題型07A: 線性條件

０７A**０１**【 元智83工工[3] 】

[是非題]

Let T: $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ and $T(x)=Ax+b$, where b is a nonzero vector in $\mathbb{R}^m$, Then T is a linear transformation. (2%)

【解】×, 解說如下:

$T(o)+T(o)=b+b=2b$, $T(o+o)=T(o)=b$

不滿足線性條件.

０７A**０２**【 台大79資工[4](ii) 】

(Yes or No question and explain the reason:)

(ii) Let T:$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined as

$$T\left(x_1, x_2\right)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+[e, f]$$

Then T is a linear transformation.

【勘誤】行向量與列向量無法相加. T的定義應修正為

$$T\left(x_1, x_2\right)=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} e \\ f \end{bmatrix}$$

【解】 NO

在$[e, f]\neq[0, 0]$ 時， $T((0, 0))=[e, f]$,

$T(2(0, 0))=[e, f]\neq 2T((0, 0))$

$\therefore$ T不為linear transformation (綜線CH7定理2)

０７A**０３**【 中央86資工[1](d) 】

[是非論證題]

(d) The geometric operations: translation, rotation, and scaling are linear transformation.

【解】(d) False.

translation(平移)將零向量映到非零向量, 所以不是線性映射.

０７A**04** 【 師大84資教[8] 】

A mapping M define $M(x)=(x_1^2+x_2^2)^{1/2}$.　Is M a linear transformation ?

【解】No.

$M((1,0)+(0,1))=M(1,1)=\sqrt{2}$,　$M(1,0)+M(0,1)=1+1=2$

不合於線性條件.

０７A**05** 【 元智83電資[1] 】

Let $\varphi:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ be a mapping defined by

$$\varphi(x,y)=\begin{cases}(x,y) & \text{if } xy>0\\(2x,2y) & \text{if } xy\le 0\end{cases}$$

(ⅰ) Is φ linear ?

(ⅱ) If so, prove it, otherwise, explain why not.　(10%)

【分析】由綜線CH7定理6即知φ不是線性映射. 但本題須依定義證明φ不滿足線性條件

【解】(ⅰ) No!

(ⅱ) φ並不滿足線性條件. 例如：

$\varphi(2,1)+\varphi(-1,2)=(2,1)+(-2,4)=(0,5)$,

$\varphi((2,1)+(-1,2))=\varphi(1,3)=(1,3)$,　兩者不等.

０７A**06** 【 中正85資工[4] 】

Let $T(a+bx+cx^2)=(a+b)+(a+c)x+(b+c)x^2$.　Prove that T： $\mathbb{R}_2[x]\longrightarrow\mathbb{R}_2[x]$ is a linear mapping.　Also, find that $T^{-1}:\mathbb{R}_2[x]\longrightarrow\mathbb{R}_2[x]$.

【解】$T(k_1(a_1+b_1x+c_1x^2)+k_2(a_2+b_2x+c_2x^2))$

$=T((k_1a_1+k_2a_2)+(k_1b_1+k_2b_2)x+(k_1c_1+k_2c_2)x^2)$

$=(k_1a_1+k_2a_2+k_1b_1+k_2b_2)+(k_1a_1+k_2a_2+k_1c_1+k_2c_2)x+(k_1b_1+k_2b_2+k_1c_1+k_2c_2)x^2$

$=k_1(\,(a_1+b_1)+(a_1+c_1)x+(b_1+c_1)x^2\,)+k_2(\,(a_2+b_2)+(a_2+c_2)x+(b_2+c_2)x^2\,)$

$=k_1T(a_1+b_1x+c_1x^2)+k_2T(a_2+b_2+c_2x^2)$

∴ T為線性映射.

令 $T(a+bx+cx^2)=p+qx+rx^2$, 比較係數得： $p=a+b,\ q=a+c,\ r=b+c.$

∴ $(p+q+r)/2=a+b+c,$

∴ $a=(p+q-r)/2,\quad b=(p-q+r)/2,\quad c=(-p+q+r)/2$

$$\therefore\ T^{-1}(p+qx+rx^2)=\frac{p+q-r}{2}+\frac{p-q+r}{2}x+\frac{-p+q+r}{2}x^2\ .$$

０７A07 【 元智81工工[1] 】

若 $\varphi:\ \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ 為一linear transformation, 使

$\varphi(1,1,0)=(3,1)$, $\varphi(1,0,1)=(\pi,0)$, $\varphi(0,1,1)=(\sqrt{2},\ 2)$, 求 $\varphi(x,y,z)$.

【解】令 $(x,y,z)=\alpha(1,1,0)+\beta(1,0,1)+\gamma(0,1,1)$,

得 $$\begin{cases}\alpha+\beta = x\\ \alpha+\gamma = y\\ \beta+\gamma = z\end{cases}$$

可解出 $\alpha=\dfrac{x+y-z}{2}$, $\beta=\dfrac{x-y+z}{2}$, $\gamma=\dfrac{-x+y+z}{2}$

$\therefore\ \varphi(x,y,z)=\varphi(\alpha(1,1,0)+\beta(1,0,1)+\gamma(0,1,1))$

$=\alpha\,\varphi(1,1,0)+\beta\,\varphi(1,0,1)+\gamma\,\varphi(0,1,1)$ (綜線CH7定義1)

$$=\frac{x+y-z}{2}(3,1)+\frac{x-y+z}{2}(\pi,0)+\frac{-x+y+z}{2}(\sqrt{2},\ 2)$$

$$=\left(\frac{3+\pi-\sqrt{2}}{2}x+\frac{3-\pi+\sqrt{2}}{2}y+\frac{-3+\pi+\sqrt{2}}{2}z\ ,\frac{-1}{2}x+\frac{3}{2}y+\frac{3}{2}z\right)$$

０７A08 【 元智80工工[1] 】

Let P_2 = the space of all real polynomials with degree 2 or less. If T: $P_2 \longrightarrow P_2$ is known to be linear and $T(x+1)=x$, $T(x-1)=1$, $T(x^2)=0$, find $T(2+3x-x^2)$.

