題型09A: 內積的性質

０９A**01** 【 大同82資工[10] 】

Let V be a (real or complex) vector space over F. An inner product $\langle \cdot , \cdot \rangle$: $V\times V \longrightarrow F$ is a mapping which assign each pair of vector $u, v\in V$ a scalar, i.e., $\langle u,v \rangle \in F$.

Which of the following statements is not the property of an inner product ?

(a) $\langle u,u\rangle \geq 0$; and $\langle u,u\rangle = 0$ if and only if $u = o$;

(b) $\langle au_1 + bu_2, v\rangle = a\langle u_1, v\rangle + b\langle u_2, v\rangle$;

(c) $\langle u,v\rangle = \overline{\langle v,u\rangle}$;

(d) none.

【解】選d (綜線CH9定義2)

０９A**02** 【 中原85工工[2] 】

Consider the Euclidean space $\mathbb{R}^n$. Let $u=(u_1, u_2, ..., u_n)$ and $v=(v_1, v_2, ..., v_n)$ be two vectors in $\mathbb{R}^n$.

(a) Give the definitions of the inner product $u \cdot v$ and the norm $\| u \|$. (5%)

(b) Write down the Cauchy–Schwarz inequality in $\mathbb{R}^n$. (5%)

(c) Use (b) to show $ab \leq \dfrac{a^2+b^2}{2}$, for any real number a,b. (5%)

(d) Please write down the Triangle inequality and prove it. (10%)

【解】(a) $u \cdot v = u_1v_1 + u_2 v_2 + ... + u_nv_n$ (綜線CH1定義3)

$$\| u \| = \sqrt{u \cdot u} = \sqrt{u_1^2 + u_2^2 + ... + u_n^2}$$

(b) $(u \cdot v)^2 \leq \| u \|^2 \| v \|^2$, 即 (綜線CH1定理6)

$$(u_1v_1 + ... + u_nv_n)^2 \leq (u_1^2 + ... + u_n^2)(v_1^2 + ... + v_n^2)$$

也可寫成 $| u \cdot v | \leq \| u \| \| v \|$

(c) $2ab = (a,b) \cdot (b,a) = | (a,b) \cdot (b,a) |$

$$\leq \| (a,b) \| \| (b,a) \| = \sqrt{a^2+b^2} \sqrt{a^2+b^2} = a^2+b^2$$

$\therefore \quad ab \leq \dfrac{a^2+b^2}{2}$

(d) 三角不等式為 $\| a+b \| \leq \| a \| + \| b \|$, 證明如下: (綜線CH1定理7)

$$\begin{aligned} \| a+b \|^2 &= (a+b)\cdot(a+b) = a\cdot a + 2a\cdot b + b\cdot b \\ &= \| a \|^2 + 2a\cdot b + \| b \|^2 \\ &\leq \| a \|^2 + 2\| a \| \, \| b \| + \| b \|^2 \qquad (\text{由(b)}) \\ &= (\| a \| + \| b \|)^2 \end{aligned}$$

兩邊開平方即得證.

09A03 【 大同84資工[4](b) 】
(b) If x and y are vectors, show that $(1/2)(|x+y|^2+|x-y|^2)=|x|^2+|y|^2$

【參考章節】綜線CH9定理8.

【解】(b) $|x+y|^2=(x+y)^T(x+y)=x^Tx+x^Ty+y^Tx+y^Ty$

$|x-y|^2=(x-y)^T(x-y)=x^Tx-x^Ty-y^Tx+y^Ty$

$(1/2)(|x+y|^2+|x-y|^2)=x^Tx+y^Ty=|x|^2+|y|^2$.

09A04 【 交大80資科[2] 】
Let $C[-\pi, \pi]$ be the real vector space of continuous realvalued function on $[-\pi, \pi]$. For two functions f and g in $C[-\pi, \pi]$ define the inner product by $(f, g)=\int_{-\pi}^{\pi} f(t)g(t)dt.$ (A) Write out the Schwarz inequality in this case. (B) Find the angle between cos$x$ and sin$x$. (C) Find an orthonormal basis for the subspace spanned by 1, cos$x$, and sin$x$.

