題型08A: 核空間與值域

０８A**０１**【 交大86資料[4] 】

A linear transformation T of $\mathbb{R}^3$ into $\mathbb{R}^2$ has the value

$T(1, 1, 1)=(2, 2)$, $T(0, 1, 1)=(0, 1)$, $T(0, 0, 1)=(-1, 1)$

Find the null space of T.

【解】令 $(x, y, z)=\alpha(1,1,1)+\beta(0,1,1)+\gamma(0,0,1)$,

可解得 $\alpha=x$, $\beta=y-x$, $\gamma=z-y$.

$\therefore$ $T(x, y, z)=xT(1,1,1)+(y-x)T(0,1,1)+(z-y)T(0,0,1)$

$=x(2, 2)+(y-x)(0, 1)+(z-y)(-1, 1)=(2x+y-z, x+z)$

設 $T(x,y,z)=(0,0)$, 即 $\begin{cases} 2x+y-z=0 \\ x+z=0 \end{cases}$ (綜線CH8定義5)

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$

$\therefore$ null space of $T=\{(-t, 3t, t) \mid t\in\mathbb{R}\}$

０８A**０２**【 雲技84電資Y[1] 】

The linear mapping $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1+x_2 \\ x_2+x_3 \end{bmatrix}$$

Let S be the subspace of $\mathbb{R}^3$ spanned by $e_2=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

(a) Find the ker(L)(Kernel of L)? and the $L(S)$ (image of S)? (8%)

(b) Find the matrix representation of L with respect to the ordered bases $[u_1, u_2, u_3]$ and $[b_1, b_2]$ where

$$u_1=\begin{bmatrix}1\\1\\0\end{bmatrix},\ u_2=\begin{bmatrix}1\\0\\1\end{bmatrix},\ u_3=\begin{bmatrix}0\\1\\1\end{bmatrix}\ \text{and}\ \ b_1=\begin{bmatrix}1\\0\end{bmatrix},\ b_2=\begin{bmatrix}1\\1\end{bmatrix}\quad (7\%)$$

【解】(a)

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\in \mathrm{Ker}L \iff L(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix})=\begin{bmatrix}0\\0\end{bmatrix}$$ (綜線CH8定義5)

$$\iff x_1+x_2=0,\quad x_2+x_3=0$$

$$\therefore\ \mathrm{Ker}L=\left\{\begin{bmatrix}-t\\t\\-t\end{bmatrix}\middle|\ t\in\mathbb{R}\right\}$$

$$L(S)=\{L(te_2)\mid t\in\mathbb{R}\}$$ (綜線CH8定義1)

$$=\{tL(e_2)\mid t\in\mathbb{R}\}=\left\{t\begin{bmatrix}1\\1\end{bmatrix}\middle|\ t\in\mathbb{R}\right\}$$

(b) (請參閱題型07B)

$$L(u_1)=\begin{bmatrix}2\\1\end{bmatrix}=1\begin{bmatrix}1\\0\end{bmatrix}+1\begin{bmatrix}1\\1\end{bmatrix}$$

$$L(u_2)=\begin{bmatrix}1\\1\end{bmatrix}=0\begin{bmatrix}1\\0\end{bmatrix}+1\begin{bmatrix}1\\1\end{bmatrix}$$

$$L(u_1)=\begin{bmatrix}1\\2\end{bmatrix}=(-1)\begin{bmatrix}1\\0\end{bmatrix}+2\begin{bmatrix}1\\1\end{bmatrix}$$

∴ 矩陣表示為 $\begin{bmatrix}1 & 0 & -1\\1 & 1 & 2\end{bmatrix}$ (綜線CH7定義9)

０８A**03** 【 交大83資料[1] 】

Let M_2 be the real vector space of (2×2)–dimensional matrices and P_3 be the real vector space of polynomials of degrees less than 3. Let T be a linear transformation from M_2 to P_3 and be defined as

$$T\left(\begin{bmatrix}a & b\\c & d\end{bmatrix}\right)=(a+c)+(b+d)x+(c+d)x^2$$

Also, let

$$B=\left\{\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix};\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix};\begin{bmatrix}0 & 0\\1 & 1\end{bmatrix};\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right\}$$

and $C=\{1+x;\ 1+x^2;\ x+x^2\}$ be ordered bases for M_2 and P_3 respectively. Answer the following questions.

(a) What is the coordinate vector of $\begin{bmatrix}2 & 1\\3 & 4\end{bmatrix}$ with respect to B ? (2%)

(b) Find a basis for the image space of T. (2%)

(c) Find a basis for the null space of T. (3%)

(d) What is the matrix representation of T with respect to B and C ? (6%)

【分析】集合以表列式表示時, 各元素通常是以逗點隔開. 本題以分號隔開較罕見. 不過這可能是要強調"ordered". 作答時仍以考題的符號爲準.

【解】(a) (本小題屬於題型06D.)

令 $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} = a\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + d\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

比較各對應位置得 $\begin{cases} b+d=2 \\ a+d=1 \\ a+b+c+d=3 \\ b+c+d=4 \end{cases}$

解得 $a=-1, b=0, c=2, d=2$.

$\therefore$ $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ 對B的座標向量爲 $\begin{bmatrix} -1 \\ 0 \\ 2 \\ 2 \end{bmatrix}$. (綜線CH6定義28)

(b) 令

$D=\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$,

$E=\{ 1 ; x ; x^2 \}$.

依T的定義:

$T(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix})=1$, $T(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})=x$, $T(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix})=1+x^2$, $T(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix})=x+x^2$

$\therefore$ T對D,E的矩陣表示爲

$$X=\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&1\end{bmatrix}$$ (綜線CH7定義9)

對X做行運算(column operation):

$$\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&1\end{bmatrix}\overset{c}{\sim}....\sim\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$$

∴ X的column space為$\mathbb{R}^3$,以$\left\{\begin{bmatrix}1\\0\\0\end{bmatrix};\begin{bmatrix}0\\1\\0\end{bmatrix};\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$為基底.

∴ T的image spacce為P_3，以$\{1;x;x^2\}$ 為基底.

(c) 接(b), 對X做列運算(row operation):

$$\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&1\end{bmatrix}\overset{r}{\sim}\begin{bmatrix}1&0&0&-1\\0&1&0&1\\0&0&1&1\end{bmatrix}$$

∴ X的null space為 $\left\{\begin{bmatrix}t\\-t\\-t\\t\end{bmatrix}\middle|\, t\in\mathbb{R}\right\}$, 以 $\left\{\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}\right\}$為基底.

$\therefore$ T的null space 以 $\left\{\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\right\}$ 為基底.

(d) (本小題屬於題型07C.)