【解】令 $2+3x-x^2=\alpha(x+1)+\beta(x-1)+\gamma x^2$

比較係數得 $2=\alpha-\beta$, $3=\alpha+\beta$, $-1=\gamma$,

上式可解得 $\alpha=5/2$, $\beta=1/2$, $\gamma=-1$.

$T(2+3x-x^2)=\alpha T(x+1)+\beta T(x-1)+\gamma T(x^2)=\alpha\cdot x+\beta\cdot 1+\gamma\cdot 0=(5/2)x+(1/2)$

０７A**09** 【 清大79資料[2] 】

Construct a linear transformation of $\mathbb{R}^4$ into $\mathbb{R}^4$ which has a null–space spanned by (1, 2, 3, 4) and (1, 1, 2, 2).

【解】1°

$$\begin{bmatrix}1&2&3&4\\1&1&2&2\end{bmatrix}\sim\begin{bmatrix}0&1&1&2\\1&1&2&2\end{bmatrix}\sim\begin{bmatrix}0&1&1&2\\1&0&1&0\end{bmatrix}\sim\begin{bmatrix}1&0&1&0\\0&1&1&2\end{bmatrix}$$

$$\because\ \begin{vmatrix}1&0&1&0\\0&1&1&2\\0&0&1&0\\0&0&0&1\end{vmatrix}\neq 0,$$

$\therefore$ { (1, 0, 1, 0), (0, 1, 1, 2), (0, 0, 1, 0), (0, 0, 0, 1) }

爲$\mathbb{R}^4$ 的基底. (綜線CH12習題16.1)

2°$\therefore$ 可令線性映射 $T:\mathbb{R}^4\longrightarrow\mathbb{R}^4$，滿足

$T(1, 0, 1, 0)=(0, 0, 0, 0)$

$T(0, 1, 1, 2)=(0, 0, 0, 0)$

$T(0, 0, 1, 0)=(0, 0, 1, 0)$

$T(0, 0, 0, 1)=(0, 0, 0, 1)$ (綜線CH7定理3)

$\because$ $(a, b, c, d)=a(1,0,1,0)+b(0,1,1,2)+(c-a-b)(0,0,1,0)+(d-2b)(0,0,0,1)$

$\therefore T(a, b, c, d)=aT(1,0,1,0)+bT(0,1,1,2)+(c-a-b)T(0,0,1,0)+(d-2b)T(0,0,0,1)$

$=(c-a-b)(0, 0, 1, 0)+(d-2a)(0, 0, 0, 1)$

$=(0, 0, c-a-b, d-2b)$

3°以下證明T合於所求,

即欲證 $\ker T = \operatorname{span}\{(1, 2, 3, 4), (1, 1, 2, 2)\}$.

$$
\begin{aligned}
& (a, b, c, d) \in \ker T \\
\Longleftrightarrow\quad & \begin{cases} c-a-b=0 \\ d-2b=0 \end{cases} \\
\Longleftrightarrow\quad & (a, b, c, d)=(a, b, a+b, 2b) \\
\Longleftrightarrow\quad & (a, b, c, d)=a(1, 0, 1, 0)+b(0, 1, 1, 2) \\
\Longleftrightarrow\quad & (a, b, c, d)\in \operatorname{span}\{(1, 0, 1, 0), (0, 1, 1, 2)\} \\
\Longleftrightarrow\quad & (a, b, c, d)\in \operatorname{span}\{(1, 2, 3, 4), (1, 1, 2, 2)\}
\end{aligned}
$$

題型07B: 矩陣表示

０７B**０１**【 成大81資工甲乙[5] 】

Let $L：\mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined for $x=[x_1, x_2, x_3]^t$ by

$$L(x)=\begin{bmatrix} 3x_1-x_2+2x_3 \\ 2x_1+4x_2-x_3 \end{bmatrix}$$

Let $S=\{v_1, v_2, v_3\}$ and $T=\{w_1, w_2\}$, where $v_1=[1, 1, 0]^t$, $v_2=[1, 0, 1]^t$, $v_3=[1, 1, 1]^t$, $w_1=[1, 1]^t$ and $w_2=[-1, 0]^t$

(a) Determine the matrix A of L with respect to bases S and T. (5%)

(b) Compute $L(x)$ for $x=[4, 2, 1]^t$. (5%)

【解】(a)

$$L(v_1)=\begin{bmatrix} 2 \\ 6 \end{bmatrix}, L(v_2)=\begin{bmatrix} 5 \\ 1 \end{bmatrix}, L(v_3)=\begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

令 $\begin{cases} L(v_1)=aw_1+bw_2 \\ L(v_2)=cw_1+dw_2 \\ L(v_3)=ew_1+fw_2 \end{cases}$

則得 $\begin{cases} 2=a-b \\ 6=a, \end{cases}$ $\begin{cases} 5=c-d \\ 1=c, \end{cases}$ $\begin{cases} 4=e-f \\ 5=e \end{cases}$