【解】(A) Schwarz inequality為 $|(f, g)|^2 \leq (f, f)(g, g)$ (綜線CH9定理9)

即 $\left| \int_{-\pi}^{\pi} f(t)g(t)dt \right|^2 \leq \int_{-\pi}^{\pi} f(t)^2 dt \int_{-\pi}^{\pi} g(t)^2 dt$

(B) 令 $f(x)=\cos x$, $g(x)=\sin x$, 所求夾角為 θ,

則 $\cos\theta=\dfrac{(f,g)}{(f,f)^{1/2}(g,g)^{1/2}}$ (綜線CH9定義4)

$(f,g)=(\cos x,\sin x)=\int_{-\pi}^{\pi}\cos x\sin x\,dx=\ldots=0$

$\therefore\ \cos\theta=0$, $\therefore\ \theta=\pi/2$

(C) 令 $f_1(x)=1,\quad f_2(x)=\cos x,\quad f_2(x)=\sin x$

由(B)部分已知 $(f_2,f_3)=0$

又 $(f_1,f_2)=\int_{-\pi}^{\pi}\cos x\,dx=0,\qquad (f_1,f_3)=\int_{-\pi}^{\pi}\sin x\,dx=0$

$\therefore\ f_1,f_2,f_3$ 已兩兩正交

$\|f_1\|=(f_1,f_1)^{1/2}=\left(\int_{-\pi}^{\pi}1\,dx\right)^{1/2}=\sqrt{2\pi}$

$\|f_2\|=(f_2,f_2)^{1/2}=\left(\int_{-\pi}^{\pi}\cos^2x\,dx\right)^{1/2}=\sqrt{\pi}$

$\|f_3\|=(f_3,f_3)^{1/2}=\left(\int_{-\pi}^{\pi}\sin^2x\,dx\right)^{1/2}=\sqrt{\pi}$

$\therefore\ \left\{\dfrac{1}{\sqrt{2\pi}},\ \dfrac{1}{\sqrt{\pi}}\cos x,\ \dfrac{1}{\sqrt{\pi}}\sin x\right\}$ 為orthonormal basis.

０９Ａ**05** 【 師大83資教[4] 】

Given an orthonormal set S in the first order continuous space $C[-\pi,\pi]$,

$$S=\left\{1/\sqrt{2\pi},\ (1/\sqrt{\pi})\cos x,\ (1/\sqrt{\pi})\cos 2x,\ (1/\sqrt{\pi})\cos 3x,\ (1/\sqrt{\pi})\cos 4x\right\},$$

with inner product defined by

$$\langle f,g\rangle=\int_{-\pi}^{\pi}f(x)g(x)dx,$$

determine the values of

(a) $\int_{-\pi}^{\pi} \sin^4 x \cos 2x \, dx$, (b) $\int_{-\pi}^{\pi} \sin^4 x \cos 4x \, dx$ (16%)

【分析】(甲) 本題可以直接積分求解, 但計算複雜.

題目已給出orthonormal basis, 利用內積的性質可迅速求出.

(乙) $\cos^2\theta = \dfrac{1+\cos 2\theta}{2}$, $\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

【解】

$$\sin^4 x = (\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = (1/4)(1-2\cos 2x + \cos^2 2x)$$

$$= (1/4)\left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) = 3/8 - (1/2)\cos 2x + (1/8)\cos 4x$$

(a) $\int_{-\pi}^{\pi} \sin^4 x \cos 2x \, dx = \langle \sin^4 x , \cos 2x \rangle$

$= \langle 3/8 - (1/2)\cos 2x + (1/8)\cos 4x , \cos 2x \rangle$

$= (3/8)\langle 1,\cos 2x\rangle - (1/2)\langle \cos 2x,\cos 2x\rangle + (1/8)\langle \cos 4x,\cos 2x\rangle$

$= (3/8)\cdot 0 - (1/2)\pi \left\langle \dfrac{\cos 2x}{\sqrt{\pi}} , \dfrac{\cos 2x}{\sqrt{\pi}} \right\rangle + (1/8)\cdot 0 = -\pi/2$

$\int_{-\pi}^{\pi} \sin^4 x \cos 4x \, dx = \langle \sin^4 x , \cos 4x \rangle$

$= \langle 3/8 - (1/2)\cos 2x + (1/8)\cos 4x , \cos 4x \rangle$

$= (3/8)\langle 1,\cos 4x \rangle - (1/2)\langle \cos 2x,\cos 4x \rangle + (1/8)\langle \cos 4x,\cos 4x \rangle$

$= (3/8)\cdot 0 - (1/2)\cdot 0 + (1/8)\pi \left\langle \dfrac{\cos 4x}{\sqrt{\pi}} , \dfrac{\cos 4x}{\sqrt{\pi}} \right\rangle = \pi/8$

０９A**06** 【 交大79資料[10] 】

Let T^3 be the real vector space of polynomials of degree strictly less than 3; define the inner product between the polynomials

f and g in T^3 by $(f, g) = \int_0^1 f(t)g(t)dt$. Find the angle between t and t^2-t+1.

