,t\text{爲純量}\right\}$$

$$\therefore \text{ 基底可取爲 } \left\{\begin{bmatrix} -4 \\ 0 \\ -1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -2 \\ 0 \\ 0 \\ 3 \end{bmatrix}\right\}$$

０６A１５ 【 交大83資工[1] 】

Consider the system of equations that represent the intersection of three planes

Q_1： $x-y+2z=1$

Q_2： $2x+2z=1$

Q_3： $x-3y+4z=2$

(a) Describe explicitly the solution. (2%)

(b) Can you represent the equation of Q_2 as the linear combination of equations of Q_1 and Q_3 ? If so, find the linear combination $Q_2 = \alpha Q_1 + \beta Q_3$. (2%)

(c) Modify the equation of Q_3 such that the system has no solution . (1%)

【解】(a) 由列運算:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & -3 & 4 & 2 \end{array}\right] \xrightarrow{(-2)(-1)} \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & -2 & 2 & 1 \end{array}\right] \xrightarrow{(1)} \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 2 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1/2 \\ 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \quad Q_1 \cap Q_2 \cap Q_3$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \begin{array}{l} x = 1/2 - t \\ y = -1/2 + t \\ z = t ,\ t \in \mathbb{R} \end{array} \right\} = \left\{ \begin{bmatrix} 1/2 - t \\ -1/2 + t \\ t \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$$

(b) 前述列運算以Q_1 , Q_2 , Q_3 表示，即:

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} \sim \begin{bmatrix} Q_1 \\ Q_2 - 2Q_1 \\ Q_3 - Q_1 \end{bmatrix} \sim \begin{bmatrix} Q_1 \\ Q_2 - 2Q_1 \\ Q_3 + Q_2 - 3Q_1 \end{bmatrix} \sim \cdots$$

$\therefore \quad Q_3 + Q_2 - 3Q_1 = O \qquad \therefore \quad Q_2 = 3Q_1 - Q_3$

$\therefore \quad \alpha = 3,\ \beta = -1$.

(c) 將Q_3 改爲 $x - 3y + 4z = 1$, 則(a)中的列運算變成

$$\left[\begin{array}{ccc|c}1 & -1 & 2 & 1\\ 2 & 0 & 2 & 1\\ 1 & -3 & 4 & 1\end{array}\right] \begin{array}{c}(-2)(-1)\\ \leftarrow\\ \leftarrow\end{array} \sim \left[\begin{array}{ccc|c}1 & -1 & 2 & 1\\ 0 & 2 & -2 & -1\\ 0 & -2 & 2 & 0\end{array}\right] \begin{array}{c}(1)\\ \leftarrow\end{array} \sim \left[\begin{array}{ccc|c}1 & -1 & 2 & 1\\ 0 & 2 & -2 & -1\\ 0 & 0 & 0 & -1\end{array}\right]$$

則此方程組無解. (註:只須將Q_3 的常數項改爲不等於2的數就可以了.)

０６Ａ１６【 交大82資工[2](a) 】

Given matrix A with row vectors $r_1=(1, 2, 4)$, $r_2=(2, 1, 3)$, and $r_3=(4, -1, 1)$,

(a) apply Gaussian Elimination to matrix A to

ⅰ. verify if r_1 , r_2 , and r_3 are linearly independent.

ⅱ. determine α_1 and α_2 such that $r_3=\alpha_1 r_1+\alpha_2 r_2$.

(You are not supposed to solve $\alpha_1 r_1+\alpha_2 r_2=r_3$ for α_1 and α_2) (4%)

【解】(a) ⅰ.

$$\begin{bmatrix}1 & 2 & 4\\ 2 & 1 & 3\\ 4 & -1 & 1\end{bmatrix} \begin{array}{c}(-2)\ (-4)\\ \leftarrow\\ \leftarrow\end{array} \sim \begin{bmatrix}1 & 2 & 4\\ 0 & -3 & -5\\ 0 & -9 & -15\end{bmatrix} \begin{array}{c}(-3)\\ \leftarrow\end{array} \sim \begin{bmatrix}1 & 2 & 4\\ 0 & -3 & -5\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore$ r_1, r_2, r_3 並非 linear independent. (綜線CH6定理23)

ⅱ. 由列運算的過程可知:

$[0 \quad 0 \quad 0] = [0 \quad -9 \quad -15] + (-3)[0 \quad -3 \quad -5]$

$= (r_3+(-4)r_1)+(-3)(r_2+(-2)r_1) = 2r_1-3r_2+r_3$

$\therefore$ $r_3= -2r_1+3r_2$

題型06B: 基底理論

０６B**01** 【 台大86資工[1](a) 】

[是非題]

(a) $B=\{A_1, A_2, \ldots, A_k\}$ generates $\mathbb{R}_{n\times n}$, 則 $k\geq n^2$.

【解】True. (綜線CH6定理19a)

０６B**02** 【 中央85資工[2](d) 】

[是非論證題]

(d) A basis of a vector space is a maximal independent set and a minimal spanning set.

【解】(d) T, (綜線CH6定理19a)

對向量空間V, 設B爲V的基底, X爲獨立集, Y爲生成集.

∵ 基底爲生成集, ∴ $\#(B)\geq\#(X)$ (綜線CH6定理18)

∵ 基底爲獨立集, ∴ $\#(B)\leq\#(Y)$ (綜線CH6定理18)

０６B**03** 【 大同80資工[3](f) 】

True or False:

If $x_1, x_2, \ldots\ldots, x_m$ span a subspace S, then $\dim S=m$.

【解】(f) False, 反例如下:

$$\text{取} m=3,\ x_1=\begin{bmatrix}1\\0\\0\end{bmatrix}, x_2=\begin{bmatrix}0\\1\\0\end{bmatrix}, x_3=\begin{bmatrix}1\\1\\0\end{bmatrix},$$

$$S=\left\{\begin{bmatrix}x\\y\\0\end{bmatrix}\,\middle|\, x, y\in\mathbb{R}\right\}$$

則 x_1, x_2, x_3 span S, 但 $\dim S = 2$.

06B**04** 【 台大77資工[5](i) 】
True or false, with counterexample if false: If vectors $x_1, x_2, \ldots, x_n$ span a subspace S, then $\dim S = n$.

【解】 False. 同上題.

06B**05** 【 元智85工工甲[1](2) 】
Please provide detailed answers for the following questions. (2) Suppose that V is a vector space of dimension 7 and W is a subspace of dimension 5, try to argue for " True or False " in the following statements: (a) Every basis for W can be extended to a basis for V by adding any 2 more vectors in V. (10%) (b) Every basis for V can be reduced to a basis for W by removing 2 vectors. (10%)

【解】(2a) False , 單單"任意"補兩向量未必能成爲V的基底. 例如

$V = \mathbb{R}^{1\times 7}$,

$W = \{ (x_1, x_2, x_3, x_4, x_5, 0, 0) \mid 各 x_i \in \mathbb{R} \}$.

{ (1,0,0,0,0,0,0), (0,1,0,0,0,0,0), (0,0,1,0,0,0,0), (0,0,0,1,0,0,0), (0,0,0,0,1,0,0) }
是W的基底, 但加入(0,0,0,0,0,1,1), (0,0,0,0,0,2,2) 後並不成爲V的基底.

(註: 本題若將any 2 more 改爲 some 2 more 就成立. 可利用獨立集擴編定理(綜線CH6定理21)或Steinize代換定理(綜線CH6定理17)加以證明.)

(2b) False. 如(2a)的V和W.

{ (1,0,0,0,0,0,1), (0,1,0,0,0,0,1), (0,0,1,0,0,0,1), (0,0,0,1,0,0,1), (0,0,0,0,1,0,1), (0,0,0,0,0,1,1), (0,0,0,0,0,0,1) } 是V的基底, 但其中每個向量都不在W中, 所以不可能刪成W的基底.

06B**06** 【 清大75資科[3](1) 】
(1) Let $P_3[x]$ be the linear space of polynomials of degree 3 or less. Let $p_1(x) = 1 + x + x^3$, $p_2(x) = 1 + x^2$ and $p_3(x) = 1$.

(a) Is $\{ p_1 , p_2 , p_3 \}$ linearly independent ? Prove your answer.

(b) Is the linear span of $\{ p_1 , p_2 , p_3 \}$ equal to $P_3 [x]$?

【解】(1) (a) Yes.

[prove]

若 $ap_1(x)+bp_2(x)+cp_3(x)=0$, $a, b, c\in\mathbb{R}$,

(註: 此式為x的恆等式, 等號右邊的0是零多項式.)

則 $a(1+x+x^3)+b(1+x^2)+c(1)=0$

比較係數, 得 $a+b+c=0, a=0, b=0, a=0$

解得 $a=b=c=0$

故得證 $\{p_1, p_2, p_3 \}$ 線性獨立. (綜線CH6定義9)

(b) No.

[證明]

由$P_3[x]=\{ a_0+a_1x+a_2x^2+a_3x^3 \mid a_0, a_1, a_2, a_3\in\mathbb{R} \}$

可知 $\dim P_3[x]=4$.

但 $\dim \text{span}\{p_1, p_2, p_3\}\leq 3$

∴ 可判知 $\text{span}\{p_1, p_2, p_3\}\neq P_3 [x]$

[另證]

The linear span of $\{p_1, p_2, p_3\}$ is

$\{ap_1(x)+bp_2(x)+cp_3(x) \mid a, b, c\in\mathbb{R} \}$

$=\{(a+b+c)+ax+bx^2+ax^3 \mid a, b, c\in\mathbb{R} \}$

For the polynomial $x^3\in P_3[x]$,

suppose that $x^3=(a+b+c)+ax+bx^2+ax^3$, for some a, b, c,

then $\begin{cases} a+b+c=0 \\ a=0 \\ b=0 \\ a=1 \end{cases}$.

It is a contrdiction.

∴ $x^3\notin$ the linear span of $\{p_1, p_2, p_3 \}$.

06B07 【清大82工工[3]】

Show that the vectors

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ \text{and}\quad a_3 = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

form a basis for E^3. Supposing that a_2 is replaced by $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$,

indicate whether the new set of vectors still form a basis in E^3.

【解說】E^3 就是$\mathbb{R}^3$，常稱爲Euclidean space.