解出 $a=6, b=4, c=1, d=-4, e=5, f=1$

$\therefore \begin{bmatrix} 6 & 1 & 5 \\ 4 & -4 & 1 \end{bmatrix}$ 爲所求。 (綜線CH7定義9)

(b) 依本題之定義,

$$L(x)=\begin{bmatrix}3\cdot 4-2\ +2\cdot 1\\ 2\cdot 4+4\cdot 2-1\end{bmatrix}=\begin{bmatrix}12\\15\end{bmatrix}$$

０７B**02**【 元智82電資[14] 】

Let $T\begin{bmatrix}1\\0\\0\\0\end{bmatrix}=\begin{bmatrix}3\\4\\5\end{bmatrix}$, $T\begin{bmatrix}0\\1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\1\\0\end{bmatrix}$, $T\begin{bmatrix}0\\0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\3\\6\end{bmatrix}$, $T\begin{bmatrix}0\\0\\0\\1\end{bmatrix}=\begin{bmatrix}9\\4\\1\end{bmatrix}$.

(a). Write down the matrix A such that corresponds to the linear transformation T.

(b). Determine a basis for the null space and nullity of T.

(c). Determine the range of T.

(d). Calculate $T\begin{bmatrix}6\\9\\-5\\1\end{bmatrix}$

【解】(a).

$$T\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=T(x\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+y\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+w\begin{bmatrix}0\\0\\0\\1\end{bmatrix})$$

$$=xT\begin{bmatrix}1\\0\\0\\0\end{bmatrix}+yT\begin{bmatrix}0\\1\\0\\0\end{bmatrix}+zT\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+wT\begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

$$=x\begin{bmatrix}3\\4\\5\end{bmatrix}+y\begin{bmatrix}1\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\3\\6\end{bmatrix}+w\begin{bmatrix}9\\4\\1\end{bmatrix}.$$

$$=\begin{bmatrix}3&1&0&9\\4&1&3&4\\5&0&6&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$$

$\therefore$ 所求爲 $\begin{bmatrix}3&1&0&9\\4&1&3&4\\5&0&6&1\end{bmatrix}$

(b). 執行列運算:

$$\begin{bmatrix}3&1&0&9\\4&1&3&4\\5&0&6&1\end{bmatrix}\begin{matrix}(-1)\\ \leftarrow\\ \end{matrix}\sim\begin{bmatrix}3&1&0&9\\1&0&3&-5\\5&0&6&1\end{bmatrix}\begin{matrix}\leftarrow\\(-3)\ (-5)\\ \leftarrow\end{matrix}$$

$$\sim \;....\; \sim \begin{bmatrix} 1 & 0 & 0 & 33/9 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -26/9 \end{bmatrix}$$

得

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (-33/9)t \\ 0 \\ (26/9)t \\ t \end{bmatrix} = (t/9) \begin{bmatrix} -33 \\ 0 \\ 26 \\ 9 \end{bmatrix}$$

∴ 可取 $\{[-33 \quad 0 \quad 26 \quad 9]^T\}$爲 T 的null space的基底 .

T的nullity爲1 .

(c). $T: \mathbb{R}^{4\times1} \longrightarrow \mathbb{R}^{4\times1}$,

rank(T)＝4–nullity(T)＝4–1＝3

∴ dim [range of T]＝3＝dim [codomain of T]

∴ [range of T]＝[codomain of T]＝$\mathbb{R}^{3\times1}$

(d). 由(a).,

$$T \begin{bmatrix} 6 \\ 9 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 9 \\ 4 & 1 & 3 & 4 \\ 5 & 0 & 6 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 9 \\ -5 \\ 1 \end{bmatrix} = \begin{bmatrix} 36 \\ 22 \\ 1 \end{bmatrix}$$

０７B**03** 【 交大86資料[2] 】

Let $b_1 = [1\ 1\ 1]^T$, $b_2 = [1\ 1\ 0]^T$ and $b_3 = [1\ 0\ 0]^T$ and L be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ as $L(x) = x_1 b_1 + x_2 b_2 + (x_1 + x_2) b_3$.

Find the matrix A representing L with respect to the bases $[e_1, e_2]$ and $[b_1, b_2, b_3]$.

【分析】本題b_1 , b_2 , b_3 的確實值只是障眼法，並不須用到.

【解】 $L(e_1) = L([1,0]^T) = b_1 + b_3$, $\quad L(e_2) = L([0,1]^T) = b_2 + b_3$

$$\therefore\ A=\begin{bmatrix}1 & 0\\ 0 & 1\\ 1 & 1\end{bmatrix}$$ (綜線CH7定義9)

０７B**04** 【 中正85資工[1] 】

Find the 4 by 4 matrix A that represents a cyclic permutation: each vector such that $(x_1, x_2, x_3, x_4)^T$ is transformed to $(x_2, x_3, x_4, x_1)^T$ by A. What are the effects of A^2 and A^3 ? Show that $A^3 = A^{-1}$.

【解】

$$\begin{bmatrix}x_2\\ x_3\\ x_4\\ x_1\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix},\quad \therefore\ A=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\end{bmatrix}$$

$$A^2\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=A\begin{bmatrix}x_2\\ x_3\\ x_4\\ x_1\end{bmatrix}=\begin{bmatrix}x_3\\ x_4\\ x_1\\ x_2\end{bmatrix}$$

$$A^3\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}=A\begin{bmatrix}x_3\\ x_4\\ x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_4\\ x_1\\ x_2\\ x_3\end{bmatrix}$$

$$\therefore\ A^2=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix}=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}$$

$$\therefore\ A^4=A^2A^2=\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}\begin{bmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{bmatrix}=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=I$$

$\therefore\ A^3A=AA^3=I \qquad \therefore\ A^3=A^{-1}$. (綜線CH2定義10)

０７B**05** 【 交大78資工[5] 】

Let V be a vector space over a field. Let T be a transformation of V^3 into itself with respect to the canonical basis $\varepsilon_1, \varepsilon_2, \varepsilon_3$.