由D描寫B, 令 $P=\begin{bmatrix}0&1&0&1\\1&0&0&1\\1&1&1&1\\0&1&1&1\end{bmatrix}$

由E描寫C, 令$Q=\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}$

則T對B,C的矩陣表示為$Q^{-1}XP$ (綜線CH7定理24)

$$=\begin{bmatrix}1&1&0\\1&0&1\\0&1&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&0&1&1\end{bmatrix}\begin{bmatrix}0&1&0&1\\1&0&0&1\\1&1&1&1\\0&1&1&1\end{bmatrix}=\ldots=\begin{bmatrix}1/2&1/2&0&1\\1/2&3/2&1&1\\1/2&1/2&1&1\end{bmatrix}$$

(d) [另解] 將B中向量經T映射後以C展開 (綜線CH7定義9)

$T(\begin{bmatrix}0&1\\1&0\end{bmatrix})=1+x+x^2$, $T(\begin{bmatrix}1&0\\1&1\end{bmatrix})=2+x+2x^2$

$T(\begin{bmatrix}0&0\\1&1\end{bmatrix})=1+x+2x^2$, $T(\begin{bmatrix}1&1\\1&1\end{bmatrix})=2+2x+2x^2$

令 $\begin{cases} 1+x+x^2=a_{11}(1+x)+a_{21}(1+x^2)+a_{31}(x+x^2) \\ 2+x+2x^2=a_{12}(1+x)+a_{22}(1+x^2)+a_{32}(x+x^2) \\ 1+x+2x^2=a_{13}(1+x)+a_{23}(1+x^2)+a_{33}(x+x^2) \\ 2+2x+2x^2=a_{14}(1+x)+a_{24}(1+x^2)+a_{34}(x+x^2) \end{cases}$

經比較各次方係數，解得

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 & 1 \\ 1/2 & 3/2 & 1 & 1 \\ 1/2 & 1/2 & 1 & 1 \end{bmatrix}$$

即為所求.

０８A**04**【 交大81資料[2] 】

Let T be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(a) Is T onto?

(b) Find a basis for the image space.

(c) Find a basis for the null space.

【解】(a)

令$A=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$，經列運算化為 $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ rank$A=2$ (綜線CH6定理23)

$\therefore$ $\dim \text{Im}(T)=2<\dim\mathbb{R}^3$

$\therefore$ $\text{Im}(T)\neq\mathbb{R}^3$ (綜線CH6定理22a)

∴ T不是onto

(b) ImT＝column space of A

因列運算不改變各行的獨立性, (綜線CH6定理24)

而 $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ 的1, 2行線性獨立,

所以A的第1, 2行線性獨立. (綜線CH6定理24)

∴ 可取 $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ 爲image space的基底

(c) 由(a)中之列運算可知null space爲

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \begin{array}{l} x_1 = -t \\ x_2 = -t, \; t \in \mathbb{R} \\ x_3 = t \end{array} \right\} = \left\{ t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \middle| \; t \in \mathbb{R} \right\}$$

∴ 可取 $\left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right\}$ 爲null space的基底

０８A**05** 【 交大79工工[13] 】

Let L: $\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be defined by $L(x, y) = (x+y, x-y, x+2y)$

(a) Find a basis for range$L$.

(b) Is L onto ?

【參考章節】CH8定義5,範例6

【解】 (a) range$L = \{ (x+y, x-y, x+2y) \mid x, y \in \mathbb{R} \} = \{ x(1, 1, 1) + y(1, -1, 2) \mid x, y \in \mathbb{R} \}$

$\therefore$ { (1, 1, 1), (1, –1, 2) } 生成 range$L$.

而 (1, 1, 1), (1, –1, 2) 不成比例, 所以形成獨立集. (綜線CH6定理10要訣1)

$\therefore$ { (1, 1, 1), (1, –1, 2) } 爲 range$L$ 之基底.

(b) No.

$\dim \text{range}L = \#\{(1,1,1),(1,-1,2)\} = 2$

$\therefore$ range$L \neq \mathbb{R}^3$, 即L不爲onto.

０８A**06** 【 清大75資料[2] 】

Let $T: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be the linear transformation defined by

$T(x, y, z, w) = (x-2y+z-w,\ 3x-2z+3w,\ 5x-4y+w)$

(1) Find bases for the kernel and image of T.

(2) Find the value of a if $(1, 4, a) \in \text{Im}(T)$, the image of T.

【解】(1)(甲)

$$\text{Ker}T = \left\{ (x,y,z,w) \;\middle|\; \begin{array}{l} x-2y+z-w=0 \\ 3x-2z+3w=0 \\ 5x-4y+w=0 \end{array} \right\}$$ (綜線CH8定義5①)

由列運算:

$$\begin{bmatrix} 1 & -2 & 1 & -1 \\ 3 & 0 & -2 & 3 \\ 5 & -4 & 0 & 1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & -2/3 & 1 \\ 0 & 1 & -5/6 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \begin{cases} x = (2/3)t_1 - t_2 \\ y = (5/6)t_1 - t_2 \\ z = t_1 \\ w = t_2 \end{cases} \quad ; t_1, t_2 \in \mathbb{R}$$

$\therefore$ $\text{Ker}T = \{ ((2/3)t_1 - t_2, (5/6)t_1 - t_2, t_1, t_2) \mid t_1, t_2 \in \mathbb{R} \}$

$= \{ t_1(2/3, 5/6, 1, 0) + t_2(-1, -1, 0, 1) \mid t_1, t_2 \in \mathbb{R} \}$

{(2/3 , 5/6 , 1,0),(–1,–1,0,1) }生成 Ker$T$, 且線性獨立,

$\therefore$ It is a basis for Ker$T$. (綜線CH6定義16)

(乙) $T(x, y, z, w) = x(1,3,5) + y(-2,0,-4) + z(1,-2,0) + w(-1,3,1)$

$$\therefore\ \text{Im}T = \text{the row space of } \begin{bmatrix} 1 & 3 & 5 \\ -2 & 0 & -4 \\ 1 & -2 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$ (綜線CH5定義16②)

$$\begin{bmatrix} 1 & 3 & 5 \\ -2 & 0 & -4 \\ 1 & -2 & 0 \\ -1 & 3 & 1 \end{bmatrix} \underset{\cdots\cdots}{\overset{\text{列運算}}{\sim}} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore\ \text{Im}T = \text{the row space of } \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(2) [解法1]

由 (1),

$(1,4,a) \in \text{Im}T \Longleftrightarrow \exists\, t_1, t_2$, 使得 $(1,4,a) = t_1(1,0,2) + t_2(0,1,1)$

比較各分量, $1 = t_1$, $4 = t_2$, $a = 2t_1 + t_2$

$\therefore\ a = 6$

[解法2]

$$(1,4,a) \in \text{Im}T \Longleftrightarrow \text{聯立方程式} \begin{cases} x-2y+z-w=1 \\ 3x-2z+3w=4 \\ 5x-4y+w=a \end{cases} \text{有解}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 1 \\ 3 & 0 & -2 & 3 & 4 \\ 5 & -4 & 0 & 1 & a \end{array}\right] \overset{\text{列運算}}{\sim \cdots\cdots \sim} \left[\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 1 \\ 0 & 6 & -5 & 6 & 1 \\ 0 & 0 & 0 & 0 & a-6 \end{array}\right]$$

有解 $\Longleftrightarrow a-6=0$ (綜線CH3定理10)

$\therefore\ a=6$

【加強演練】

設線性映射 $T:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ 定義爲 $T(x,y,z)=(x+y+z,\ 2x+y-z,\ x-2z)$,
若 $(a,b,c)\in \mathrm{Im}T$, 求 a,b,c 之間的關係式.

Ans: 所求之關係式爲$a-b+c=0$.

０８A**07** 【 台大79資工[4](i) 】

(Yes or No question and explain the reason:)

(i) There is a linear transformation $T:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ such that T maps the unit circle ($x^2+y^2=1$) to the line $x=y$.