【解】 $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 3 \neq 0$,

$\therefore \{a_1, a_2, a_3\}$ 爲 E^3 中之獨立集。 (綜線CH6定理14)

而 $\dim E^3 = 3$

$\therefore \{a_1, a_2, a_3\}$ 爲 E^3 之基底。 (綜線CH6定理22)

$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 5 \\ 0 & 1 & 3 \end{vmatrix} = -2 \neq 0$, $\therefore$ a_2 換成 $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ 之後仍爲基底。

06B08 【清大79工工[5]】

A basis for a vector space V is a linearly independent spanning set for V.

(a) Show that the infinite set $\{1, t, t ,\dots\}$ is a basis for the real vector space R of all polynomials. (10%)

(b) Determine whether the following set can be a basis for $\mathbb{R}^3$, if yes, why? if not, construct a basis from it. (15%)

$\{ [\ \pi^2, 2.35, \sqrt{\pi}\ ]^T, [-3.289, \pi+2, 1]^T \}$

【解】(a) 令 $S=\{1, t, t^2, ...\}$, 欲證S生成R, 且線性獨立:

1°$\forall f(t)\in R$,

$\exists k\in\{0, 1, 2, ...\}$, $\exists a_0, a_1, ..., a_k\in\mathbb{R}$, 使得 $f(t)=a_0+a_1t+a_2t^2+....+a_kt^k$

∴ S為R的spanning set.

2°若實數$c_0, c_1, ..., c_r$ 使得

$c_0+c_1t+c_2t^2+....+c_rt^r=0$ (零多項式)

比較係數可得$c_0=0, c_1=0, ..., c_r=0$

∴ S為linearly independent set.

(b) No. 因為$\mathbb{R}^3$ 的基底必恰含3個向量. (綜線CH6定理19)

令 $B=\{[\pi^2, 2.35, \sqrt{\pi}\ ]^T, [-3.289, \pi+2, 1]^T, [0, 1, 0]^T\}$.

$$\begin{vmatrix} \pi^2 & -3.289 & 0 \\ 2.35 & \pi+2 & 1 \\ \sqrt{\pi} & 1 & 0 \end{vmatrix} = -\begin{vmatrix} \pi^2 & -3.289 \\ \sqrt{\pi} & 1 \end{vmatrix} = -(\pi^2+3.289\sqrt{\pi})\neq 0$$

∴ B為含有3向量的線性獨立集 (綜線CH6定理14)

∴ B為$\mathbb{R}^3$ 的基底. (綜線CH6定理22①)

06B**09** 【 交大79工工[1] 】

是非題:

Every set of linearly independent vectors in $\mathbb{R}^3$ contains three vectors

【解】非

【討論】反例: $\{(1, 0, 0), (0, 1, 0)\}$ 線性獨立, 但只含兩向量.

06B**10** 【 清大81工工[7.2] 】

Let $S=\{v_1, v_2, ..., v_k\}$ be a set of vectors in a vector space V.

《複選》

(a) If each vector in V can be expressed uniquely as a linear combination of vectors in S, then S is an independent set.

(b) If each vector in V can be expressed uniquely as a linear combination of vectors in S, then S is a basis of V.

(c) If S is independent, each vector in V can be expressed uniquely as a linear combination of vectors in S.

(d) For each i and $v_i \neq o$, the subset $\{v_i\}$ is linearly independent.

(e) none of the above.

【編註】本題(d)應修改爲 “ For each i with $v_i \neq o$, ... ”

【解】選(a)(b)(d).

分別解說如下:

(a)(b)(c) :

S生成V $\Longleftrightarrow$ V中每個向量都可表成S內向量的線性組合

S獨立 $\Longleftrightarrow$ V中向量表成S內向量的線性組合頂多有一種表法

(綜線CH6定義28要訣1,2)

由以上性質可知(b)成立，當然(a)也成立.

若只知S獨立, V中的向量還未必能表成S內向量的線性組合，所以(c)不成立.

(d) 若 $kv_i = o$, 則必定$k = 0$,

否則兩邊左乘$1/k$將得出$v = o$, 與已知條件$v_i \neq o$矛盾.

$\therefore$ $\{v_i\}$線性獨立 (綜線CH6定義9)

06B11 【 交大80資工[1](g) 】

True (T) or False (F): (1 for each)

(g) Any independent subset of V is a basis for V.

【解】(g) False. 反例如下:

取$V = \mathbb{R}^3$, $S = \{ (1,0,0), (0,1,0) \}$,

則S爲independent subset of V但S不爲V之basis.

06B12【 元智81電資[3] 】

Let $\{v_1 , v_2\}$ be a basis for a vector space V over the real field $\mathbb{R}$ and let $\{w_1 , w_2\}$ be a subset of V such that $v_1, v_2 \in L(w_1, w_2)$, where

$L(w_1, w_2)=\{ c_1w_1+c_2w_2 \mid c_1, c_2\in \mathbb{R} \}$.

Show that $\{w_1, w_2\}$ is a basis for V.

【分析】 $\{w_1, w_2\}$可組成基底, 所以是生成集.

又因$\dim V=2$, 即可推知$\{w_1, w_2\}$也是基底. (綜線CH6定理22)

【解】 $\because\ v_1, v_2 \in L(w_1, w_2)$

$\therefore\ \exists h_1, h_2, h_3, h_4$ 滿足$v_1=h_1w_1+h_2w_2$, $v_2=h_3w_1+h_4w_2$

$\forall\ u\in V$,

$\because\ \{v_1, v_2\}$是基底,

$\therefore\ \exists\ k_1, k_2$ 使得 $u=k_1v_1+k_2v_2$ (綜線CH6定義16,28)

$\therefore\ u=k_1(h_1w_1+h_2w_2)+k_2(h_3w_1+h_4w_2)=(k_1h_1+k_2h_3)w_1+(k_1h_2+k_2h_4)w_2$

$\therefore\ \{w_1, w_2\}$生成v (綜線CH6定義1)

而 $\dim V=2$

$\therefore\ \{w_1, w_2\}$ 爲V之基底. (綜線CH6定理22②)

０６B**13** 【 交大83資料[7] 】

Let C_B be the coordinate isomorphism with respect to an ordered basis B from the real vector space V to $\mathbb{R}^P$, Prove that $\{v_1, v_2, ..., v_r\}$ is linearly independent in V if only if $\{C_B(v_1), ..., C_B(v_r)\}$ is linearly independent in $\mathbb{R}^P$ ($\mathbb{R}^P$ is the real vector space of all real $p\times 1$column matrices). (7%)

【分析】(甲) 本題請參閱綜線CH8定理11b.

(乙) 若T爲一對一的線性映射, 則有“$v=o \Longleftrightarrow T(v)=o$”.

(丙) 本題對一對一的線性映射即已成立. 證法也相同.

【解】若 $\{C_B(v_1),...,C_B(v_r)\}$ 線性相關,

則存在不全爲零的係數$t_1, t_2, ..., t_r$, 使得

$t_1C_B(v_1)+t_2C_B(v_2)+....+t_rC_B(v_r)=o$

$\therefore\ C_B(t_1v_1+t_2v_2+....+t_rv_r)=o$

$\therefore\ t_1v_1+t_2v_2+....+t_rv_r=o$ (綜線CH8定理7)

即 $v_1, v_2, ..., v_r$ 線性相關 .

反之, 若 $v_1, v_2, \ldots, v_r$ 線性相關 .

則存在不全爲零的係數$t_1, t_2, \ldots, t_r$, 使得

$$t_1v_1+t_2v_2+\ldots+t_rv_r=o$$

$\therefore\quad C_B(t_1v_1+t_2v_2+\ldots+t_rv_r)=o$

$\therefore\quad t_1C_B(v_1)+t_2C_B(v_2)+\ldots+t_rC_B(v_r)=o$

即 $C_B(v_1), C_B(v_2), \ldots, C_B(v_r)$ 線性相關 .

０６Ｂ**１４**【 台大80資工[2] 】

[True or False Problem]

Let V be a vector space with $\dim(V)=3$. Let $\beta=\{u, v, w\}$ and $\beta'=\{u+v, u+w, v+w\}$ where $u, v, w\in V$. Then β is a basis for V if and only if β' is a basis for V.

【相關考題】請參閱中正80資工[4].

【解】 True, 證明如下:

1°先求出β與β'的關係:

令$\beta'=\{u', v', w'\}$, 其中

$$\begin{cases} u'=u+v & \ldots\ldots ① \\ v'=u+w & \ldots\ldots ② \\ w'=v+w & \ldots\ldots ③ \end{cases}$$

①, ②, ③再除以2得

$\therefore\ u+v+w=(1/2)(u'+v'+w')$ ………④

$$\therefore\ \begin{cases} u=u'/2+v'/2-w'/2 & (④-③) \\ v=u'/2-v'/2+w'/2 & (④-②) \\ w=-u'/2+v'/2+w'/2 & (④-①) \end{cases}$$

2°若β爲V的基底, 欲證β'爲V的基底:

[生成]

$\forall v\in V$,

$\because\ \{u, v, w\}$ 生成V,

$\therefore\ \exists$純量a_1, a_2, a_3, 使$v=a_1u+a_2v+a_3w$

$$v=a_1(u'+v'-w')/2+a_2(u'-v'+w')/2+a_3(-u'+v'+w')/2$$

$$=\frac{a_1+a_2-a_3}{2}u'+\frac{a_1-a_2+a_3}{2}v'+\frac{-a_1+a_2+a_3}{2}w'$$

∴ β'生成V (綜線CH6定義1)

[獨立] 若 $au'+bv'+cw'=0$

則 $a(u+v)+b(u+w)+c(v+w)=0$

即 $(a+b)u+(a+c)v+(b+c)w=0$

∵ $\{u, v, w\}$ 獨立,

$$\therefore \begin{cases} a+b=0 \\ a+c=0 \\ b+c=0 \end{cases}$$ (綜線CH6定義)

可解得 $a=0, b=0, c=0$

∴ β'獨立. (綜線CH6定義9)

3°若β'爲V的基底, 依2°之方式同法可證得β爲V的基底.