The matrix of the transformation is $\begin{bmatrix}1&1&-2/3\\-1&0&4/3\\1&2&-1\end{bmatrix}$

Let $\beta_i=T^i(\varepsilon_1)$, i=1, 2, 3 Are β_1, β_2 and β_3 linearly independent?

【解】 $\beta_1=T^1(\varepsilon_1)=\begin{bmatrix}1&1&-2/3\\-1&0&4/3\\1&2&-1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\-1\\1\end{bmatrix}$

$\beta_2=T^2(\varepsilon_1)=T(T(\varepsilon_1))$

$$=T(\beta_1)=\begin{bmatrix}1 & 1 & -2/3\\ -1 & 0 & 4/3\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}=\begin{bmatrix}-2/3\\ 1/3\\ -2\end{bmatrix}$$

$\beta_3=T^3(\varepsilon_1)=T(T^2(\varepsilon_1))$

$$=T(\beta_2)=\begin{bmatrix}1 & 1 & -2/3\\ -1 & 0 & 4/3\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}-2/3\\ 1/3\\ -2\end{bmatrix}=\begin{bmatrix}-1\\ -2\\ 2\end{bmatrix}$$

試解方程式 $x\beta_1+y\beta_2+z\beta_3=0$:

$$x\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}+y\begin{bmatrix}-2/3\\ 1/3\\ -2\end{bmatrix}+z\begin{bmatrix}-1\\ -2\\ 2\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

即 $\begin{cases}x-(2/3)y-z=0\\ -x+(1/3)y-2z=0\\ x-2y+2z=0\end{cases}$

經列運算:

$$\begin{bmatrix}1 & -2/3 & -1\\ -1 & 1/3 & -2\\ 1 & -2 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

即 $x=0,\quad y=0,\quad z=0$

∴ $\beta_1,\ \beta_2,\ \beta_3$ 爲linear independent (綜線CH6定義9②)

０７B**06** 【 交大83資工[4] 】

Let L be the operator on P_3 (P_3 denotes the set of all polynomials of degree less than 3) defined by $L(p(x))=xp'(x)+p''(x)$ where $p'(x)$ denotes the derivative of $p(x)$ and $p''(x)$

denotes the second derivative of $p(x)$.

(a) Find the matrix A representing L with respect to ordered basis $[1, x, x^2]$.

(b) Find the matrix B representing L with respect to ordered basis $[1, x, 1+x^2]$.

(c) Find the matrix S such that $B=S^{-1}AS$.

(d) If $p(x)=a_0+a_1x+a_2(1+x^2)$, calculate $L^n(p(x))$. (10%)

【解】 (a) $L(1)=x\cdot(1)'+(1)''=0=0\cdot 1+0\cdot x+0^2\cdot x$

$L(x)=x\cdot(x)'+(x)''=x=0\cdot 1+1\cdot x+0\cdot x^2$

$L(x^2)=x\cdot(x^2)'+(x^2)''=2x^2+2=2\cdot 1+0\cdot x+2\cdot x^2$

$$\therefore\ A=\begin{bmatrix}0&0&2\\0&1&0\\0&0&2\end{bmatrix}$$ (綜線CH7定義9)

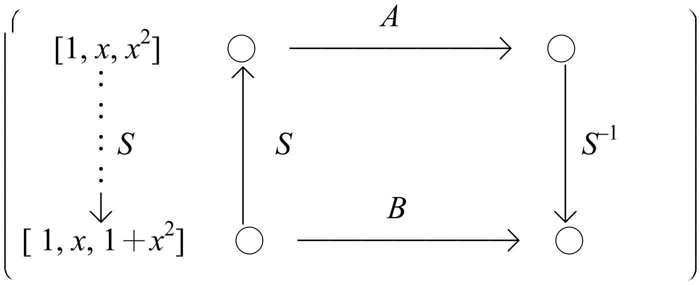
(b) $L(1)=0=0\cdot 1+0\cdot x+0\cdot(1+x^2)$

$L(x)=x=0\cdot 1+1\cdot x+0\cdot(1+x^2)$

$L(1+x^2)=x\cdot(1+x^2)'+(1+x^2)''=2x^2+2=0\cdot 1+0\cdot x+2(1+x^2)$

$$\therefore\ B=\begin{bmatrix}0&0&0\\0&1&0\\0&0&2\end{bmatrix}$$

(c)



以$[1, x, x^2]$ 描述 $[1, x, 1+x^2]$, 取

$$S=\begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix},$$ (綜線CH6定理33)

則$B=S^{-1}AS$. (綜線CH7定理19)

(d) 設$\mathscr{B}=[1, x, 1+x^2]$.

$[L^n(p(x))]_{\mathscr{B}}=[L^n]_{\mathscr{B}}[p(x)]_{\mathscr{B}}$ (綜線CH7定理15)

$=[L]_{\mathscr{B}}^n[p(x)]_{\mathscr{B}}=B^n[p(x)]_{\mathscr{B}}$

$$=\begin{bmatrix}0&0&0\\0&1&0\\0&0&2^n\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}=\begin{bmatrix}0\\a_1\\2^na_2\end{bmatrix}$$

$\therefore\ L^n(p(x))=a_1x+2^na_2(1+x^2)$

０７Ｂ**07** 【 大同85資工[2] 】

Let L be the operator on P_3 defined by $L(p(x))=xp'(x)+p''(x)$, where P_n denotes the set of all polynomials of degree less than n.