【解】令 $A=\{(x,y)\mid x^2+y^2=1\}=\{(\cos\theta,\sin\theta)\mid \theta\in\mathbb{R}\}$, $B=\{(x,y)\mid x=y\}$

對linear transformation $T:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ 而言:

$T(A)\subseteq B$是可能的, 例如:

設 $T(x,y)=(x+y,\ x+y)$, 則

$T(\cos\theta,\sin\theta)=(\cos\theta+\sin\theta,\ \cos\theta+\sin\theta)\in B$

$\therefore\ T(A)\subseteq B$

但$T(A)=B$是不可能的, 證明如下:

設$T(x,y)=(ax+by,\ cx+dy)$, $K=\max\{\,|a|,|b|,|c|,|d|\,\}$

則$T(\cos\theta,\sin\theta)=(a\cos\theta+b\sin\theta,\ c\cos\theta+d\sin\theta)$

$\therefore\ \|T(\cos\theta,\sin\theta)\|=\sqrt{(a\cos\theta+b\sin\theta)^2+(c\cos\theta+d\sin\theta)^2}$

$\le\sqrt{(2k)^2+(2k)^2}=2\sqrt{2}\,k$

$\because\ \|(3k,3k)\|=3\sqrt{2}\,k>2\sqrt{2}\,k$

$\therefore$ $(3k, 3k) \notin T(A)$,

但 $(3k, 3k) \in B$,

$\therefore$ $T(A) \neq B$

[註: 本題題意有點模糊, 但應該是指$T(A)=B$, 所以建議以NO作答]

題型08B: 秩的求算

０８Ｂ01 【 交大86資料[5] 】

Find the rank of the matrix $A=\begin{bmatrix} 5 & 4 & 6 & 7 \\ -2 & 1 & 2 & -1 \\ 0 & 0 & 1 & 3 \\ 3 & 5 & 9 & 9 \end{bmatrix}$

【解】對A做列運算 (爲避開分數運算，不使用標準的高斯消去法)

$$\begin{bmatrix} 5 & 4 & 6 & 7 \\ -2 & 1 & 2 & -1 \\ 0 & 0 & 1 & 3 \\ 3 & 5 & 9 & 9 \end{bmatrix}\begin{matrix} \\ (1) \\ \\ (-1) \end{matrix} \sim \begin{bmatrix} 0 & 0 & -1 & -3 \\ -2 & 1 & 2 & -1 \\ 0 & 0 & 1 & 3 \\ 3 & 5 & 9 & 9 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 6 & 11 & 8 \\ 0 & 0 & 1 & 3 \\ 3 & 5 & 9 & 9 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 6 & 11 & 8 \\ 3 & 5 & 9 & 9 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{matrix} (-3) \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 6 & 11 & 8 \\ 0 & -13 & -24 & -15 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ rank$A=3$. (綜線CH6定理23)

08B**02** 【 朝陽85工工[7] 】

Let $A=\begin{bmatrix}1 & 2 & 1 & 3\\ 2 & 1 & -4 & -5\\ 7 & 8 & -5 & -1\\ 10 & 14 & -2 & 8\end{bmatrix}$. Find the rank and nullity of A.

【解】 $\begin{bmatrix}1 & 2 & 1 & 3\\ 2 & 1 & -4 & -5\\ 7 & 8 & -5 & -1\\ 10 & 14 & -2 & 8\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 1 & 3\\ 0 & -3 & -6 & -11\\ 0 & -6 & -12 & -22\\ 0 & -6 & -12 & -22\end{bmatrix}\sim\begin{bmatrix}1 & 2 & 1 & 3\\ 0 & -3 & -6 & -11\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

$\therefore$ rank$A=2$ (綜線CH6定理23)

$\therefore$ nullity$A=4-$rank$A=2$ (綜線CH8定理8)

08B**03** 【 成大84資工[1] 】

Compute the ranks of the following matrices. (10%)

(1) $A=\begin{bmatrix}3 & 1 & 2 & 5\\ 1 & 2 & -1 & 2\\ 4 & 3 & 1 & 7\end{bmatrix}$ (2) $B=\begin{bmatrix}3 & 1 & 2 & 5\\ 1 & 2 & -1 & 2\\ 4 & 3 & 1 & 1\end{bmatrix}$

【解】

(1) $\begin{bmatrix}3 & 1 & 2 & 5\\ 1 & 2 & -1 & 2\\ 4 & 3 & 1 & 7\end{bmatrix}$ (–3) (–4) $\sim\begin{bmatrix}0 & -5 & 5 & -1\\ 1 & 2 & -1 & 2\\ 0 & -5 & 5 & -1\end{bmatrix}\sim\begin{bmatrix}1 & 2 & -1 & 2\\ 0 & -5 & 5 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$

$\therefore$ rank$A=2$

(2) $\begin{bmatrix} 3 & 1 & 2 & 5 \\ 1 & 2 & -1 & 2 \\ 4 & 3 & 1 & 1 \end{bmatrix}$ (–3) (–4) $\sim \begin{bmatrix} 0 & -5 & 5 & -1 \\ 1 & 2 & -1 & 2 \\ 0 & -5 & 5 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -5 & 5 & -1 \\ 0 & 0 & 0 & -6 \end{bmatrix}$

$\therefore$ rank$A = 3$

0 8 B **04** 【 交大79工工[14] 】

Find the rank of the matrix $\begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 0 & 1 & -3 & 2 & 3 \\ 2 & 3 & 1 & 4 & -1 \\ -1 & 2 & 2 & 2 & -5 \\ 3 & 1 & -1 & 2 & 4 \end{bmatrix}$

【解】 $\begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 0 & 1 & -3 & 2 & 3 \\ 2 & 3 & 1 & 4 & -1 \\ -1 & 2 & 2 & 2 & -5 \\ 3 & 1 & -1 & 2 & 4 \end{bmatrix}$ (–2)(1)(–3) $\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 0 & 1 & -3 & 2 & 3 \\ 0 & -1 & 3 & -2 & -3 \\ 0 & 4 & 1 & 5 & -4 \\ 0 & -5 & 2 & -7 & 1 \end{bmatrix}$ (1) (–4)(5)

$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 0 & 1 & -3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 13 & -3 & -16 \\ 0 & 0 & -13 & 3 & 16 \end{bmatrix} \sim\ldots\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 1 \\ 0 & 1 & -3 & 2 & 3 \\ 0 & 0 & 13 & -3 & -16 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ rank爲3 (綜線CH6定理23)

題型08C: 秩的性質

０８Ｃ**01** 【 淡江80資工[1] 】

證明對任意矩陣, 列秩＝行秩.

【解】請參閱綜線CH8定理13, 此處不再重覆.

０８Ｃ**02** 【 清大80工工[2] 】

Given $A=\begin{bmatrix} 1 & 2 & -6 \\ -3 & 4 & 7 \\ 2 & 4 & 3 \end{bmatrix}$ to show that for a nonsingular matrix,

rank＝dimension of row space＝dimension of column space.

【分析】rank＝dimension of row space＝dimension of column , (綜線CH8定理13)

這對一切矩陣(可逆或不可逆, 方陣或非方陣)都成立.

但本題只是針對所給定的A驗證這個性質.

【解】 對A執行列運算:

$$\begin{bmatrix} 1 & 2 & -6 \\ -3 & 4 & 7 \\ 2 & 4 & 3 \end{bmatrix} \begin{matrix} (3)\ (-2) \\ \leftarrow \\ \leftarrow \end{matrix} \sim \begin{bmatrix} 1 & 2 & -6 \\ 0 & 10 & -11 \\ 0 & 0 & 15 \end{bmatrix}$$

∴ [1, 2, –6], [0, 10, –11], [0, 0, 15]

形成A的row space的basis (綜線CH6定理23)

∴ " dimension of row space of A " 爲3 (綜線CH6定理19)

對A執行行運算:

$$\begin{bmatrix} 1 & 2 & -6 \\ -3 & 4 & 7 \\ 2 & 4 & 3 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 10 & -11 \\ 2 & 0 & 15 \end{bmatrix} \overset{c}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 10 & 0 \\ 2 & 0 & 15 \end{bmatrix}$$

(−2) → column 2, (6) → column 3; (11/10) → column 3

$\therefore \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 15 \end{bmatrix}$ 形成A的column space的basis.