０６Ｂ１５ 【 台大83資工[2] 】

[複選題]

Let V be a finite–dimension vector space over a field F, and $\beta=\{x_1,\dots,x_n\}$ is a subset of V. Then, β is a basis of V if :

(1) β is linearly independent and span(β)$=V$,

(2) β is linearly independent and $n\geq\dim(V)$,

(3) for all $x\in V$, there exist $a_i\in F$, $i=1,\dots,n$, such that $x=a_1x_1+\dots+a_nx_n$,

(4) $\beta'=\{x_1+x_2, x_2+x_3, x_3+x_4, \dots, x_n+x_1\}$ is a basis for V.

【分析】述及$\beta=\{x_1, \dots , x_n\}$時通常暗示 $x_1, \dots , x_n$ 相異. (綜線CH6定義28)

【解】 (1)(2)

【討論】(1) 此爲定義. (綜線CH6定義16)

(2) ∵ β獨立

∴ $n\leq\dim V$ (綜線CH6定理18)

再由已知條件得 $n=\dim V$

∴ β爲基底. (綜線CH6定理22①)

(3) 所述條件只是生成集.

(4) 以$n=2$為例,

$\beta'=\{x_1+x_2, x_2+x_1\}=\{x_1+x_2\}$

若β'是V的基底, 則V變成一維空間. 這時只要x_1與x_2相異就使得

$\beta=\{x_1, x_2\}$ 不能是V的基底.

【加強演練】

對$n\times n$矩陣 $$\begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ & & \cdots\cdots & & & \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$$ 求行列式.

[解] 對最後一行降階展開(綜線CH4定理11)可得知:

n為奇數時行列式為2, n為偶數時行列式為0.

０６Ｂ１６【 台大85資工[1] 】

[複選題]

Let V be a finite–dimensional vector space. Which of the following are true?

(1) If W_1 and W_2 are subspaces of V, then $W_1\cap W_2$ and $W_1\cup W_2$ are also subspaces of V.

(2) Let $\beta=\{x_1, x_2, ..., x_n\}$ be a subset of V, and let $\beta'=\{x_1+2x_2, x_2+2x_3, ..., x_n+2x_1\}$. Then, β is linearly independent if and only if β' is linearly independent.

(3) Let $\beta=\{x_1, x_2, ..., x_n\}$ be a subset of V. Then, β is a basis of V if $V=\text{span}(\beta)$.

(4) Let W_1 and W_2 be subspaces of V, and $V=W_1+W_2$. Then,
$\dim(V)=\dim(W_1)+\dim(W_2)$.

【解】選(2)

【討論】

(1) False.

$W_1\cap W_2$仍為子空間, 但通常$W_1\cup W_2$不再是子空間. (綜線CH5定理22,24).

(2) True. (本題應假設$n\geq 2$)

本題較複雜,以下分段加以解說:

1°將 β'中的向量依序命名為 $y_1, y_2, ..., y_n$.

設$n\times n$矩陣$P=[p_{ij}]$ 定義如下:

$$p_{ij}=\begin{cases} 1, & i=j \\ 2, & i=j+1 \text{或“} i=1 \text{且} j=n\text{”} \\ 0, & \text{其它} \end{cases}$$

則 $y_j=\sum_{i=1}^{n} p_{ij}x_i, j=1,2,...,n.$ (已知條件)

2°

$$\det P=\begin{vmatrix} 1 & 0 & 0 & ... & 0 & 2 \\ 2 & 1 & 0 & ... & 0 & 0 \\ 0 & 2 & 1 & ... & 0 & 0 \\ \multicolumn{6}{c}{\cdots\cdots\cdots\cdots} \\ 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & ... & 2 & 1 \end{vmatrix}$$

(依第一列降階, 見綜線CH4定理11)

$$=1\cdot\begin{vmatrix} 1 & 0 & ... & 0 & 0 \\ 2 & 1 & ... & 0 & 0 \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots} \\ 0 & 0 & ... & 1 & 0 \\ 0 & 0 & ... & 2 & 1 \end{vmatrix}+(-1)^{n+1}2\begin{vmatrix} 2 & 1 & 0 & ... & 0 \\ 0 & 2 & 1 & ... & 0 \\ \multicolumn{5}{c}{\cdots\cdots\cdots\cdots} \\ 0 & 0 & 0 & ... & 1 \\ 0 & 0 & 0 & ... & 2 \end{vmatrix}$$

$$= 1 + (-1)^{n+1}2^n \neq 0$$

$\therefore$ P可逆. (綜線CH4定理17)

3°令 $Q=P^{-1}=[q_{jk}]$

對$k=1,2,...,n,$

$$\sum_{j=1}^{n} q_{jk}y_j = \sum_{j=1}^{n} q_{jk}\left(\sum_{i=1}^{n} p_{ij}x_i\right)$$

$= \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij}\, q_{jk}\, x_i$ (分配律)

$= \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}\, q_{jk}\, x_i$ (改變加的方式)

$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p_{ij}\, q_{jk} \right) x_i$ (分配律)

$= \sum_{i=1}^{n} \delta_{ik}\, x_i = x_k$ (綜線CH2定義10)

4°若 $x_1, x_2, \ldots, x_n$ 線性獨立, 我們想證 $y_1, \ldots, y_n$ 線性獨立:

若 $\sum_{j=1}^{n} s_j\, y_j = o$,

即 $\sum_{j=1}^{n} s_j \left(\sum_{i=1}^{n} p_{ij}\, x_i \right) = o$

$\therefore \sum_{j=1}^{n} \sum_{i=1}^{n} p_{ij}\, s_j\, x_i = o$ (分配律)

$\therefore \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij}\, s_j\, x_i = o$ (改變加的方式)

$\therefore \sum_{i=1}^{n} \left(\sum_{j=1}^{n} p_{ij}\, s_j \right) x_i = o$ (分配律)

因x_1 , ,,, x_n 線性獨立,

$\therefore \forall i = 1,2,\ldots,n,\ \sum_{j=1}^{n} p_{ij}\, s_j = 0$. (綜線CH6定義9)

由P可逆得知 $s_1 = s_2 = \ldots = s_n = 0$. (Cramer's rule)

5°若$y_1, y_2, \ldots, y_n$ 線性獨立, 依4°之法可證$x_1, \ldots, x_n$線性獨立.

(3) False.

生成集未必是基底.

本小題若加上dim$V = n$的條件就會成立. (綜線CH6定理22)

(4) False.

應該是 $\dim(V) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$. (綜線CH6定理25)

０６Ｂ17【 清大75資料[5](3) 】

Prove or disprove the following statements. If U and V are subspaces of $\mathbb{R}^7$ and $\dim U = \dim V = 4$, then $U \cap V \neq \{o\}$

【解】(3)(prove)

$7 = \dim \mathbb{R}^7$

$\geq \dim(U+V)$ (CH5定理27, CH6定理22③)

$= \dim U + \dim V - \dim(U \cap V)$ (綜線CH6定理25)

$= 4 + 4 - \dim(U \cap V)$

$\therefore \dim(U \cap V) \geq 4 + 4 - 7 = 1$

$\therefore U \cap V \neq \{o\}$

題型06C: 求算基底

０６Ｃ**０１**【 大同85資工[1] 】

Let

$$A=\begin{bmatrix}3 & 9 & 0 & 6\\1 & 3 & 1 & 6\\4 & 12 & 3 & 20\end{bmatrix},$$

find

(a) dimension and a basis for the row space of A. (3%)

(b) dimension and a basis for the column space of A. (2%)

(c) dimension and a basis for the null space of A. (3%)

(d) rank and nullity of A. (2%)

【解】先對A做列運算:

$$\begin{bmatrix}3 & 9 & 0 & 6\\1 & 3 & 1 & 6\\4 & 12 & 3 & 20\end{bmatrix}\sim\begin{bmatrix}0 & 0 & -3 & -12\\1 & 3 & 1 & 6\\0 & 0 & -1 & -4\end{bmatrix}\sim\ldots\sim\begin{bmatrix}1 & 3 & 0 & 2\\0 & 0 & 1 & 4\\0 & 0 & 0 & 0\end{bmatrix}$$

(a) A 的row space 的dimension是2, 可取基底如下:

$\{[1\ \ 3\ \ 0\ \ 2],[0\ \ 0\ \ 1\ \ 4]\}$ (綜線CH6定理23)

(b) A的column space 的dimension是2, 可取基底如下:

$$\left\{\begin{bmatrix}3\\1\\4\end{bmatrix},\begin{bmatrix}0\\1\\3\end{bmatrix}\right\}$$ (綜線CH6定理24)

(c) A的null space如下: (綜線CH3範例7)

$$\left\{ \begin{bmatrix} -3s-2t \\ s \\ -4t \\ t \end{bmatrix} \middle| \, s,t\text{為純量} \right\} = \left\{ s\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \middle| \, s,t\text{為純量} \right\}$$

此空間維度為2, 可取基底如下:

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -4 \\ 1 \end{bmatrix} \right\}$$

(d) rank 為 2, nullity 為 2.

06C**02** 【中原85工工[4]】

Consider a 4–equation linear system $AX=b$ with 5 variables, where

$$A=\begin{bmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}, X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, b=\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

(a) How many solutions in this system ? Show your reason. (5%)

(b) Find a basis for the null space of A. How can you represent a general solution of $AX=b$. (10%)

(c) What columns of A can form a basis of the column space of A. (5%)

(d) Find a basis for the row space of A. (5%)

(e) What is the definition of a basis of $\mathbb{R}^4$? What columns of A form a basis for $\mathbb{R}^4$?

Show your reasons. (5%)

【分析】本題(e)錯誤: 由A的columns不能取得$\mathbb{R}^4$ 的基底.

【解】(a) (請參閱題型03A) 對分隔矩陣 $[\,A \mid b\,]$ 做列運算:

$$\left[\begin{array}{ccccc|c} 2 & 2 & -1 & 0 & 1 & 2 \\ -1 & -1 & 2 & -3 & 1 & -1 \\ 1 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \sim \cdots\cdots \sim \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\because$ rank$[\,A \mid b\,]=$rank$A=3 < A$的寬度

$\therefore$ 此方程組有無限多解. (綜線CH3定理10)

(b) 由(a)得知此方程組之通解為

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1-s-t \\ s \\ -t \\ 0 \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$ (綜線CH3範例7)

A的null space可取基底為 $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

(c) A的第1,3,4行形成A的column space 的基底, 即

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$ (綜線CH6定理24)

(d) A的row space可取基底爲 (綜線CH6定理23)

$\{[1 \quad 1 \quad 0 \quad 0 \quad 1], [0 \quad 0 \quad 1 \quad 0 \quad 1], [0 \quad 0 \quad 0 \quad 1 \quad 0]\}$

(e) $\mathbb{R}^4$ 的基底是線性獨立的生成集.