(a) Find the matrix A representing L with respect to $[1,x,x^2]$. (2%)

(b) Find the matrix B representing L with respect to $[1,x,1+x^2]$. (2%)

(c) Find the matrix S such that $B=S^{-1}AS$. (3%)

(d) If $p(x)=a_0+a_1x+a_2(1+x^2)$, calculate $L^n(p(x))$. (3%)

【解】本題與上題完全相同.

０７Ｂ**08** 【 清大78工工[6] 】

(ⅰ) Let V equal P^3, the space of polynomials of degree strictly less than 3, and let W equal the analogous space P^4 . Define T from V to W so that for each polynomial f in V, $T(f)$ is that polynomial in W whose value at t equals $tf(t)+\{f(t)-f(0)\}/t$

Find the matrix representation of T with respect to the ordered bases

$B=\{1; 1+t; 1+t+t^2\}$ for V,

and

$C=\{1;\ 1-t;\ 1+2t+t^2\ ;\ 1-3t+3t^2-t^3\}$ for W. (10%)

(ii) Let P^4 be as above, and let T be the linear transformation from P^4 to P^4 that takes each polynomial f to the derivative of $tf(t)$. Use the ordered basis $\{1;t;t^2;t^3\}$ for P^4 as both domain and range to find a matrix representation for T. (10%)

【解】(i) 令 $f_1(t)=1,\ T(f_1)=F_1$

則 $F_1(t)=tf_1(t)+(f_1(t)-f_1(0))/t=t+(1-1)/t=t$

令 $f_2(t)=1+t,\quad T(f_2)=F_2$

則 $F_2(t)=t(1+t)+((1+t)-1)/t2=t(1+t)+1=1+t+t^2$

令 $f_3(t)=1+t+t^2,\quad T(f_3)=F_3$

則 $F_3(t)=t(1+t+t^2)+(t+t^2)/t=t(1+t+t^2)+1+t=1+2t+t^2+t^3$

$F_1(t)=t=1-(1-t)=1\cdot 1+(-1)(1-t)+0(1+2t+t^2)+0(1-3t+3t^2-t^3)$

$F_2(t)=1+t+t^2=(1+2t+t^2)-t=(1+2t+t^2)+(1-t)-1$

$=(-1)1+1(1-t)+1(1+2t+t^2)+0(1-3t+3t^2-t^3)$

$F_3(t)=1+2t+t^2+t^3=-(1-3t+3t^2-t^3)+4t^2-t+2$

$=-(1-3t+3t^2-t^3)+4(1+2t+t^2)-9t-2$

$=-(1-3t+3t^2-t^3)+4(1+2t+t^2)+9(1-t)-11$

$=(-11)\cdot 1+9(1-t)+4(1+2t+t^2)+(-1)(1-3t+3t^2-t^3)$

$$\therefore \begin{bmatrix} 1 & -1 & -11 \\ -1 & 1 & 9 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$ 爲所求. (綜線CH7定義9)

(ii) 令$f_1(t)=1,\ Tf_1=F_1,$ 則 $F_1(t)=(tf_1(t))'=t'=1$

令$f_2(t)=t,\ Tf_2=F_2,$ 則 $F_2(t)=(tf_2(t))'=(t^2)'=2t$

令$f_3(t)=t^2,\ Tf_3=F_3,$ 則 $F_3(t)=(tf_3(t))'=(t^3)'=3t^2$

令$f_4(t)=t^3,\ Tf_4=F_4,$ 則 $F_4(t)=(tf_4(t))'=(t^4)'=4t^3$

$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$ 爲所求　　(綜線CH7定義9)

題型07C: 矩陣表示的變換

０７C**01** 【 大同86資工[2] 】

Let L be the linear transformation on $\mathbb{R}^3$ defined by

$L(x)=(2x_1-x_2-x_3, 2x_2-x_1-x_3, 2x_3-x_1-x_2)^T$

and let A be the matrix representing L with respect to the standard basis $[e_1, e_2, e_3]$.

If $u_1=(1, 1, 0)^T$, $u_2=(1, 0, 1)^T$ and $u_3=(0, 1, 1)^T$.

(a) Find A. (4%)

(b) Find the transition matrix U corresponding to a change of basis from $[u_1, u_2, u_3]$ to $[e_1, e_2, e_3]$ (4%)

(c) Determine the matrix B representing L with respect to $[u_1, u_2, u_3]$ (4%)

【解】(a) (請參閱題型07B)

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)=\begin{bmatrix} 2x_1-x_2-x_3 \\ 2x_2-x_1-x_3 \\ 2x_3-x_1-x_2 \end{bmatrix}=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\therefore\ A=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$ (綜線CH7定理15)

(b) (請參閱題型06D)

$$\begin{cases} u_1=1\cdot e_1+1\cdot e_2+0\cdot e_3 \\ u_2=1\cdot e_1+0\cdot e_2+1\cdot e_3 \\ u_3=0\cdot e_1+1\cdot e_2+1\cdot e_3 \end{cases}$$

$$\therefore\ U=\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$$ (綜線CH6定義33)

(c) $$B=U^{-1}AU=\ \ldots\ =\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$$

【討論】此題B恰好等於A .