$\therefore$ " dimension of column space of A " 為3

此二dimension之值相等.

０８Ｃ03 【 中正85資工[3] 】

Let A be an $m \times n$ matrix. Show that if P is an invertible $m \times m$ matrix, then rank(PA)＝rank(P).

【勘誤】本題應是 rank(PA)＝rank(A). 此為明顯之筆誤，考生仍應詳加作答.

【解】rank(PA)≤rank(A)

rank(A)＝rank($P^{-1}PA$)≤rank(PA)

$\therefore$ rank(PA)＝rank(A)

０８Ｃ04 【 中央85資工[4] 】

Give two algorithms to find invertible matrix (you shall not use determinant).

【分析】本題要求提出兩種判定矩陣是否可逆的演算法，但禁用行列式.

以下列出幾種"演算法"供讀者選答. (綜線CH8定理17)

(1) 對$n \times n$矩陣A， A可逆 $\Longleftrightarrow$ rank$A$＝n

可經列運算化到梯形，這時的非零列列數就是 rank$A$.

(2) 對$n \times n$矩陣A，可解方程式Ax＝o.

A可逆 $\Longleftrightarrow$ Ax＝o無非零解.

經列運算將A化到梯形,

“$n>$非零列數”就是有非零解, “$n=$非零列數”就是無非零解.

(3) 對$n\times n$矩陣A, 可解方程式$xA=o$.

解此方程式須對A^{T}作列運算或對A作行運算.

A可逆 $\Longleftrightarrow$ $xA=o$ 無非零解.

(4) 對的$n\times n$矩陣A, 可試求 Col$A$ 的基底.

A可逆 $\Longleftrightarrow$ Col$A$的基底內含n個向量.

本題題意有些模糊, 並未明示(1)與(2)算同一種還是兩種. 爲安全起見, 最好不要同時選寫(1)(2).

【解】不方的矩陣一定不可逆.

對方陣A, ...(讀者依上面所列各型自行選答)

０８C**05** 【 中原85工工[1] 】

Let A be an $n\times n$ matrix. We have known that A is invertible is equivalent to that $\det(A)\neq 0$. Please write down another five statements that are equivalent to that A is invertible.

【解】方陣可逆的等價條件很多, 請參閱綜線CH8定理17.

０８C**06** 【 交大84資料[1] 】

Suppose that A is a $p\times p$ matrix of rank 1. Show that $A^2=tA$ for some number t.

【分析】請參閱綜線CH8定理16b.

【解】∵ rank$A=1$,

∴ A可經列運算化爲矩陣 $\begin{bmatrix} Y \\ O_{(p-1)\times p} \end{bmatrix}$,其中$Y$爲1×p矩陣 (綜線CH6定理3).

∴ 存在p×p矩陣P使得 $A=P\begin{bmatrix} Y \\ O_{(p-1)\times p} \end{bmatrix}$ (綜線CH3定理17)

令 $P=\begin{bmatrix} X & Z \end{bmatrix}$，其中$X$為$p\times 1$矩陣，$Z$為$p\times(p-1)$矩陣.

則 $A=\begin{bmatrix} X & Z \end{bmatrix}\begin{bmatrix} Y \\ O_{(p-1)\times p} \end{bmatrix}=XY+ZO=XY$ (綜線CH3定理8)

因YX為1×1矩陣，可視為數. 令$t=YX$，則

$A^2=(XY)(XY)=X(YX)Y=X[t]Y=t(XY)=tA$

【加強演練】

以 $A=\begin{bmatrix} 2 & 3 \\ 8 & 12 \end{bmatrix}$ 為例追蹤本題的證明過程.

[解]

$$A=\begin{bmatrix} 2 & 3 \\ 8 & 12 \end{bmatrix}\xrightarrow{(-4)}\sim\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}$$

$$\therefore\ A=\begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}=\begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \end{bmatrix}$$ (綜線CH2定理8)

$$\therefore\ A^2=\begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \end{bmatrix}$$

$$=\begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} 14 \end{bmatrix}\begin{bmatrix} 2 & 3 \end{bmatrix}=14\begin{bmatrix} 1 \\ 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \end{bmatrix}=14A$$

題型08D: 秩的應用I;

０８D**01** 【 中央83資工[6] 】

Let A and B be two $n\times n$ square matrices.

(a) Show that rank(AB) is less than or equal to the minimum of rank(A) and rank(B). (8%)

(b) Use the result of (a) to show that if AB is invertible then both A and B are invertible. (7%)

【分析】本題(b)部份若去掉"方陣"的條件, 就不成立.

【解】(a) 以下用CSP(X)代表矩陣X的column space, 用RSP(X)代表X的row space.

$\because$ CSP(AB)是CSP(A)的子空間. (綜線CH5定理21a)

$\therefore$ rank(AB)＝dim CSP(AB) (綜線CH8定義13)

$\leq$dim CSP(A)＝rank(A) (綜線CH6定理22a)

$\because$ RSP(AB)是RSP(B)的子空間. (綜線CH5定理21a)

$\therefore$ rank(AB)＝dim RSP(AB)

$\leq$dim RSP(B)＝rank(B) (綜線CH6定理22a)

$\therefore$ rank(AB)$\leq$min{rank$A$, rank$B$}

(2) 若AB可逆,

則 rank(AB)＝n (綜線CH8定理17)

$\therefore$ rank$A\geq$rank(AB)＝n

$\therefore$ rank$B\geq$rank(AB)＝n

而 rank(A), rank(B)都不超過n (綜線CH8定理15)

$\therefore$ rank$A$＝n, rank$B$＝n .

$\therefore$ A, B都可逆. (綜線CH8定理17)

０８D**02** 【 大同82資工[二2] 】

If A is an $m\times n$ matrix, and B is an $n\times m$ matrix with $n<m$. Prove that AB is not invertible.

【解】 rank(AB) $\leq$ rank(A) (綜線CH8定理16)

$\leq$ n (綜線CH8定理15)

$<$ m

但AB是$m\times m$矩陣, ∴ AB不可逆。 (綜線CH8定理17)

08D03【 交大79資料[6] 】

Let A be an $m\times n$ matrix and B be an $n\times m$ matrix where $m\geq n$. Prove that $\text{DET}(A\times B)=0$

【勘誤】本題須將$m\geq n$修正為$m>n$

【解】 $\text{rank}(A\times B)\leq \text{rank}(A)$ (綜線CH8定理16①)

$\leq n$ (綜線CH8定理15②)

$<m$ (已知條件)

又$A\times B$為$m\times m$矩陣, (綜線CH2定義4)

∴$A\times B$不可逆, (綜線CH8定理17)

∴$\text{DET}(A\times B)=0$ (綜線CH4定理17)

【討論】本題未修正時不成立. 反例如下:

取 $m=n=2, A=B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, $\text{DET}(A\times B)=\text{DET}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\neq 0$

08D04【 清大78資料[4] 】

Let A be a 4 by 3 matrix and B a 3 by 4 matrix. Prove that $\det(AB)=0$.

【解】同上題.

08D05【 大同80資工[3](h) 】

True or False:

(h) If a square matrix A has independent columns, so does A^2 .

【解】(h) True, 證明如下:

A為具有獨立行的方陣

∴ A可逆 (綜線CH8定理17)

∴ A^2 可逆 (綜線CH2定理12)

∴ A^2 具有獨立行 (綜線CH8定理17)

０８D**06** 【 台大77資工[5](iv)& 】

(True or false, with counterexample if false:)

(ⅳ) If a square matrix A has independent columns, so does A^2.