A的5個columns生成A的culumn space, 但A的culumn space只有三維, 所以A的column不可能選出四維空間$\mathbb{R}^4$ 的基底. (綜線CH6定理19a)

06C03 【交大84資工[1]】

Let $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & -2 & -17 & 16 \\ 3 & 2 & -1 & -4 \end{bmatrix}$

(a) Find a basis for $N(A)$.

(b) Find a basis for the row space of A.

(c) Compute dim($N(A)$).

(d) Compute dim(row space of A).

【解】對A作列運算:

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 3 & -2 & -17 & 16 \\ 3 & 2 & -1 & -4 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ 可取 $\{[1 \quad 0 \quad -3 \quad 0], [0 \quad 1 \quad 4 \quad 0], [0 \quad 0 \quad 0 \quad 1]\}$

爲 row space of A 的基底. ...... Answer of (b)

$\therefore$ dim(row space of A) $= 3$ . ... Answer of (d)

前述列運算同時也用來解齊次方程式$Ax=o$, 得出

$$x=\begin{bmatrix}3t\\-4t\\t\\0\end{bmatrix}, t\text{為任意常數}.$$

∴ 可取 $\{[3\ \ -4\ \ 1\ \ 0]^T\}$ 為 $N(A)$ 的基底. .... Answer of (a)

∴ $\dim(N(A))=1$ . .... Answer of (c) #

０６Ｃ**04** 【 大同84資工[3] 】

Let B be the matrix $\begin{bmatrix}1&-1&-1&2\\-2&4&0&-3\\1&-2&0&5\end{bmatrix}$

(a) Find the basis for the nullspaces, column space, and row space of B.

(b) Write the general solution to the equation $Bx=\begin{bmatrix}1\\-1\\4\end{bmatrix}$ in the form of a particular solution plus an arbitrary member of the nullspace of B.

【解】(a) 對B做列運算:

$$\begin{bmatrix}1&-1&-1&2\\-2&4&0&-3\\1&-2&0&5\end{bmatrix}\sim\ ....\ \sim\begin{bmatrix}1&0&-2&0\\0&1&-1&0\\0&0&0&1\end{bmatrix}$$

row space 的基底可取為

$\{[1\ \ 0\ \ -2\ \ 0], [0\ \ 1\ \ -1\ \ 0], [0\ \ 0\ \ 0\ \ 1]\}$ (綜線CH6定理23)

colume space 的基底可取爲

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \right\}$$

(綜線CH6定理24)

null space 爲 $\left\{ \begin{bmatrix} 2t \\ t \\ t \\ 0 \end{bmatrix} \middle| \, t\text{爲純量} \right\}$ (綜線CH3範例7)

null space 的基底可取爲 $\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

(b) (請參閱題型03A)

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ -2 & 4 & 0 & -3 & -1 \\ 1 & -2 & 0 & 5 & 4 \end{array}\right] \sim \,....\, \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right]$$

通解爲 $\begin{bmatrix} -1+2t \\ t \\ t \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2t \\ t \\ t \\ 0 \end{bmatrix}$ ， 其中

$\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ 為一特解, $\begin{bmatrix} 2t \\ t \\ t \\ 0 \end{bmatrix}$ 為B的null space中的任意成員.

06C**05** 【 清大84工工[8] 】

The matrix below are row equivalent.

$$A=\begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix},\ B=\begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find rank$A$ and dim Nul$A$. (5%)

(b) Find bases for Col$A$ and Row$A$. (5%)

(c) Find a basis for Nul$A$. (5%)

【解】(a) rank$A$＝rank$B$＝2 . (綜線CH8定理14)

dim Nul$A$＝4–rank$A$＝2 (綜線CH8定理8)

(b) Row$A$的基底可取為 { [1 0 –1 5], [0 –2 5 –6] }(綜線CH6定理23)

Col$A$的基底可取為 $\left\{ \begin{bmatrix} 1 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \right\}$. (綜線CH6定理24)

(c) 繼續對B做列運算:

$$\begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore\ \text{Nul}A = \left\{ \begin{bmatrix} s+5t \\ 5s/2-3t \\ s \\ t \end{bmatrix} \middle|\ s,t\text{為純量} \right\}$$

$$= \left\{ (s/2)\begin{bmatrix} 2 \\ 5 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \middle|\ s,t\text{為純量} \right\}$$

Nul$A$的基底可取為 $\left\{ \begin{bmatrix} 2 \\ 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

０６C**06** 【 交大82資工[1] 】

Consider the homogeneous system $Ax = o$ where

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & -3 \\ 3 & 6 & 4 & -1 & 2 \\ 4 & 8 & 5 & 1 & -1 \\ -2 & -4 & -3 & 3 & -5 \end{bmatrix}$$

(A) What is the dimension of the null space of A, $N(A)$? (4%)

(B) Find a basis for $N(A)$ and write the general solution of $Ax = o$ in terms of the basis. (3%)

(C) Find a $b(\neq o)$ such that $Ax = b$ has an infinite number of solutions and write the general solution of $Ax = b$ using the basis of $N(A)$. (3%)

【解】先對A進行列運算:

$$A=\begin{bmatrix}1&2&1&2&-3\\3&6&4&-1&2\\4&8&5&1&-1\\-2&-4&-3&3&-5\end{bmatrix}\sim\cdots\sim\begin{bmatrix}1&2&0&9&-14\\0&0&1&-7&11\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

(a) $\dim N(A)=5-\text{rank}A$ (綜線CH8定理8)

$=5-2=3$ (綜線CH6定理23)

(b) $Ax=o$ 的通解爲

$x_1=-2t_2-9t_4+14t_5,\ x_2=t_2,\ x_3=7t_4-11t_5,\ x_4=t_4,\ x_5=t_5$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=t_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+t_4\begin{bmatrix}-9\\0\\7\\1\\0\end{bmatrix}+t_5\begin{bmatrix}14\\0\\-11\\0\\1\end{bmatrix}$$

∴ 取$N(A)$的基底爲

$$\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-9\\0\\7\\1\\0\end{bmatrix},\begin{bmatrix}14\\0\\-11\\0\\1\end{bmatrix}\right\}$$

(c) 令

$$x_0=\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix},\quad 取\ b=Ax_0=\begin{bmatrix}1\\3\\4\\-2\end{bmatrix}$$

則 $Ax=b$ 的通解為 (綜線CH3定理2)

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}+t_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+t_4\begin{bmatrix}-9\\0\\7\\1\\0\end{bmatrix}+t_5\begin{bmatrix}14\\0\\-11\\0\\1\end{bmatrix}$$

06C**07** 【交大81資工[3](a)】

Let

$$A=\begin{bmatrix}1&-2&1&1\\-1&3&0&2\\0&1&1&2\end{bmatrix}$$

(a) (3%) Find individually a basis for the column space, the row space, and the null space of A.

【解】(a) 對A做列運算:

$$\begin{bmatrix}1&-2&1&1\\-1&3&0&2\\0&1&1&2\end{bmatrix}\sim\begin{bmatrix}1&-2&1&1\\0&1&1&3\\0&1&1&2\end{bmatrix}\sim\begin{bmatrix}1&-2&1&1\\0&1&1&3\\0&0&0&-1\end{bmatrix}$$

∴ A的第1, 2, 4行線性獨立 (綜線CH6定理24)

$\therefore$ 可取 $\left\{\begin{bmatrix}1\\-1\\0\end{bmatrix},\begin{bmatrix}-2\\3\\1\end{bmatrix},\begin{bmatrix}1\\2\\2\end{bmatrix}\right\}$ 為column space 的基底.

(綜線CH6定理24)

繼續做列運算:

$$\sim\begin{bmatrix}1&-2&1&1\\0&1&1&3\\0&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&-2&1&0\\0&1&1&0\\0&0&0&1\end{bmatrix}\sim\begin{bmatrix}1&0&3&0\\0&1&1&0\\0&0&0&1\end{bmatrix}$$

$\therefore$ 可取 $\{[1\ \ 0\ \ 3\ \ 0],[0\ \ 1\ \ 1\ \ 0],[0\ \ 0\ \ 0\ \ 1]\}$
為row space 的基底. (綜線CH6定理23)

$$\text{NS}(A)=\left\{\begin{bmatrix}p\\q\\r\\s\end{bmatrix}\middle|\ \begin{matrix}p+3r=0\\q+r=0\\s=0\end{matrix}\right\}=\left\{\begin{bmatrix}-3r\\-r\\r\\0\end{bmatrix}\middle|\ r\in\mathbb{R}\right\}$$

$\therefore$ 可取 $\left\{\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}\right\}$ 為null space 的基底.

06C**08** 【 交大80資料[1](BCD) 】

$$A=\begin{bmatrix}0&0&4&2\\2&4&6&2\\-4&-8&-10&-2\end{bmatrix}$$

(A) Find a L_0U_0–decomposition $A = P^t L_0 U_0$.

(B) Find a basis for the column space of A.

(C) Find a basis for the row space of A.

(D) Find a basis for the null space of A.

【解】 (A) 本小題詳細過程請參閱題型03E.

$$P=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\quad U_0=\begin{bmatrix}2&4&6&2\\0&0&2&2\\0&0&0&-2\end{bmatrix},\ L_0=\begin{bmatrix}1&0&0\\-2&1&0\\0&2&1\end{bmatrix}.$$

(B) 由(A)部份得知A經列運算化為U_0 .