這是由於A取為$-U^2+4I$，以致$AU=UA$. (綜線CH定理16③)

$\therefore\ U^{-1}AU=A$.

０７C**02** 【 清大85資料[1] 】

Let $\mathcal{C}[a,b]$ be the vector space of all continuous functions on $[a,b]$ with $(f+g)(x)=f(x)+g(x)$, and $(f\cdot g)(x)=f(x)\cdot g(x)$.Let V be the subspace of $\mathcal{C}[a,b]$ spanned by $\{1, e^{-x}, e^{x}\}$ and let D be the differentiation operator on V.

(a) Find the transition matrix S representing the change of coordinates from $[1, e^{-x}, e^{x}]$ to $[1, \sinh(x), \cosh(x)]$, where $\sinh(x)=(e^{x}-e^{-x})/2$ and $\cosh(x)=(e^{x}+e^{-x})/2$.

(b) Find the matrix A representing D with respect to $[1, e^{-x}, e^{x}]$.

【勘誤】本題 $(f\cdot g)(x)=f(x)\cdot g(x)$ 應更正為 $(c\cdot g)(x)=c\cdot g(x)$.

【解】(a) 令

$$\begin{cases}1 = s_{11}\cdot 1+ s_{21}\sinh x+ s_{31}\cosh x\\ e^{-x}= s_{12}\cdot 1 + s_{22}\sinh x+ s_{32}\cosh x\\ e^{x} = s_{13}\cdot 1 + s_{23}\sinh x+ s_{33}\cosh x\end{cases}$$

解方程式可得出 a_{ij}，即

$$\begin{cases}1 = 1\cdot 1 + 0\cdot\sinh x + 0\cdot\cosh x\\ e^{-x}= 0\cdot 1 + (-1)\cdot\sinh x + 1\cdot\cosh x\\ e^{x} = 0\cdot 1 + 1\cdot\sinh x + 1\cdot\cosh x\end{cases}$$

$$\therefore \quad S=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 1\\ 0 & 1 & 1\end{bmatrix}$$ (綜線CH6定義33)

(b) (請參閱題型07B)

$$\begin{cases} D(1) = 0 = 0\cdot 1 + 0\cdot e^{-x} + 0\cdot e^{x} \\ D(e^{-x}) = -e^{-x} = 0\cdot 1 + (-1)\cdot e^{-x} + 0\cdot e^{x} \\ D(e^{x}) = e^{x} = 0\cdot 1 + 0\cdot e^{-x} + 1\cdot e^{x} \end{cases}$$

$$\therefore \quad A=\begin{bmatrix}0 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}$$ (綜線CH7定義9)

【加強演練】

接本題,

(c) Find the matrix B representaing D with respect to [1, sinh$x$, cosh$x$].

(d) How to compute B through A ?

[解] (c) $D(\sinh x)=D((e^{x}-e^{-x})/2)=(e^{x}+e^{-x})/2=\cosh x$

$D(\cosh x)=D((e^{x}+e^{-x})/2)=(e^{x}-e^{-x})/2=\sinh x$

$$\because \begin{cases} D(1) = 0 = 0\cdot 1 + 0\cdot \sinh x + 0\cdot \cosh x \\ D(\sinh x) = \cosh x = 0\cdot 1 + 0\cdot \sinh x + 1\cdot \cosh x \\ D(\cosh x) = \sinh x = 0\cdot 1 + 1\cdot \sinh x + 0\cdot \cosh x \end{cases}$$

$$B=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$$ (綜線CH7定義)

(d)

$$\left(\begin{array}{ccc} [\,1\,,e^{-x}\,,e^{x}\,] & \circ \xrightarrow{\quad A \quad} \circ \\ \uparrow \cdot S & S\downarrow \qquad\qquad \downarrow S \\ [\,1,\sinh x,\cosh x\,] & \circ \xrightarrow[\quad B \quad]{} \circ \end{array}\right)$$

$B=SAS^{-1}$ (綜線CH7定理19)

(讀者請自行驗證上式) #

０７C**03** 【 元智83電資[2] 】

Let $P^n(\mathbb{R})=\{a_0+a_1x+\ldots+a_nx^n \mid a_i\in\mathbb{R},\ i=0,1,\ldots,n\}$.Let $T:P^2(\mathbb{R})\longrightarrow P^3(\mathbb{R})$ be a linear mapping defined by

$$T(f(x))=xf(x)+(f(x)-f(0))/x,\quad \forall f(x)\in P^2(\mathbb{R}).$$

Let $B=(1, 1+x, 1+x+x^2)$ and $C=(1, 1-x, 1+2x+x^2, 1-3x+3x^2-x^3)$ be bases of $P^2(\mathbb{R})$ and $P^3(\mathbb{R})$, respectively. Find the matrix $[T]_B^C$ of T relative to bases B and C. (10%)

【分析】本題的矩陣表示的符號與綜合線性代數的符號上下相反. (綜線CH7定義9)

【解】設$S_2=(1, x, x^2)$, $S_3=(1,x,x^2,x^3)$.

$$\begin{cases} T(1)=x\cdot 1+(1-1)/x=x=0\cdot 1+1\cdot x+0\cdot x^2+0\cdot x^3 \\ T(x)=x\cdot x+(x-0)/x=x^2+1=1\cdot 1+0\cdot x+1\cdot x^2+0\cdot x^3 \\ T(x^2)=x\cdot x^2+(x^2-0)/x=x^3+x=0\cdot 1+1\cdot x+0\cdot x^2+1\cdot x^3 \end{cases}$$

$\therefore$ T對S_2 , S_3 的矩陣表示爲 $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (綜線CH7定義9)

(依綜線CH6定理33, 由觀察得知:)

由S_2到B的描述矩陣為 $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

由S_3到C的描述矩陣為 $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

∴ 所求矩陣表示為 (綜線CH7定理24)

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -11 \\ -1 & 1 & 9 \\ 0 & 1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$$

【另解】利用CH7定義9直接求算, 計算的複雜度與第一種解法相同.