【解】true. 證明同上題.

０８D**07** 【 大同80資工[3](b)& 】

True or False:

(b) If matrix A is invertible and its rows are in reverse order in matrix B, then B is invertible.

【解】(b) True, 證明如下:

設A為$n\times n$矩陣

$\because$ A可逆 $\therefore$ A的n個列形成獨立集 (綜線CH8定理17)

$\therefore$ B的n個列也形成獨立集

$\therefore$ B可逆 (綜線CH8定理17)

０８D**08** 【 交大81資工[2](c) 】

Decide the correctness of the following inductions. You should concisely explain the reasons for the correct ones and the incorrect ones.

(c) (3%) Suppose that the dimension of matrix A is $p\times q$ with $p<q$, we can conclude that matrix A cannot possibly have a left–inverse, but must have a right–inverse.

【參考章節】(c) 綜合線性代數定理16a

【解】(c) Incorrect. 反例如下:

令 $A=\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}$, 則$A$滿足$p<q$的條件, 但$A$並無右反:

$$\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}a&d\\b&e\\c&f\end{bmatrix}=\begin{bmatrix}a&d\\0&0\end{bmatrix}\neq I$$

０８D**09** 【 成大85資工[2] 】

Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{R})$, $n > m \geq 1$.

(ⅰ) Show that A need not have a right inverse. (6%)

(ⅱ) Suppose that A does have a right inverse. Show that A has more than one right inverse. (10%)

【解】(ⅰ) 例如對 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, A可能的right inverse爲3×2矩陣.

對任意 $B = \begin{bmatrix} p & s \\ q & t \\ r & u \end{bmatrix}$, $AB = \begin{bmatrix} p & s \\ 0 & 0 \end{bmatrix} \neq I$

(ⅱ) 已知 $\exists B$使得$AB = I$

∵ $\mathrm{rank}A \leq m < n$

∴ $\dim \ker A = n - m > 0$

∴ $\exists v \neq o$ 使得 $Av = o$.

令 $C = [\, v \quad v \quad \ldots \quad v \,]$, 由$m$個$v$排成的$n\times m$矩陣,

則 $AC = O$ (綜線CH2定理6)

∴ $A(B+C) = AB + O = I$

∴ $B+C$也是A的right inverse.

【討論】對任意常數k, $B + kC$都是A的right inverse. 也就是說:

$m < n$的$m\times n$矩陣一旦有右反, 就會有無限多個右反.

但$n\times n$矩陣最多只能有一個右反. (綜線CH8定理17,CH2定理11)

題型08E: 映射定理

０８Ｅ**01** 【 交大79工工[17] 】

If $L: V \longrightarrow W$ is a linear transformation, then $\dim(\ker L)+\dim(\mathrm{range}L)=\dim V$.

【解】請參閱綜線CH8定理8.

０８Ｅ**02** 【 元智83工工[10] 】

[是非題]

For any m by n matrix A, the rank of the null space of A plus the rank of the column space of A is n.

【解】○，此為定理. (綜線CH8定理8)

０８Ｅ**03** 【 交大84資工[2] 】

Let T be a linear transformation from V to W.

(a) Given that $\dim(V)=\dim(\text{range of } T)=2$, find dim(null space of T).

(b) Given that $\dim(\text{null space of } T)=\dim(\text{range of } T)=2$, find $\dim(V)$.

(c) Given that $\dim(V)=7$, $\dim(W)=5$, and $\mathrm{rank}(T)=5$, find nullity(T).

(d) Given that $\dim(V)=5$, $\dim(W)=1$, and $T\neq O$, find rank(T) and nullity(T).

【解】(a) $\dim(\text{null space of } T)$

$=\dim(V)-\dim(\text{range of } T)$ (綜線CH8定理8)

$=2-2=0$

(b) $\dim(V)=\dim(\text{null space of } T)+\dim(\text{range of } T)=2+2=4$.

(c) $\mathrm{nullity}(T)=\dim(V)-\mathrm{rank}(T)=7-5=2$. (綜線CH8定理8)

(d) 由$T\neq O$得知range of T 不是零空間,

$\therefore \dim(\text{range of } T)\geq 1$.

而 $\dim(\text{range of } T) \leq \dim W = 1$,

$\therefore \dim(\text{range of } T) = 1$. 即 $\mathrm{rank}T=1$.

$\mathrm{nullity}T=\dim V-\mathrm{rank}T=5-1=4$. (綜線CH8定理8)

０８Ｅ**04** 【 台大85資工[2] 】

[複選題]

Let V and W be both finite-dimensional vector spaces (over the same field). Which of the following are true.

(1) V is isomorphic to W if and only if $\dim(V)=\dim(W)$.

(2) If $\dim(V)=n$ and $\dim(W)=m$, then the vector space $L(V,W)$, the class of linear transformations from V to W, is finite-dimensional with dimension nm.

(3) Let $T: V\longrightarrow W$ be linear. If $\dim(V)<\dim(W)$, then T can not be one-to-one.

(4) Let $T: V\longrightarrow W$ be linear. Then, $\text{nullity}(T)+\text{rank}(T)=\dim(V)$.

【解】選(1)(2)(4)

【討論】

(1) True. 此爲定理. (綜線CH8定理28b)

(2) True. 此爲定理. (綜線CH8定理25)

(3) False. 舉反例如下:

令$V=\mathbb{R}^2$, $W=\mathbb{R}^3$, $T(x, y)=(x, y, 0)$. 則$\dim(V)<\dim(W)$, 但T爲一對一.

本題若改爲 $\dim V>\dim W$ 則可成立. (綜線CH8定理10a)

(4) True. 此爲重要定理. (綜線CH8定理8)

０８Ｅ**05** 【 中央86資工[1](e) 】

[是非論證題]

(e) If a linear transformation $T: \mathbb{R}^n\longrightarrow\mathbb{R}^m$ is one-to-one, then $n=m$.

【解】(e) False. 例如

$T: \mathbb{R}^2\longrightarrow\mathbb{R}^3$, $T(x, y)=(x, y, 0)$爲one-to-one, 但$n\neq m$. (參閱綜線CH8定理10a)

０８Ｅ**06** 【 中正84資工[1](e) 】

True or False:

(e) Let $T: V\longrightarrow W$ be a linear mapping and $\dim(V)<\infty$, $\dim(W)<\infty$.

If $T: V\longrightarrow W$ is one-to-one, then $\dim(V)=\dim(W)$.

【解】(e) False. (同上題)

０８E**07** 【 清大81工工[7.3] 】

Let $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation with standard matrix representation A.《複選》

(a) The dimension of the range of T is equal to the row rank of A.

(b) If $m > n$, then $\ker(T) \neq \mathbb{R}^n$.

(c) If $m < n$, then $\ker(T) \neq \{ o \}$.

(d) $\dim(\text{Ker}(T)) + \dim(\text{range}(T)) = m$.

(e) None of the above.

【解】選(a)(c). 分別解說如下:

(a) rank定理指出 column rank 與 row rank 相等， (綜線CH8定理13)

而 range(T)＝column space of A (綜線CH8習題5.1)

∴ dim(range(T))＝column rank of A

＝row rank of A (綜線CH8定義12)

(b) 即使 $m > n$，仍可令 T 爲零映射而使 $\ker(T) = \mathbb{R}^n$

(c) 假設 $\ker(T) = \{ o \}$,

則 $\dim \mathbb{R}^n = \dim\ker(T) + \dim \text{range}(T)$ (綜線CH8定理8)

$= \dim \text{range}(T)$

$\leq \dim \mathbb{R}^m$ (綜線CH6定理22a)

於是導致$n \leq m$, 此與已知條件矛盾.