∵ U_0 的第一, 三, 四行線性獨立,

∴ A的第一, 三, 四行線性獨立 (綜線CH6定理24)

而 $\dim(\text{CSP}A)=\dim(\text{RSP}A)=\dim(\text{RSP}U_0)=3$

∴ A的第一, 三, 四行為A的行空間的基底, 即 (綜線CH6定理24)

$$\left\{\begin{bmatrix}0\\2\\-4\end{bmatrix},\begin{bmatrix}4\\6\\-10\end{bmatrix},\begin{bmatrix}2\\2\\-2\end{bmatrix}\right\}$$

(C) 對A做列運算. 接續(A)已做的部分,

$$A\overset{r}{\sim}\begin{bmatrix}0&0&0&-2\\2&4&6&2\\0&0&2&2\end{bmatrix}\overset{r}{\sim}\cdots\overset{r}{\sim}\begin{bmatrix}1&2&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

∴ 可取 $\{[1\ \ 2\ \ 0\ \ 0],[0\ \ 0\ \ 1\ \ 0],[0\ \ 0\ \ 0\ \ 1]\}$ 為A的row space的基底 (綜線CH8定理23)

(D) null space of $A=\{x \mid Ax=0\}$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle| \begin{array}{r} x_1 + 2x_2 \qquad = 0 \\ x_3 \quad = 0 \\ x_4 = 0 \end{array} \right\} = \left\{ \begin{bmatrix} -2t \\ t \\ 0 \\ 0 \end{bmatrix} \middle| t\text{為任意常數} \right\}$$

∴ 可取 $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ 為A的null space的基底.

06C09 【 交大81資料[4](bc) 】

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 3 & 4 \\ 0 & 2 & 1 & 1 \\ 0 & 4 & 2 & 5 \\ 0 & 6 & 3 & 1 \end{bmatrix}$$

(a) Find a $P^TL_0U_0$ decomposition for matrix A.

(b) Find a basis for the null space of A.

(c) Find a basis for the row space of A.

【解】(a) [詳情請參閱題型03E]

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, L_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 3 & 3 & -2/3 & 1 \end{bmatrix}, U_0 = \begin{bmatrix} 3 & 2 & 3 & 4 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) 由(a)知A可經列運算化為U_0，以下再繼續計算:

$$U_0=\begin{bmatrix}3&2&3&4\\0&2&1&1\\0&0&0&3\\0&0&0&0\end{bmatrix}\sim\ldots\sim\begin{bmatrix}1&0&2/3&0\\0&1&1/2&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

$\therefore$ null space of $A=\{x \mid Ax=o\}$

$$=\left\{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\middle|\begin{array}{l}x_1=(-2/3)t\\x_2=(-1/2)t\\x_3=t\\x_4=0\end{array},\ t\in\mathbb{R}\right\}=\left\{t\begin{bmatrix}-2/3\\-1/2\\1\\0\end{bmatrix}\middle|\ t\in\mathbb{R}\right\}$$

$\therefore$ 可取 $\left\{\begin{bmatrix}-4\\-3\\6\\0\end{bmatrix}\right\}$ 爲null space的基底.

(c) 由(b)知可取 $\{(1, 0, 2/3, 0), (0, 1, 1/2, 0), (0, 0, 0, 1)\}$ 爲row space of A的基底.

(綜線CH6定理23)

06C**10** 【交大82資料[2]】

Let A be the following matrix.

$$A=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&1&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}2&0&0&0\\-1&-1&0&0\\-1&1&2&0\\3&0&2&-1\end{bmatrix}\begin{bmatrix}1&-1&0&1\\0&1&-2&-1\\0&0&1&1\\0&0&1&1\end{bmatrix}$$

Find the rank and a basis for each of the following vector spaces

(A) row space of A; (B) column space of A;
(C) image space of A; (D) null space of A.

【勘誤】本題的rank應更正爲dimension.

向量空間有dimension, 無rank. 矩陣有rank, 無dimension.

【解】(A)

$$A\text{可經列運算化爲}\begin{bmatrix}1 & -1 & 0 & 1\\0 & 1 & -2 & -1\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{bmatrix}\text{，再繼續化爲}\begin{bmatrix}1 & -1 & 0 & 1\\0 & 1 & -2 & -1\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

∴ row space of A 的 dimension 爲 3, 且可取 (綜線CH6定理23)

$\{[1\ \ -1\ \ 0\ \ 1], [0\ \ 1\ \ -2\ \ -1], [0\ \ 0\ \ 1\ \ 1]\}$ 爲基底.

(B)

$$A=\begin{bmatrix}0 & 0 & 1 & 0\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 0 & 0\\-1 & -1 & 0 & 0\\-1 & 1 & 2 & 0\\3 & 0 & 2 & -1\end{bmatrix}\begin{bmatrix}1 & -1 & 0 & 1\\0 & 1 & -2 & -1\\0 & 0 & 1 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 1 & 0\\1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & -2 & 0 & 2\\-1 & 0 & 2 & 0\\-1 & 2 & 0 & 0\\3 & -3 & 1 & 4\end{bmatrix}=\begin{bmatrix}-1 & 2 & 0 & 0\\2 & -2 & 0 & 2\\-1 & 0 & 2 & 0\\3 & -3 & 1 & 4\end{bmatrix}$$

dim(column space of A) = dim(row space of A) = 3 .

另外, 由(A)小題得知當A經列運算化爲梯形矩陣後第1,2,3行線性獨立,

(綜線CH6定理24①)

∴ A的第1,2,3行線性獨立, (綜線CH6定理24②)

$$\therefore \text{ 可取 } \left\{ \begin{bmatrix} -1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\} \text{爲}A\text{的column space 的基底.}$$

(綜線CH6定理24③)

(C) image space即A的 column space,

∴ 本小題答案與上個小題相同.

(D) 接第(A)小題,

$$A\text{可經列運算化爲} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\text{，再繼續化爲}$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\text{，最後變成} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ null space of A

$$= \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \middle| \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \middle| \begin{array}{l} x=-2t, \\ y=-t, \\ z=-t, \\ w=t, \quad t\in\mathbb{R} \end{array} \right\} = \left\{ \begin{bmatrix} -2t \\ -t \\ -t \\ t \end{bmatrix} \middle| \; t\in\mathbb{R} \right\}$$

$\therefore$ null space of A 的 dimension 爲1， 且可取 $\left\{ \begin{bmatrix} -2t \\ -t \\ -t \\ t \end{bmatrix} \right\}$ 爲基底.

【加強演練】將本題的A化爲$P^{\mathrm{T}}LU$分解式.

【解】

令 $P^{\mathrm{t}} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ， 則

$$A = P^{\mathrm{t}} \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= P^{\mathrm{t}} \begin{bmatrix} 2 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 3 & 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$=P^t\begin{bmatrix}2&0&0&0\\-1&-1&0&0\\-1&1&2&0\\3&0&1&-1\end{bmatrix}\begin{bmatrix}1&-1&0&1\\0&1&-2&-1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

０６C**１１**【 清大77資料[8] 】

Let $V=\mathbb{R}^n$ and $W=\{(a_1, a_2, \dots, a_n)\in V \mid a_1+a_2+\dots+a_n=o\}$

Find a basis of W over $\mathbb{R}$.

【解】 $a_1+a_2+\dots+a_n=0$

$$\therefore\begin{cases}a_1=-t_2-\dots-t_n\\a_2=t_2\\\quad\dots\dots\\a_n=t_n\qquad\qquad, \quad t_2,\dots,t_n\in\mathbb{R}\end{cases}$$

$\therefore W=\{(-t_2-t_3-\dots-t_n, t_2, t_3, \dots, t_n) \mid t_2,\dots,t_n\in\mathbb{R}\}$

$=\{t_2(-1,1,0,0,\dots,0)+t_3(-1,0,1,0,\dots,0)+\dots+t_n(-1,0,0,0,\dots,1) \mid t_2,\dots,t_n\in\mathbb{R}\}$

$\therefore$ $\{(-1,1,0,0,\dots,0), (-1,0,1,0,\dots,0), \dots, (-1,0,0,0,\dots,1)\}$ 生成W

(綜線CH6定義1③及要訣2)

以下證明獨立： (綜線CH6定義16,定義9②)

若 $\alpha_2(-1,1,0,0,\dots,0)+\alpha_3(-1,0,1,0,\dots,0)+\dots\dots+\alpha_n(-1,0,0,0,\dots,1)=(0,0,0,0,\dots,0)$

則
$$\begin{cases}-\alpha_2-\alpha_3-\dots-\alpha_n=0\\\alpha_2=0\\\alpha_3=0\\\quad\dots\dots\\\alpha_n=0\end{cases}$$

$\therefore$ $\alpha_2=\alpha_3=\dots=\alpha_n=0$

故得證$\{(-1,1,0,0,\dots,0), (-1,0,1,0,\dots,0), \dots, (-1,0,0,0,\dots,1)\}$爲$W$的基底.

【加強演練】

Let $V=\mathbb{R}^4$,

$W=\{(x_1, x_2, x_3, x_4) \mid x_1+x_2+x_3+x_4=0\} \cap \{(x_1, x_2, x_3, x_4) \mid x_1-x_2+x_3-x_4=0\}$

Find a basis of W over $\mathbb{R}$.

[解]

令 $A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$, 則 $W=\text{Ker}A$.

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{cases} x_1+x_3=0 \\ x_2+x_4=0 \end{cases}$$

$$\therefore \begin{cases} x_1= -t_3 \\ x_2= -t_4 \\ x_3= t_3 \\ x_4= t_4 \end{cases} ;\ t_3, t_4 \in R$$

讀者自證{ (–1, 0, 1, 0), (0, –1, 0, 1) } 是W的基底.

０６Ｃ**12** 【 交大83資料[6] 】

Show that the two sets S_1 and S_2 below span the same subspace of $\mathbb{R}^3$ ($\mathbb{R}^3$ is the real vector space of all geometrical vectors in three–dimensional physical space).

$$S_1=\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} \text{ and } S_2=\left\{ \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\} \quad (6\%)$$

【解】將S_1 內的向量排成矩陣, 再作行運算:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$

$$\therefore\ \operatorname{span}S_1 = \text{column space}(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}) = \text{column space}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix})$$

將S_2內的向量排成矩陣，再作行運算:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ -1 & 1 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 0 & 1 \\ -3 & 2 \\ -3 & 1 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\therefore\ \operatorname{span}S_1 = \text{column space}(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \end{bmatrix})$$

$\therefore\ \operatorname{span}S_1 = \operatorname{span}S_2.$

０６Ｃ**13** 【 交大85資料[4] 】

[複選題]

In the following, which pairs span the same subspace of $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -5 \end{bmatrix} \right\}$

(b) $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}2\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}\right\}$

(c) $\left\{\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}2\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right\}$

(d) $\left\{\begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-3\end{bmatrix}\right\}$ and $\left\{\begin{bmatrix}3\\1\\-2\end{bmatrix}, \begin{bmatrix}-1\\3\\-4\end{bmatrix}\right\}$

【解】選(a)(b)(d).