$T(1) = x \cdot 1 + (1-1)/x = x$

$T(1+x) = x \cdot (1+x) + (1+x-1)/x = 1 + x + x^2$

$T(1+x+x^2) = x \cdot (1+x+x^2) + (x+x^2)/x = 1 + 2x + x^2 + x^3$

設所求矩陣表示為 $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{bmatrix}$.

則 (由綜線CH7定義9)

$x = a_1 \cdot 1 + a_2 \cdot (1-x) + a_3 \cdot (1+2x+x^2) + a_4 \cdot (1-3x+3x^2-x^3)$

$1+x+x^2 = b_1 \cdot 1 + b_2 \cdot (1-x) + b_3 \cdot (1+2x+x^2) + b_4 \cdot (1-3x+3x^2-x^3)$

$$1+2x+x^2+x^3 = c_1 \cdot 1 + c_2 \cdot (1-x) + c_3 \cdot (1+2x+x^2) + c_4 \cdot (1-3x+3x^2-x^3)$$

以上各式比較係數解連立方程式即可解出各未知數.

０７Ｃ**04** 【 清大75資科[3](2) 】

(2) Let $T(p) = p'' - 3p' + 4p$, where p' and p'' are the first and the second derivative of p respectively. Show that T is a linear transformation on $P_2[x]$ and find its matrix representation with respect to the basis $\{1, x, x^2\}$.

【解】(2) １° $\forall \alpha, \beta \in \mathbb{R}, \forall\, p(x), q(x) \in P_2[x]$

$$T(\alpha p(x) + \beta q(x))$$

$$= (\alpha p(x)+\beta q(x))'' - 3(\alpha p(x)+\beta q(x))' + 4(\alpha p(x)+\beta q(x))$$

$$= \alpha(p''(x) - 3p'(x) + 4p(x)) + \beta(q''(x) - 3q'(x) + 4q(x))$$

$$= \alpha T(p(x)) + \beta T(q(x))$$

$\therefore$ T is a linear transformation on $P_2[x]$ (綜線CH7定義1)

２° $T(1) = (1)'' - 3(1)' + 4 \cdot (1) = 4 = 4 \cdot 1 + 0 \cdot x + 0 \cdot x^2$

$T(x) = (x)'' - 3(x)' + 4 \cdot x = -3 + 4x = -3 \cdot 1 + 4 \cdot x + 0 \cdot x^2$

$T(x^2) = (x^2)'' - 3(x^2)' + 4 \cdot x^2 = 2 - 6x + 4x^2 = 2 \cdot 1 + (-6) \cdot x + 4 \cdot x^2$

$\therefore$ The matrix representation with respect to the basis $\{1, x, x^2\}$ is

$$\begin{bmatrix} 4 & -3 & 2 \\ 0 & 4 & -6 \\ 0 & 0 & 4 \end{bmatrix}$$ (綜線CH7定義9)

題型07D: 相似矩陣

07D01 【 元智81電資[4] 】

Let V be a 2–dimensional vector space over $\mathbb{R}$ and $T: V \longrightarrow V$ be a linear map. Let $W=\{v_1, v_2\}$ and $W'=\{v_1', v_2\}$ be two bases of V. Prove that the matrix representation $[T]_W$ of T relative to W and the matrix representation $[T]_{W'}$ of T relative to W' are similar.

【參考章節】綜線CH7定理21a.

【解】1° ∵ $\{v_1, v_2\}$ 是基底,

∴ 可令 $v_1'=p_1v_1+p_3v_2$, $v_2'=p_2v_1+p_4v_2$,

∵ $\{v_1', v_2'\}$ 是基底,

∴ 可令 $v_1=q_1v_1'+q_3v_2'$, $v_2=q_2v_1'+q_4v_2'$,

$$\therefore \begin{cases} v_1=q_1(p_1v_1+p_3v_2)+q_3(p_2v_1+p_4v_2)=(q_1p_1+q_3p_2)v_1+(q_1p_3+q_3p_4)v_2 \\ v_2=q_2(p_1v_1+p_3v_2)+q_4(p_2v_1+p_4v_2)=(q_2p_1+q_4p_2)v_1+(q_2p_3+q_4p_4)v_2 \end{cases}$$

$$\therefore \begin{cases} q_1p_1+q_3p_2=1 \quad , \quad q_1p_3+q_3p_4=0 \\ q_2p_1+q_4p_2=0 \quad , \quad q_2p_3+q_4p_4=1 \end{cases}$$ (綜線CH6定義28要訣2)

$$\therefore \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

∴ $\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$ 為可逆矩陣. (綜線CH3定理19)

2° 設 $\begin{bmatrix} T \end{bmatrix}_W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\begin{bmatrix} T \end{bmatrix}_{W'} = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$,

則 $Tv_1'=T(p_1v_1+p_3v_2)=p_1Tv_1+p_3Tv_2$ (綜線CH7定義1)

$=p_1(av_1+cv_2)+p_3(bv_1+dv_2)$ (綜線CH7定義9)