(d) 題目中等式的右邊應該是定義域的dimension, 也就是n.

０８E**08** 【 清大81資料[17] 】

Let the null space of a matrix $A \in \mathbb{R}^{3\times 4}$ have dimension 1. What is the rank of matrix A ?

(a) 1 (b) 2 (c) 3 (d) 4 (e) 0

【解】選(c).

rank$A$＝4–1＝3 (綜線CH8定理8)

０８E**09** 【 台大83資工[3] 】

[複選題]

Let V and W be finite-dimension vector spaces over the field F, and T：$V \longrightarrow W$ is a linear map. T is one-to-one if :

(1) null space $N(T)=\{o\}$,

(2) T is a projection map ,

(3) T maps every linearly independent subset of V into a linearly independent subset of W ,

(4) $R(T)+N(T)=V$,

(5) $\dim(V)=\dim(W)$ and T is onto.

【解】 選(1)(3)(5).

【討論】(1) 此為定理. (綜線CH8定理7)

(2) 反例如下:

設T：$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$，定義如$T(x,y)=(x,0)$.

則T為projection，但並非一對一.

(3) 此為定理. (綜線CH8定理33)

(4) 反例同(2).

$R(T)=x$軸, $N(T)=y$軸. $R(T)+N(T)=\mathbb{R}^2$，但T並非一對一.

(5) T onto 即 $R(T)=W$,

$\therefore$ $\dim N(T)=\dim V-\dim R(T)$ (綜線CH8定理8)

$=\dim V-\dim W=0$

$\therefore$ $N(T)=\{o\}$.

$\therefore$ T為一對一. (綜線CH8定理7)

０８Ｅ１０ 【 清大77資料[6](a) 】

(Prove or disprove)

Let V and W be two finite dimensional vector spaces and T be a near transformation from V to W. If $\dim(V)>\dim(W)$ then there is a nonzero vector v_0 in V such that $Tv_0=o$.

【解】(a) [prove]

由$\dim V=\dim \mathrm{Ker}T+\dim \mathrm{Im}T$, (綜線CH8定理8)

及 $\dim V>\dim W$ (已知條件)

$\geq \dim \mathrm{Im}T$ (綜線CH6定理22(3))

可得知 $\dim \mathrm{Ker}T>0$

$\therefore$ Ker$T \neq \{o\}$ (綜線CH6定理19要訣2)

取$v_0 \in$ Ker$T \setminus \{o\}$

則v_0 是V中滿足 $Tv_0 = o$ 的非零向量 (綜線CH8定義5①)

【加強演練】

Let V and W be two finite dimensional vector spaces and T be a linear transformation from V to W. If dim$V <$ dim$W$, then there is a vector w_0 in W such that for each v in V, $Tv \neq w_0$

[解] 由dim$V =$ dim Ker$T +$ dim ImT

可知 dim Im$T \leq$ dim$V$

再由 dim$V <$ dim$W$

可知dim Im$T <$ dim$W$

$\therefore$ Im$T \neq W$

取 $w_0 \in W \setminus$ ImT 即可

08E**11** 【 交大78資工[2] 】

Let V and W be the two vector spaces over $\mathbb{R}$ of which the dimensions are n and m, respectively. Let L be a linear mapping from V to W. Can L have an inverse if

(a) $n < m$? (b) $n > m$? (c) $n = m$?

Explain your answers.

【要訣】函數的"可逆"通常是指一對一且映成. (綜線CH8定義27及定理28要訣(1))

【解】(a) $n < m$時, L不可能有inverse

(pf) 若L有inverse , 則L為onto $\therefore$ Im$L = W$ (映成的定義)

dim$V =$ dimKer$L +$ dimIm$L \geq$ dimIm$L =$ dim$W$ (綜線CH8定理8)

即 $n \geq m$, 與已知條件矛盾

(b) $n > m$時, L不可能有inverse

(pf) 若L有inverse, 則L為one-to-one , $\therefore$ Ker$L = \{o\}$,

$\therefore$ dimKer$L = 0$ (綜線CH8定理7)

$\therefore$ dim$V =$ dim Ker$L +$ dim Im$L =$ dim Im$L \leq$ dim$W$

即 $n \leq m$, 與已知條件矛盾

(c) 由(a), (b) 可知; L有inverse必須在$n = m$的情形下才有可能.

$n=m$時, L有inverse是可能的 (例如$L:\mathbb{R}^2\longrightarrow\mathbb{R}^2$,
$L(x, y)=(-x, y)$, 則L有inverse)
但單單$n=m$並不能推論L有inverse(例如若$L:\mathbb{R}^2\longrightarrow\mathbb{R}^2$,
$L(x, y)=(0, 0)$, 則L無inverse)

【加強演練】

(a) 接上題, 在$n<m$時, 試舉一對一但不映成的例子.

(b) 接上題, 在$n>m$時, 試舉映成但不一對一的例子.

[解] (a) 令$L:\mathbb{R}^2\longrightarrow\mathbb{R}^3$, $L(x, y)=(x, y, 0)$ 即可.

(b) 令$L:\mathbb{R}^3\longrightarrow\mathbb{R}^2$, $L(x, y, z)=(x, y)$ 即可.

０８E12 【 交大79工工[5] 】

是非題:

Let L : $\mathbb{R}^n\longrightarrow\mathbb{R}^n$ be a linear transformation defined by $L(X)=AX$, for X in $\mathbb{R}^n$. Then A is singular if and only if $\ker L=\{o_V\}$.

【解】非

【討論】本題剛好講反了, 應爲 A is non-singular if and only if $\ker L=\{o_V\}$.

證明如下:

A non-singular(可逆) (綜線CH2定義10)

$\Longleftrightarrow$ L 可逆 (綜線CH8定理29, CH7定義9要訣3)

$\Longleftrightarrow$ L 爲一對一 (綜線CH8定理11)

$\Longleftrightarrow$ $\ker L=\{o_V\}$ (綜線CH8定理7)

反例隨便取都可以.

０８E13 【 交大79工工[10] 】

是非題:

Let $L:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ be the linear transformation defined by $L(X)=AX$, for X in $\mathbb{R}^n$.
Then L is onto if and only if A is nonsingular.

【解】是

【討論】證明如下:

A non-singular(可逆) (綜線CH2定義10)

$\Longleftrightarrow$ L 可逆 (綜線CH8定理29, CH7定義9要訣3)

$\Longleftrightarrow$ L 爲映成 (綜線CH8定理11)

０８Ｅ**１４**【 交大82資工[3](b,c) 】

Determine true or false for the following statements. For each statements, you can obtain 2 points for a correct answer, –1 point for a wrong answer, and 0 point for a blank answer.

(b) Let $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation then the dimension of the kernel of T and the dimension of the range of T sum to n.

(c) Let $A\in\mathbb{R}^{m\times n}$. If the matrix A has a right inverse, then the column vectors of A must be linearly independent.

【解】(b) True. 此爲定理。 (綜線CH8定理8)

(c) False. 解說如下:

$$\text{取 } A=\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\text{, 則 } A\text{有right inverse }\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}.$$

但A的三個columns卻線性相關。

本題若將right inverse改爲left inverse則爲true:

行獨立 $\Longleftrightarrow$ $\ker A=\{o\}$ (綜線CH6定理15)

$\Longleftrightarrow$ $\operatorname{rank}A=n$ (綜線CH8定理8)

$\Longleftrightarrow$ 有左反 (綜線CH8習題17.2)

０８Ｅ**１５**【 大同82資工[4] 】

Let V be an n-dimensional ($n>0$) vector space over the field F and let T be a linear operator on V. Let R_T and N_T denote the range and the null space of T, respectively. Suppose that $R_T=N_T$ Then which of the following statements is wrong ?