【說明】將各向量排成列向量, 並以列運算化至列簡梯陣即可判定是否相同.

(a)

$$\begin{bmatrix}1&3&-7\\2&-1&0\\3&-1&-1\\4&-3&2\end{bmatrix} \sim \begin{bmatrix}1&3&-7\\0&-7&14\\0&-10&20\\0&-15&30\end{bmatrix} \sim \begin{bmatrix}1&3&-7\\0&1&-2\\0&0&0\\0&0&0\end{bmatrix} \sim \begin{bmatrix}1&0&-1\\0&1&-2\\0&0&0\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}1&-1&1\\1&1&-3\\1&2&-5\end{bmatrix} \sim \begin{bmatrix}1&-1&1\\0&2&-4\\0&3&-6\end{bmatrix} \sim \begin{bmatrix}1&0&-1\\0&1&-2\\0&0&0\end{bmatrix}$$

$\therefore$ 兩個子空間相同.

(b)

$$\begin{bmatrix}1&0&-1\\1&1&0\\0&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&-1\\0&1&1\\0&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}2&1&-1\\1&2&1\end{bmatrix}\sim\begin{bmatrix}0&-3&-3\\1&2&1\end{bmatrix}\sim\begin{bmatrix}0&1&1\\1&0&-1\end{bmatrix}$$

∴ 兩個子空間相同.

(c)

$$\begin{bmatrix}1&0&-1\\1&1&0\\0&1&1\end{bmatrix}\sim \text{同上} \sim\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}$$

$$\begin{bmatrix}2&1&-1\\1&-1&0\end{bmatrix}\sim\begin{bmatrix}0&3&-1\\1&-1&0\end{bmatrix}\sim\begin{bmatrix}0&1&-1/3\\1&-1&0\end{bmatrix}\sim\begin{bmatrix}1&0&-1/3\\0&1&-1/3\end{bmatrix}$$

∴ 兩個子空間不同.

(d)

$$\begin{bmatrix}2&-1&1\\1&2&-3\end{bmatrix}\sim\begin{bmatrix}0&-5&7\\1&2&-3\end{bmatrix}\sim\begin{bmatrix}1&2&-3\\0&1&-7/5\end{bmatrix}\sim\begin{bmatrix}1&0&-1/5\\0&1&-7/5\end{bmatrix}$$

$$\begin{bmatrix}3&1&-2\\-1&3&-4\end{bmatrix}\sim\begin{bmatrix}0&10&-14\\-1&3&-4\end{bmatrix}\sim\begin{bmatrix}1&-3&4\\0&1&-7/5\end{bmatrix}\sim\begin{bmatrix}1&0&-1/5\\0&1&-7/5\end{bmatrix}$$

∴ 兩個子空間相同.

０６Ｃ**14** 【 交大85工工[2] 】

Find the dimension of the subspace $\mathbb{R}^4$ spanned by X_1 , X_2 , X_3 , and X_4

$$X_1=\begin{bmatrix}1\\2\\-1\\0\end{bmatrix},X_2=\begin{bmatrix}2\\5\\-3\\2\end{bmatrix},X_3=\begin{bmatrix}2\\4\\-2\\0\end{bmatrix},X_2=\begin{bmatrix}3\\8\\-5\\4\end{bmatrix}$$

【解】將X_1 , X_1 , X_3 , X_4 拼成矩陣A, 再做列運算.

$$A=\begin{bmatrix}1&2&2&3\\2&5&4&8\\-1&-3&-2&-5\\0&2&0&4\end{bmatrix}\sim\begin{bmatrix}1&2&2&3\\0&1&0&2\\0&-1&0&-2\\0&2&0&4\end{bmatrix}\sim\begin{bmatrix}1&2&2&3\\0&1&0&2\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

所求子空間的維度

$=\dim \text{CSP}A=\text{rank}A=2$. (綜線CH8定理13,CH6定理23)

０６Ｃ１５ 【 中央86資工[4] 】

Find bases for Col$A$, Row$A$, Nul$A$, and Nul$A$, where

$$A=\begin{bmatrix}0&6&6&3\\1&2&1&1\\4&1&-3&4\\1&3&2&0\end{bmatrix}$$

【解】對A做列運算: (綜線CH3範例4b)

$$\begin{bmatrix} 0 & 6 & 6 & 3 \\ 1 & 2 & 1 & 1 \\ 4 & 1 & -3 & 4 \\ 1 & 3 & 2 & 0 \end{bmatrix} \sim \cdots\cdots \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ Row$A$的基底可取爲

$\{[1 \quad 0 \quad -1 \quad 0], [0 \quad 1 \quad 1 \quad 0], [0 \quad 0 \quad 0 \quad 1]\}$. (綜線CH6定理23)

由前述列運算可知 $\text{Nul}A = \left\{ \begin{bmatrix} t \\ -t \\ t \\ 0 \end{bmatrix} \middle| \, t \text{ 爲純量} \right\}$ (綜線CH3範例7)

∴ Nul$A$的基底可取爲 $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

對A^T執行列運算:

$$\begin{bmatrix} 0 & 1 & 4 & 1 \\ 6 & 2 & 1 & 3 \\ 6 & 1 & -3 & 2 \\ 3 & 1 & 4 & 0 \end{bmatrix} \sim \cdots\cdots \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 \\ 0 & 1 & 0 & 19/7 \\ 0 & 0 & 1 & -3/7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ Col$A$的基底可取爲

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1/3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 19/7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3/7 \end{bmatrix} \right\}.$$ (綜線CH6定理23)

由前述列運算可知 $\mathrm{Nul}A^{\mathrm{T}} = \left\{ \begin{bmatrix} t/3 \\ -19t/7 \\ 3t/7 \\ t \end{bmatrix} \middle| \ t \text{ 爲純量} \right\}$ (綜線CH3範例7)

$\therefore$ $\mathrm{Nul}A^{\mathrm{T}}$的基底可取爲 $\left\{ \begin{bmatrix} 7 \\ -57 \\ 9 \\ 21 \end{bmatrix} \right\}$

０６Ｃ**16**【 中央85資工[5] 】

Find bases for Col$A$, Row$A$, Nul$A$, and NulA^{T} , where

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}.$$

【解】對A執行列運算:

$$\begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ Row$A$的基底可取爲{ [1 0 1 0 1], [0 1 −2 0 3], [0 0 0 1 −5]}

由前述列運算可知 Nul$A$＝

$$\left\{ \begin{bmatrix} -s-t \\ 2s-3t \\ s \\ 5t \\ t \end{bmatrix} \middle| \ s,t \text{ 爲純量} \right\} = \left\{ s\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \middle| \ s,t \text{ 爲純量} \right\}$$

∴ Nul$A$的基底可取爲 $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$

對A^T 執行列運算:

$$\begin{bmatrix} -2 & 1 & 3 & 1 \\ -5 & 3 & 11 & 7 \\ 8 & -5 & -19 & -13 \\ 0 & 1 & 7 & 5 \\ -17 & 5 & 1 & -3 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ Col$A$的基底可取為

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$ (綜線CH6定理23)

由前述列運算可知 $\text{Nul}A^T = \left\{ \begin{bmatrix} -2t \\ -7t \\ t \\ 0 \end{bmatrix} \middle| \ t \text{ 為純量} \right\}$

∴ $\text{Nul}A^T$ 的基底可取為 $\left\{ \begin{bmatrix} -2 \\ -7 \\ 1 \\ 0 \end{bmatrix} \right\}$

０６Ｃ**１７** 【 元智85工工甲[2] 】

Please answer the following questions.

(1) Given a matrix $A \in \mathbb{R}^{m\times n}$ Display the relationships among four fundamental subspaces associated with A. They are : the column space, the nullspace, the row space, and the left null space of A. (15%)

(2) Suppose that the dimension of column space of A is r, find the dimensions for the other three subspaces associated with A. (10%)

(3) Find a basis for each of the four subspaces of

$$A=\begin{bmatrix}0&1&2&3&4\\0&1&2&4&6\\0&0&0&1&2\end{bmatrix}=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}0&1&2&3&4\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}$$ (20%)

【解】將A的column space, row space, nullspace, left null space 記爲CSPA, RSPA, ker$A$, lker$A$. (因符號不統一, 使用前應加上符號說明.)