$=(ap_1+bp_3)v_1+(cp_1+dp_3)v_2$

$Tv_2' = T(p_2v_1 + p_4v_2) = p_2Tv_1 + p_4Tv_2 = p_2(av_1 + cv_2) + p_4(bv_1 + dv_2)$

$= (ap_2 + bp_4)v_1 + (cp_2 + dp_4)v_2$

另一方面:

$Tv_1' = a'v'_1 + c'v'_2$ (綜線CH7定義9)

$= a'(p_1v_1 + p_3v_2) + c'(p_2v_1 + p_4v_2)\ (a'p_1 + c'p_2)v_1 + (a'p_3 + c'p_4)v_2$

$Tv'_2 = b'v'_1 + d'v'_2$

$= b'(p_1v_1 + p_3v_2) + d'(p_2v_1 + p_4v_2) = (b'p_1 + d'p_2)v_1 + (b'p_3 + d'p_4)v_2$

比較係數可得: (綜線CH6習題28.1)

$ap_1 + bp_3 = a'p_1 + c'p_2$

$cp_1 + dp_3 = a'p_3 + c'p_4$

$ap_2 + bp_4 = b'p_1 + d'p_2$

$cp_2 + dp_4 = b'p_3 + d'p_4$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}$$

$$\therefore \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}^{-1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$$

故得證 $[T]_W$ 與 $[T]_{W'}$ 為similar。 (綜線CH7定義21)

０７D**02** 【 交大79工工[3] 】

是非題:

If A is similar to B, then rank$A$＝rank$B$.

【解】是

【討論】證明如下:

設$B = P^{-1}AP$, (綜線CH7定義21)

則 rank$B$＝rank$(P^{-1}AP)$＝rank(AP) (綜線CH8定理16④)

＝rank(A) (綜線CH8定理16④)

０７D**03** 【 中正82資工[4] 】

Show that the trace of a matrix Q is the summation of all eigenvalues of Q and is invariant under any orthonormal transformation. (10%)

【解】在適當的field上作三角化，即取可逆矩陣P使$P^{-1}QP=T$, T為三角矩陣.

(若Q為實數矩陣，在複數系必可三角化，見CH13定理10)

$\therefore$ $\text{tr}(Q)=\text{tr}(PTP^{-1})$

$=\text{tr}(P^{-1}PT)=\text{tr}(T)$ (綜線CH2定理28③)

$=Q$的eigenvalues的和 (綜線CH13定理10要訣3)

當Q經過orthonornmal transformation，也就是由一個orthogonal(或unitary) matrix U，將Q化為$U^{-1}QU$,

$\text{tr}(U^{-1}QU)=\text{tr}(UU^{-1}Q)=\text{tr}Q$ (綜線CH2定理28③)

【討論】 其實前述的U只需“可逆”就有本題的結果.

０７D**04** 【 中正84資工[1](cd) 】

True or False:

(c) If two matrices have the same set of eigenvalues, then they are similar.

(d) If two matrices have the same determinant, then they are similar.

【解】(c) False.

例如 $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ 與 $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ 的eigenvalues相同.

但因為rank不同，所以不相似. (綜線CH8定理16)

(d) False.

(c)的反例自動可當本小題的反例. (因行列式是特徵值的乘積.)

０７D**05** 【 交大86資工[6](e) 】

[是非倒扣題]

Any two $n\times n$ diagonalizable matrices having the same eigenvalues of the same algebraic multiplicities are similar.

【解】True.

因兩者都相似於同一個對角線矩陣.

０７D**06** 【 清大75資料[5](4) 】

Prove or disprove the following statements.

(4) If matrices A and B have the same characteristic polynomial then A and B are similar.

【解】(4) (disprove)

設 $A=\begin{bmatrix}0&0\\0&0\end{bmatrix}, B=\begin{bmatrix}0&1\\0&0\end{bmatrix}$,

則 $\det(A-\lambda I)=\lambda^2=\det(B-\lambda I)$

但因rank不同，所以不相似.

０７D**07** 【 大同82資工[5] 】

Let $A=\begin{pmatrix}0&2&1\\1&-4&0\\3&0&0\end{pmatrix}$ .which of the following matrix is similar to A?

(a) $\begin{pmatrix}3&3&3\\-6&-6&-2\\6&5&-1\end{pmatrix}$; (b) $\begin{pmatrix}0&4&2\\2&-8&0\\6&0&0\end{pmatrix}$; (c) $\begin{pmatrix}3&3&3\\6&5&-1\\-6&-6&-2\end{pmatrix}$, (d) $\begin{pmatrix}-6&-6&-2\\6&5&-1\\3&3&3\end{pmatrix}$.

【解】 選(a)

【說明】相似矩陣的trace必須相等.

tr(A)＝–4. 而(a),(b),(c),(d) 各小題的trace分別爲–4, –8, 6, 2

∴ 只有(a)小題的矩陣有機會和A相似。

【討論】假如有許多個矩陣的trace與A相等，可再利用det, rank來過濾.

０７D08 【 清大76資料[7](a) 】

Prove or disprove the following statements.

(a) $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ are similar. (5%)

【要訣】(1) A相似於B,則 rank$A$＝rank$B$ (綜線CH8定理16⑤)

(2) A相似於B $\Longleftrightarrow$ A與B的 Jordan form 相同. (綜線CH15定理14)

【解】 (a) disprove

[解法1]

$$\because \ \text{rank}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}=2 \text{ , rank}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}=3$$

∴ 不similar.

[解法2] 此二方陣已表成 Jordan form, 因 Jordan form 不同,

∴ 不相似.