(a) $T^2=O$; (b) n is even;

(c) T is singular; (d) $\dim(R_T)=\dim(N_T)=n$.

【解】 選 (d)

【說明】(a) $\forall\, v \in V$,

$T(v) \in R_T = N_T$

$\therefore\ T(T(v)) = o$

此即 $T \circ T = O$

(b) $n = \dim V = \dim R_T + \dim N_T$ (綜線CH8定理8)

$= 2\dim R_T$， 是個偶數

(c)由(b)得知 $\dim N_T = n/2 > 0$ $\therefore\ N_T \neq \{ o \}$

$\therefore$ T不可逆 (綜線CH8定理17)

(d)由(b)可知此敘述錯誤。

０８Ｅ**16** 【 清大81工工[5] 】

Let A be an $m \times m$ matrix.

(a) Show that if rank$A = m$, then $Ax = o$ has a unique solution. (5%)

(b) What is the unique solution? Explain it in terms of (a). (5%)

(c) Show that if rank$A < m$, then $Ax = o$ has infinite number of solutions. (5%)

【解】(a) 考慮 $T : \mathbb{R}^{m\times 1} \longrightarrow \mathbb{R}^{m\times 1}$, $T(x) = Ax$,

$\dim(\ker A) = m - \text{rank}\, A = m - m = 0$ (綜線CH8定理8)

$\therefore\ \ker A = \{ o \}$ (綜線CH6定理19要訣2)

$\therefore\ Ax = o$ 具有唯一解。 (綜線CH5定義19)

(b) $\because\ \ker A = \{x \mid Ax = o \}$ (綜線CH5定義19)

由 (a) 得 $\{x \mid Ax = o \} = \{ o \}$

$\therefore\ Ax = o$ 的唯一解為 o .

(c) 當 rank $A < m$ 時 ,

$\dim \ker A = m - \text{rank}\, A$ (綜線CH8定理8)

> 0

即 $\dim \{x \mid Ax = o \} > 0$ (綜線CH6定理19要訣2)

$\therefore\ \{x \mid Ax = o \}$ 為無限集合,

即 $Ax = o$ 有無限多解. (綜線CH3定理11)

０８Ｅ**１７**【 大同80資工[3](e) 】

True or False:

(e) If the columns of matrix A are linearly independent, then $Ax=b$ has exactly one solution for every b.

【解】(e) False (應該是 "最多可有一個解") 反例如下:

令$A=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$, 則$A$的行線性獨立.

但 $\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$ 無解 (綜線CH3定理10)

０８Ｅ**１８**【 中央86資工[1](c) 】

[是非論證題]

(c) The linear transformations of n linear-dependent vectors are linearly dependent.

【解】(c) True.

設$v_1, v_2, ..., v_k$ 線性相關,

則存在不全爲零的純量c_i, 使 $c_1v_1+...+c_kv_k=o$ (綜線CH6定義9)

∴ $T(c_1v_1+c_2v_2+...+c_kv_k)=o$

∴ $c_1Tv_1+c_2Tv_2+...+c_kTv_k=o$

∴ $Tv_1, Tv_2, ..., Tv_k$ 線性相關. (綜線CH6定義9)

０８Ｅ**１９**【 台大86資工[1](bc) 】

[是非題]

(b) 設$W=\{v_1, v_2 ,..., v_k\}\subseteq\mathbb{R}_n$, $A\in\mathbb{R}_{n\times n}$,

若W線性獨立, 則 $\{Av_1, Av_2 ,...,Av_k\}$ 線性獨立.

(c) 若 $\{Av_1, Av_2, \ldots, Av_k\}$ 線性獨立, 則W線性獨立.

【解】(b) False. (綜線CH8定理11b)

(c) True. (綜線CH8定理11a)

０８E**20** 【 台大80資工[3] 】

[True or False Problem]

Let V and W be finite-dimensional vector spaces over F, and T: $V \longrightarrow W$ is a function.

Then T is one-to-one if T maps linearly independent subset of V into a linearly independent subset of W.

【解】 False, 反例如下:

設$V = W = \mathbb{R}$, 爲1-dimensional vector space over $\mathbb{R}$.

T: $V \longrightarrow W$ 定義爲 $T(x) = x^2$

此時V的linearly independent subset必爲$\varnothing$或$\{r\}$, $r \neq 0$

(綜線CH6定理18, 定義8要訣5)

$T(\varnothing) = \varnothing$, 仍爲獨立集 (綜線CH6定義8要訣4)

$T\{r\} = \{r^2\}$, 仍爲獨立集 (綜線CH6定義8要訣5)

但T並非one-to-one. (例如$T(1) = T(-1)$)

【討論】本題若將“T is a function”改爲“T is a linear transformation” 則答案爲True.

詳情請參閱綜合線性代數CH8定理11b.

題型08F:　映射空間

０８F**01**【 清大81工工[8] 】

Let　$T: V \longrightarrow W$ be a linear transformation.　Prove that if T exists, then T^{-1} is also a linear transformation.

【解】對任意向量u, v 及任意純量h, k,

欲證 $T^{-1}(hu+kv)=hT^{-1}(u)+kT^{-1}(v)$: (綜線CH7定義1)

令 $x=T^{-1}(u)$, $y=T^{-1}(v)$,

則 $T(x)=u, T(y)=v$,

∵ $T(hx+ky)=hT(x)+kT(y)$, (綜線CH7定義1)

∴ $T(hx+ky)=hu+kv$,

兩邊套用 T^{-1} 得 $hx+ky=T^{-1}(hu+kv)$

即 $hT^{-1}(u)+kT^{-1}(v)=T^{-1}(hu+kv)$

０８F**02**【 大同82資工[3] 】

Let V be an n–dimensional vector space over the field F, and let W be an m–dimensional vector space over F.　Let $L(V,W)$ denote the space of linear transformations from V into W.　Then $L(V,W)$ is finite–dimensional and has dimension:

(a) $\max(m,n)$;　(b) $\min(m,n)$;　(c) $m+n$;　(d) mn.

【解】 選 d (綜線CH8定理25)

【說明】取好基底後, V可看成F^n , W可看成F^m , $L(V, W)$ 可看成$F^{m\times n}$.

０８F**03**【 清大77資科[9] 】

Let V be a finite dimentional vector space over a field F.

Let $f: V \longrightarrow F$ be a non–zero linear transformation.

(1). Find the dimention of Ker(f), the kernel of f.

(2). If $g: V \longrightarrow F$ is another linear transformation with ker(g)＝ker(f), then there is a $\lambda \in F$ such that $g=\lambda f$.

【要訣】F自己是以F爲 scalar field 的一維向量空間. (綜線CH5習題4.1)

【解】 (1) ∵ $\text{Im}f \subseteq F$,

∴ $\dim \text{Im}f \leq \dim F = 1$.

∴ $\dim \text{Im}f = 0$或1.

而 f is a non–zero linear transformation

∴ $\text{Im}f \neq \{o\}$.

∴ $\dim \text{Im}f = 1$.

∴ $\dim \text{Ker}f = \dim V - \dim \text{Im}f = \dim V - 1$ (綜線CH8定理8)

(2) Let $n = \dim V$ and let $W = \text{Ker}g = \text{Ker}f$,

Since $\dim W = \dim V - 1 = n - 1$,

we can find a basis $\{v_1, v_2, \ldots, v_n\}$ of V such that $\{v_1, \ldots, v_{n-1}\}$ is a basis of W.