(1) dim CSPA ＝ dim RSPA (綜線CH8定理13)

dim ker$A$ ＋ dim CSPA ＝ n (綜線CH8定理8)

dim lker$A$ ＋ dim RSPA ＝ m (綜線CH8定理8)

CSPA與(lker$A$)^T 在$\mathbb{R}^{m\times 1}$ 正交互補. (綜線CH11定理23)

(CSPA)^T 與lker$A$在$\mathbb{R}^{1\times m}$ 正交互補. (綜線CH11定理23)

RSPA與(ker$A$)^T 在$\mathbb{R}^{1\times n}$ 正交互補. (綜線CH11定理23)

(RSPA)^T 與ker$A$在$\mathbb{R}^{n\times 1}$ 正交互補 (綜線CH11定理23)

(2) 當 dim CSPA＝r時, 由(1)得知:

dimRSPA＝r, dim ker$A$＝n–r, dim lker$A$＝m–r

(3) 題目已給出LU分解, 但因須解nullspace, 所以對U繼續做列運算:

$$A\sim\begin{bmatrix}0&1&2&3&4\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}\sim\begin{bmatrix}0&1&2&0&-2\\0&0&0&1&2\\0&0&0&0&0\end{bmatrix}$$

$$\therefore\ \ker A=\left\{\begin{bmatrix}r\\-2s+2t\\s\\-2t\\t\end{bmatrix}\middle|\, r,s,t\in\mathbb{R}\right\}=\text{CSP}\begin{bmatrix}1&0&0\\0&-2&2\\0&1&0\\0&0&-2\\0&0&1\end{bmatrix}$$

$$\therefore\ \ker A \text{可取基底爲} \left\{ \begin{bmatrix} 1\\0\\0\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\-2\\1\\0\\0 \end{bmatrix}, \begin{bmatrix} 0\\2\\0\\-2\\1 \end{bmatrix} \right\}$$

由列運算的結果, RSPA可取基底爲

$\{[0\ \ 1\ \ 2\ \ 0\ \ -2], [0\ \ 0\ \ 0\ \ 1\ \ 2]\}$

爲求left null space, 接著對A作行運算:

(讀者請注意: 不能對U做行運算!! 見綜線CH5定理17要訣3.)

$$\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ \text{lker}A = \{[t\ \ -t\ \ t] \mid t \in \mathbb{R}\} = \{t[1\ \ -1\ \ 1] \mid t \in \mathbb{R}\}$

$\therefore\ \text{lker}A$可取基底爲 $\{[1\ \ -1\ \ 1]\}$.

$$\text{由行運算的結果, CSP}A\text{可取基底爲} \left\{ \begin{bmatrix} 1\\0\\-1 \end{bmatrix}, \begin{bmatrix} 0\\1\\1 \end{bmatrix} \right\}.$$

06C18 【 中央82資電[5] 】

Find the dimension and a basis for each of the four fundamental subspaces (column space, null space, row space, and left null space)

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix}$$

【解】對A執行列運算:

$$\begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix} \sim \cdots\cdots \sim \begin{bmatrix} 1 & 0 & 7 & 0 & -39 \\ 0 & 1 & -3 & 0 & 31 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A 的 row space 可取基底 $\{(1, 0, 7, 0, -39), (0, 1, -3, 0, 31), (0, 0, 0, 1, -7)\}$，其 dimension爲 3 .

解 $\begin{cases} x_1 + 7x_3 - 39x_5 = 0 \\ x_2 - 3x_3 + 31x_5 = 0 \\ x_4 - 7x_5 = 0 \end{cases}$

得

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -7t+39u \\ 3t-31u \\ t \\ 7u \\ u \end{bmatrix} = t\begin{bmatrix} -7 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + u\begin{bmatrix} 39 \\ -31 \\ 0 \\ 7 \\ 1 \end{bmatrix},\quad t, u\text{爲任意實數}$$

A 的 null space 可取基底 $\{(-7, 3, 1, 0, 0)^T, (39, -31, 0, 7, 1)^T\}$，其 dimension爲2 .

再對A^T 執行列運算:

$$\begin{bmatrix} 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 2 \\ 1 & 9 & 5 & 8 \\ 3 & 0 & 1 & -3 \\ 2 & 7 & 8 & 5 \end{bmatrix} \sim\cdots\cdots\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

A 的 column space 可取基底 $\{ (1, 0, 0, -1)^T, (0, 1, 0, 1)^T, (0, 0, 1, 0)^T \}$，
其 dimension為 3 .

解 $\begin{cases} x_1 - x_4 = 0 \\ x_2 + x_4 = 0 \\ x_3 = 0 \end{cases}$

得 $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} t \\ -t \\ 0 \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ ， t為任意實數

A 的 left null space 可取基底 $\{(1, -1, 0, 1)\}$，其 dimension為 1 .

０６Ｃ１９【 交大85資工[1] 】

Let P_3 denote the set of all polynomials of degree less than 3. Let S be the subspace of P_3 consisting of all polynomials $p(x)$ such that $p(0)=0$, and let T be the subspace of P_3 consisting of all polynomials $q(x)$ such that $q(1)=0$. Find bases for

(a) T (3%) (b) $S\cap T$ (3%)

【解】(a) 令 $p(x)=a+bx+cx^2$ ，

$p(x)\in T \iff p(1)=0 \iff a+b+c=0$

$\iff a=-b-c$.

$\therefore T=\{ -b-c+bx+cx^2 \mid b,c\in\mathbb{R} \}$

$=\{ b(x-1)+c(x^2-1) \mid b,c\in\mathbb{R} \}$

顯然 $x-1$ 與 x^2-1 線性獨立： (參看綜線CH6範例12)

若 $b(x-1)+c(x^2-1)=0$,

經集項得 $(-b-c)+bx+cx^2=0$

$\therefore\ b=0, c=0$

$\therefore$ T可取 $\{x-1, x^2-1\}$ 為基底.

(b) $S\cap T=\{q(x)\in T \mid q\in S\}=\{q(x)\in T \mid q(0)=0\}$

$=\{b(x-1)+c(x^2-1) \mid b,c\in\mathbb{R}, -b-c=0\}$

$=\{b(x-1)-b(x^2-1) \mid b\in\mathbb{R}\}=\{b(x-x^2) \mid b\in\mathbb{R}\}$

$\therefore$ $S\cap T$可取 $\{x-x^2\}$ 為基底.

(也可取 $\{x(x-1)\}$ 為基底.)

【另解】

(a) 兩次以內的多項式可表成如下之型:

$p(x)=a+b(x-1)+c(x-1)^2$. (Taylor's 展開式)

$p(x)\in T \iff p(1)=0 \iff a=0$

$\therefore$ $T=\{b(x-1)+c(x-1)^2 \mid b,c\in\mathbb{R}\}$

顯然 $(x-1)$ 與 $(x-1)^2$ 線性獨立. (同前法可證)

$\therefore$ T可取 $\{x-1, (x-1)^2\}$ 為基底.

(b) 仿前法可由(a)的基底再解得(b)的基底.

０６C**20** 【 清大78資科[2] 】

Let $S=\{A\in M_3(\mathbb{R}) \mid A^t=A\}$, where A^t is the transpose of A

(a) Find a basis of S over $\mathbb{R}$.

(b) If $C\in M_3(\mathbb{R})$ such that $CA=AC$ for all $A\in S$, prove that $C=\lambda I_3$ where $\lambda\in\mathbb{R}$ and I_3 is the identity matrix.

【解】(a)

設$A=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix}$,

$$A^{t}=A \Longleftrightarrow \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & k \end{bmatrix}=\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} \Longleftrightarrow d=b, g=c, h=f.$$

$$\therefore\ S=\left\{ \begin{bmatrix} a & b & c \\ b & e & f \\ c & f & k \end{bmatrix} \middle|\ a, b, c, e, f, k \in \mathbb{R} \right\}$$

$$=\left\{ a\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+b\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}+c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}+e\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right.$$

$$\left. +f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}+k\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle|\ a, b, c, e, f, k \in \mathbb{R} \right\}$$

$$令\ X=\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}$$

則X生成S. (綜線CH6定義1)

以下證明X爲線性獨立: (綜線CH6定義9)

若 $a_1\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}+a_2\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}+a_3\begin{bmatrix}0&0&1\\0&0&0\\1&0&0\end{bmatrix}+a_4\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$

$$+a_5\begin{bmatrix}0&0&0\\0&0&1\\0&1&0\end{bmatrix}+a_6\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix},$$

比較各對應位置即得:

$a_1=0, a_2=0, a_3=0, a_4=0, a_5=0, a_6=0$

故得證X為線性獨立.

∴ X為S的基底. (綜線CH6定義16)

(b)

令$C=\begin{bmatrix}c_1&c_2&c_3\\c_4&c_5&c_6\\c_7&c_8&c_9\end{bmatrix}$，且滿足$CA=AC$ for all $A\in S$.

取$A=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}$，則 $\begin{bmatrix}c_1&c_2&c_3\\c_4&c_5&c_6\\c_7&c_8&c_9\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}c_1&c_2&c_3\\c_4&c_5&c_6\\c_7&c_8&c_9\end{bmatrix}$

$$\begin{bmatrix}c_1&c_2&c_3\\c_4&c_5&c_6\\c_7&c_8&c_9\end{bmatrix}\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}c_1&c_2&c_3\\c_4&c_5&c_6\\c_7&c_8&c_9\end{bmatrix}$$

$$\therefore\ \begin{bmatrix} c_1 & 0 & 0 \\ c_4 & 0 & 0 \\ c_7 & 0 & 0 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 & c_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ c_2=0, c_3=0, c_4=0, c_7=0$

同上法, 取$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, 可得$c_3=0,c_6=0,c_7=0,c_8=0$.

$$\therefore\ C=\begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_5 & 0 \\ 0 & 0 & c_9 \end{bmatrix}$$

再取$A=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, 則 $\begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_5 & 0 \\ 0 & 0 & c_9 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_5 & 0 \\ 0 & 0 & c_9 \end{bmatrix}$

$$\therefore\ \begin{bmatrix} 0 & c_1 & 0 \\ c_5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c_5 & 0 \\ c_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ c_1=c_5$

同上法, 取$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, 可得 $c_5=c_9$.

$$\therefore\ C=\begin{bmatrix} c_9 & 0 & 0 \\ 0 & c_9 & 0 \\ 0 & 0 & c_9 \end{bmatrix}=c_9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

題型06D: 坐標

０６D01 【 成大84資工[2]

Show that the vectors (1, 1) and (–1, 2) form a basis of $\mathbb{R}$.(5%) Find the coordinate of the vector (a, b) in $\mathbb{R}^2$ with respect to the vectors (1, 1) and (–1, 2). (5%)

【解】證明獨立:

若$a(1,1)+b(-1,2)=(0,0)$, 則 $a-b=0, a+2b=0$. 可解得 $a=0, b=0$.

證明生成:

對任意$(a,b)\in\mathbb{R}^2$,

令 $(a,b)=x(1,1)+y(-1,2)$,

得 $\begin{cases} a=x-y, \\ b=x+2y. \end{cases}$

解得 $x=(2a+b)/3, y=(-a+b)/3$.

$\therefore$ (1, 1), (–1,2)生成$\mathbb{R}^2$.