(註１)

∴ $f(v_n) \neq 0$ (∵ $v_n \notin \text{Ker}f$)

Let $\lambda = g(v_n) / f(v_n)$

then
$$\begin{cases} g(v_1) = 0 = \lambda f(v_1) , \\ g(v_2) = 0 = \lambda f(v_2) , \\ g(v_{n-1}) = 0 = \lambda f(v_{n-1}) , \\ g(v_n) = \lambda f(v_n) . \end{cases}$$

therefore $g = \lambda f$. (綜線CH7定理3要訣２)

[註１] 先取 $\{v_1 ,\ldots, v_{n-1}\}$ 爲W的基底,

∵ $\{v_1, \ldots, v_{n-1}\}$ 爲有限維空間V內的獨立集,

∴ 存在 $Y \supseteq \{v_1, \ldots, v_{n-1}\}$, 使$Y$爲$V$的基底. (綜線CH6定理21)

０８F**04** 【 台大85資工[3] 】

[複選題]

Assume that all the vector spaces are finite–dimensional. Which of the following are true.

(1) If V is isomorphic to W, then V^* (dual space of V) is isomorphic to W^* (dual space of W).

(2) Every linear transformation is a linear functional.

(3) Let V be a vector space, and $x \in V$. If $\hat{x}(f) := f(x) = 0$ for all $f \in V^*$ (dual space of V), then $x = 0$.

(4) If β is a linearly independent subset of V, then $T(\beta)$ is a linearly independent subset of W.

【解】選(1)(3)

【討論】(1) True.

V isomorphic to W

$\Longrightarrow$ $\dim V = \dim W$ (綜線CH8定理28b)

$\Longrightarrow$ $\dim(V^*) = \dim(W^*)$ (綜線CH8定理25, CH7定義1⑤)

$\Longrightarrow$ V^* isomorphic to W^* (綜線CH8定理28b)

(2) False.

講反了. 應該是 "Every linear functional is a linear transformation"

(綜線CH7定義1)

(3) True.

此爲定理. (綜線附錄C定理13②)

(4) False.

反例如下:

設T：$\mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $T(x,y)=(x,0)$.

則T將獨立集 { (1,1), (–1,1) } 映至 { (1, 0) , (–1, 0) }. 不再獨立.

(請參閱綜線CH8定理11a,11b)

０８F**05** 【 大同82資工[6] 】

Let $\mathscr{B}=\{(2,1),(3,1)\}$ be an ordered basis of $\mathbb{R}^2$. We can explicitly determine the dual basis $\mathscr{B}^*=\{f_1, f_2\}$ of $\mathscr{B}$ to be:

(a) $f_1=x_1+x_2$, $f_2=x_2$; (b) $f_1=-x+3y$, $f_2=x-2y$;

(c) $f_1=x_1$; $f_2=x_2$; (d) $f_1=x-2y$, $f_2=-x+3y$.

【解】 選 (b)

【說明】題中函數的表示法不妥且符號不一致, 應該是像下列才對。

(a) $f_1(x_1, x_2)=x_1+x_2$, $f_2(x_1, x_2)=x_2$;

(b) $f_1(x_1, x_2)=-x_1+3x_2$, $f_2(x_1, x_2)=x_1-2x_2$;

(c) $f_1(x_1, x_2)=x_1$; $f_2(x_1, x_2)=x_2$;

(d) $f_1(x_1, x_2)=x_1-2x_2$, $f_2(x_1, x_2)=-x_1+3x_2$.

$\{f_1, f_2\}$ 是 $\{b_1, b_2\}$ 的 dual basis (綜線附錄C定義5)

$\Longleftrightarrow\quad f_1(b_1)=1,\quad f_1(b_2)=0,\quad f_2(b_1)=0,\quad f_2(b_2)=1$

只有(b)合乎要求.

08F06 【 成大85資工[4] 】

The vectors $v_1=(1, 1, 1)$, $v_2=(1, 1, -1)$ and $v_3=(1, -1, -1)$ form a basis of the vector space $\mathbb{C}^3$. Let $\{u_1, u_2, u_3\}$ be a dual basis of $\{v_1, v_2, v_3\}$ and let $v=(0,1,0)\in\mathbb{C}^3$. Find the inner products $\langle v,u_1\rangle$, $\langle v,u_2\rangle$ and $\langle v,u_3\rangle$.

【說明】若V爲內積空間, 則每個 $f\in V^*=\mathscr{L}(V,K)$ 都恰可配對一個$p\in V$, 使$f(v)=\langle v,p\rangle$. dual basis本來是在dual space V^* 之內, 本題將f與所對應的p視爲相同. 也就是 $\langle v_i, u_j\rangle=\delta_{ij}$. (綜線附錄C定義5)

【解】

令 $v=xv_1+yv_2+zv_3$, 即 $\begin{cases} x+y+z=0 \\ x+y-z=1 \\ x-y-z=0 \end{cases}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 0 \end{array}\right] \sim \overset{\text{列運算}}{\cdots\cdots} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

$\therefore\ x=0, y=1/2, z=-1/2$.

$\langle v,u_1\rangle=x\langle v_1, u_1\rangle+y\langle v_2, u_1\rangle+z\langle v_3, u_1\rangle=x=0$

$\langle v,u_2\rangle=x\langle v_1, u_2\rangle+y\langle v_2, u_2\rangle+z\langle v_3, u_2\rangle=y=1/2$

$\langle v,u_3\rangle=x\langle v_1, u_3\rangle+y\langle v_2, u_3\rangle+z\langle v_3, u_3\rangle=z=-1/2$

08F07 【 台大81資工[1] 】

Let V be a finite–dimensional vector space over the field F and let $\widehat{X}: V^*\longrightarrow F$ be defined by $\widehat{X}(f)=f(X)$ for every $f\in V^*$. Now let $\phi: V\longrightarrow V^{**}$ be defined by $\phi(X)=\widehat{X}$. Then ϕ is an isomorphism.

【解】請參閱綜線附錄C定理13.

０８F**08**【 台大83資工[5] 】

[複選題]

Which of the following statements is (or are) true; provided all vector spaces are finite-dimension ?

(1) Every vector space is isomorphic to its dual space.

(2) If T is an isomorphism from V onto V^*, and β is a finite ordered basis of V, then $T(\beta)=\beta$,

(3) If T is a linear transformation from V into W, then the domain of $(T^t)^t$ is V^{**} .

(4) Every vector space is the dual of some other vector space.

【解】 選(1)(3)(4)

【討論】(1) $\because\ \dim(V^*)=\dim\mathscr{L}(V,K)$ (綜線CH8定義25a)

$=(\dim V)(\dim K)$ (綜線CH8定理25③)

$=\dim V$ (綜線CH6定理19要訣3)

$\therefore$ V同構於V^* . (綜線CH8定理28b)

(2) 由上個小題, V同構於V^* .

可各自任取一組基底來製作同構映射T. (綜線CH7定理3)

β^*是β的對偶基底, T未必要將β映至β^* . (綜線CH8定義25a)

(3) T^t 表示T的對偶映射. (綜線附錄C定義14)

對$T: V\longrightarrow W$, 將有 $T^t:W^*\longrightarrow V^*$, $(T^t)^t: V^{**}\longrightarrow W^{**}$.

(4) dual space內的向量都是functional, 不是普通的向量. 照理說本小題應是False. 但對每個有限維向量空間V, V與V^{**} 形成natural isomorphic的關係. (綜線附錄C定理13)

數學上通常將V與V^{**} 視爲相等(identify). 因此V變成V^* 的dual space.