由生成的證明也順便得知

(a, b) 的坐標爲 $\begin{bmatrix} (2a+b)/3 \\ (-a+b)/3 \end{bmatrix}$

０６D02 【 交大85資工[2] 】

Let $E=[u_1 , u_2 , u_3]$ and $F=[b_1 , b_2]$ be ordered bases for vector spaces $\mathbb{R}^3$ and $\mathbb{R}^2$, respectively, where

$$u_1=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, u_2=\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, u_3=\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, b_1=\begin{bmatrix} 1 \\ -1 \end{bmatrix}, b_2=\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Let $(1,3,5)^T$ be the coordinate vector of v with respect to the basis E.

Let $x = (x_1, x_2, x_3)^T$ be a vector in $\mathbb{R}^3$ and L be a linear transformation mapping $\mathbb{R}^3$ into $\mathbb{R}^2$ defined by $L(x)=(x_1+x_2, x_1-x_3)^T$.

(a) Find $[L(v)]_F$. (6%)

(b) Determine the kernel of L and the inverse image of

$$T=\left\{ \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} \middle| \alpha \in \mathbb{R} \right\}. \qquad (4\%)$$

【分析】(1) 本題(a)必須分清楚座標與向量! 請參閱綜線CH6定義28.

(2) 本題(b)請參閱題型08A, 題型03C.

【解】(a) ∵ v的E–座標爲 $(1, 3, 5)^T$,

∴ $v=1u_1+3u_2+5u_3=(1, 0, -1)^T+3(1, 2, 1)^T+5(-1, 1, 1)^T=(-1, 11, 7)^T$

∴ $L(v)=(10, -8)^T$

令 $(10,-8)^T=a(1, -1)^T+b(2,-1)^T$,

比較分量得 $a+2b=10$, $-a-b=-8$.

解得 $a=6$, $b=2$.

$$\therefore \left[L(v)\right]_F = \begin{bmatrix} 6 \\ 2 \end{bmatrix} .$$

(b) $(x_1, x_2, x_3)^T \in L^{-1}(T)$

$\Longleftrightarrow$ $L((x_1, x_2, x_3)^T) \in T$ (綜線CH8定義1)

$\Longleftrightarrow$ $(x_1+x_2, x_1-x_3)^T \in T$

$\Longleftrightarrow$ $\exists \alpha \in \mathbb{R}$使得 $(x_1+x_2, x_1-x_3)^T=(\alpha, \alpha)$

解方程式 $\begin{cases} x_1+x_2 = \alpha \\ x_1 - x_3 = \alpha \end{cases}$

得 $x_1=\alpha+t$, $x_2=-t$, $x_3=t$; $t\in\mathbb{R}$

$\therefore$ T的inverse image為 $\left\{ \begin{bmatrix} \alpha+t \\ -t \\ t \end{bmatrix} \middle| \ \alpha, t \in \mathbb{R} \right\}$

$(x_1, x_2, x_3)^T \in \text{Ker}L$

$\Longleftrightarrow \quad L((x_1, x_2, x_3)^T) = (0, 0)^T$ (綜線CH8定義5)

$\Longleftrightarrow \quad (x_1 + x_2, x_1 - x_3)^T = (0, 0)^T$

解方程式 $\begin{cases} x_1 + x_2 \quad = 0 \\ x_1 \quad - x_3 = 0 \end{cases}$

得 $x_1 = t, x_2 = -t, \ x_3 = t \ ; \ t \in \mathbb{R}$

$\therefore \ \text{Ker}L = \left\{ \begin{bmatrix} t \\ -t \\ t \end{bmatrix} \middle| \ t \in \mathbb{R} \right\}$

０６D03 【 中原85工工[5] 】

Consider the bases $B = \{u_1, u_2\}$, $B' = \{v_1, v_2\}$ for $\mathbb{R}^2$, where

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

(a) Find the transition matrix from B to B'. (10%)

(b) Consider a vector w in $\mathbb{R}^2$, $w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Find the coordinate matrix of w respective to B'. (5%)

【解】(a)

令 $\begin{bmatrix}1\\1\end{bmatrix}=a\begin{bmatrix}1\\3\end{bmatrix}+b\begin{bmatrix}2\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}=c\begin{bmatrix}1\\3\end{bmatrix}+d\begin{bmatrix}2\\0\end{bmatrix}$,

可解得$a=1/3$, $b=1/3$, $c=1/3$, $d=-1/6$.

∴ 所求爲 $\begin{bmatrix}1/3 & 1/3\\1/3 & -1/6\end{bmatrix}$. (綜線CH6定理33)

(b) 由(a)已得知 $\begin{bmatrix}1\\1\end{bmatrix}=(1/3)\begin{bmatrix}1\\3\end{bmatrix}+(1/3)\begin{bmatrix}2\\0\end{bmatrix}$.

∴ 所求爲 $\begin{bmatrix}1/3\\1/3\end{bmatrix}$. (綜線CH6定義28)

06D**04** 【 師大84資教[7] 】

Find the transition matrix S corresponding to the change of bases from $[v_1 , v_2]$ to $[u_1 , u_2]$, where $v_1=[5, 2]^t$, $v_2=[7, 3]^t$, $u_1=[3, 2]^t$, and $u_2=[1, 1]^t$.

【解】

設 $S=\begin{bmatrix}a & b\\c & d\end{bmatrix}$, 則

$\begin{bmatrix}5\\2\end{bmatrix}=a\begin{bmatrix}3\\2\end{bmatrix}+c\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}7\\3\end{bmatrix}=b\begin{bmatrix}3\\2\end{bmatrix}+d\begin{bmatrix}1\\1\end{bmatrix}$ (綜線CH6定義33)

解方程式得 $a=3, c=-4, b=4, d=-5$, 即

$$S=\begin{bmatrix}3 & 4\\-4 & -5\end{bmatrix}$$

０６D**05** 【 大同84資工[2] 】

Let $u_1=[1, 1, 1]^T$, $u_2=[1, 2, 2]^T$ and $u_3=[2, 3, 4]^T$ and $v_1=[1, 0, 1]^T$, $v_2=[0, 1, 1]^T$, $v_3=[0, 1, 2]^T$.

(a) Find the transition matrix from $[v_1 , v_2 , v_3]$ to $[u_1 , u_2 , u_3]$

(b) If $x=2v_1+3v_2-4v_3$, determine the coordinates of x respect to $[u_1 , u_2 , u_3]$.

【解】(a) 設 transition matrix 為 $P=[\,p_{ij}]$, 則

$$\begin{cases} v_1=p_{11}u_1+p_{21}u_2+p_{31}u_3 \\ v_2=p_{12}u_1+p_{22}u_2+p_{32}u_3 \\ v_3=p_{13}u_1+p_{23}u_2+p_{33}u_3 \end{cases}$$

(綜線CH6定義33)

由 $$\begin{bmatrix}1\\0\\1\end{bmatrix}=p_{11}\begin{bmatrix}1\\1\\1\end{bmatrix}+p_{21}\begin{bmatrix}1\\2\\2\end{bmatrix}+p_{31}\begin{bmatrix}2\\3\\4\end{bmatrix}$$

解方程式得 $p_{11}=1$, $p_{21}=-2$, $p_{31}=1$.

由 $$\begin{bmatrix}0\\1\\1\end{bmatrix}=p_{12}\begin{bmatrix}1\\1\\1\end{bmatrix}+p_{22}\begin{bmatrix}1\\2\\2\end{bmatrix}+p_{32}\begin{bmatrix}2\\3\\4\end{bmatrix}$$

解方程式得 $p_{12}=-1$, $p_{22}=2$, $p_{32}=0$.

由 $$\begin{bmatrix}0\\1\\2\end{bmatrix}=p_{13}\begin{bmatrix}1\\1\\1\end{bmatrix}+p_{23}\begin{bmatrix}1\\2\\2\end{bmatrix}+p_{33}\begin{bmatrix}2\\3\\4\end{bmatrix}$$

解方程式得 $p_{13}=-2$, $p_{23}=0$, $p_{33}=1$.

$\therefore$ 所求爲 $\begin{bmatrix} 1 & -1 & -2 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(b) $x = 2v_1 + 3v_2 - 4v_3$,

$\therefore$ x 對基底 $[v_1, v_2, v_3]$ 的座標爲 $[2 \quad 3 \quad -4]^T$.

$\therefore$ x 對基底 $[u_1, u_2, u_3]$ 的座標爲

$$\begin{bmatrix} 1 & -1 & -2 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ -2 \end{bmatrix}$$ (綜線CH7定理33)

０６D**06** 【 精編加強題 】

考慮向量空間

$$V = \left\{ \begin{bmatrix} a & b & c \\ b & c & a \end{bmatrix} \middle| \, a, b, c \in \mathbb{R} \right\}$$

及基底

$$B = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\},$$

$$C = \left\{ \begin{bmatrix} 2 & 5 & 3 \\ 5 & 3 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \right\}$$

(1) 求 $x = \begin{bmatrix} 3 & 5 & 4 \\ 5 & 4 & 3 \end{bmatrix}$ 的 B 坐標及 C 坐標.

(2) 求 P 使得 $\forall v \in V$, $[v]_B = P[v]_B$

【解】(1)

$$\begin{bmatrix}3&5&4\\5&4&3\end{bmatrix}=3\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}+5\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}+4\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}$$

$\therefore [x]_B = [3, 5, 4]^T$

$$\begin{bmatrix}3&5&4\\5&4&3\end{bmatrix}=\frac{3}{5}\begin{bmatrix}2&5&3\\5&3&2\end{bmatrix}+\frac{9}{5}\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}+\frac{1}{5}\begin{bmatrix}0&1&2\\1&2&0\end{bmatrix}$$

$\therefore [x]_C = [3/5, 9/5, 1/5]^T$

(2) 將C表爲B的線性組合:

$$\begin{bmatrix}2&5&3\\5&3&2\end{bmatrix}=2\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}+5\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}+3\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}$$

$$\begin{bmatrix}1&1&1\\1&1&1\end{bmatrix}=1\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}+1\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}+1\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}$$

$$\begin{bmatrix}0&1&2\\1&2&0\end{bmatrix}=0\begin{bmatrix}1&0&0\\0&0&1\end{bmatrix}+1\begin{bmatrix}0&1&0\\1&0&0\end{bmatrix}+2\begin{bmatrix}0&0&1\\0&1&0\end{bmatrix}$$

$$\therefore P=\begin{bmatrix}2&1&0\\5&1&1\\3&1&2\end{bmatrix}$$