【解】 $(t, t^2-t+1) = \int_0^1 t(t^2-t+1)dt = 5/12$

$$\| t \|^2 = \int_0^1 t^2 dt = 1/3$$

$$\| t^2-t+1 \|^2 = \int_0^1 (t^2-t+1)^2 dt = \ldots = 7/10$$

$$\therefore \cos\theta = \frac{5/12}{\sqrt{1/3}\sqrt{7/10}} = \frac{5\sqrt{5}}{2\sqrt{2}\sqrt{3}\sqrt{7}}$$

$$\therefore \theta = \cos^{-1}\frac{5\sqrt{5}}{2\sqrt{2}\sqrt{3}\sqrt{7}}$$

０９Ａ**０７** 【 大同83資工[6] 】

Let $P_3(\mathbb{R})$ be the real vector space of polynomials of degree strictly less than 3. Define the inner product between f, g in $P_3(\mathbb{R})$ by $\langle f,g \rangle$

$$\langle f,g \rangle = \int_0^1 f(t)g(t)dt$$

(a) Find the orthogonal projection of t^2 on t. (5%)

(b) Find the angle between t^2 and t. (5%)

【解】

(a) $\langle t^2, t\rangle = \int_0^1 t^3 dt = 1/4$, $\langle t,t\rangle = \int_0^1 t^2 dt = 1/3$

所求爲$\dfrac{1/4}{1/3}t = (3/4)t$ (綜線CH1定義8,CH9定理12)

(b) $\langle t^2, t\rangle = 1/4$, $\langle t, t\rangle = 1/3$, $\langle t^2, t^2\rangle = \int_0^1 t^4 dt = 1/5$

∴ 所求夾角為 $\cos^{-1}\dfrac{1/4}{\sqrt{1/3}\sqrt{1/5}} = \cos^{-1}\dfrac{\sqrt{15}}{4}$ (綜線CH1定義3⑤)

０９A08 【 元智84電資[1] 】

Let (,) be an inner product on an $\mathbb{R}$–space V, and let $x_1, x_2, \ldots, x_t$ be nonzero vectors which are perpendicular to each other, i.e. , $(x_i, x_j) = 0\ \forall i \neq j$. Prove that $x_1, x_2, \ldots, x_t$ are linearly independent.

【解】若$c_1x_1 + c_2x_2 + \ldots + c_tx_t = o$,

$\forall i = 1, 2, \ldots, t$,

$(c_1x_1 + \ldots + c_tx_t, x_i) = (o, x_t)$,

∴ $c_1(x_1, x_i) + \ldots + c_t(x_t, x_i) = 0$ (綜線CH9定義2)

∴ $c_i(x_i, x_i) = 0$ (兩兩垂直)

而由$x_i \neq o$得知$(x_i, x_i) > 0$, (綜線CH9定義2)

∴ $c_i = 0$

故得證. (綜線CH6定義9)

０９A09 【 清大75資料[5](1) 】

Prove or disprove the following statements.

(1) An orthogonal set of non–zero vectors is linearly independent.

【分析】本題所題到的set未必是有限集, 因此證法與上題稍有不同.

【解】Prove. 證明請參閱綜線CH9定理15.

０９A10 【 交大85資料[1] 】

[複選題]

Suppose that u and v are the geometrical vectors corresponding to

$u = [4\quad -2\quad -1]^T$ and $v = [-4\quad 0\quad 3]^T$, and

let $w = [x\quad y\quad z]^T$ be the coordinates of all vectors $w = \alpha u + \beta v$

as α and β range over all real numbers. If an equation of the form $ax+by+cz=d$ is satisfied by the entries x, y, z in $w=[x \quad y \quad z]^{T}$ then (a, b, c, d) can be
(a) (3, 6, 4, 0). (b) (–6, –8, –8, 0). (c) (–1, 1, –5, 1). (d) (0, 1, 1, 1).

【解】選(b).

【說明】通常將行矩陣視為向量. 本題對此二者詳加區分, 但解題時並無不同.

由題意, $w=[x\ y\ z]^{T}=\alpha u+\beta v$.

令 $m=[a,b,c]^{T}$, 則方程式 $ax+by+cz=d$ 可表為 $m^{T}w=d$.

已知 $\forall\alpha, \beta\in\mathbb{R}$, $m^{T}w=d$

即 $\forall\alpha, \beta\in\mathbb{R}$, $m^{T}(\alpha u+\beta v)=d$

取 $\forall\alpha=\beta=0$ 即得知 $d=0$.

$\therefore$ $\alpha, \beta\in\mathbb{R}$, $m^{T}(\alpha u+\beta v)=0$

上式等價於 $m^{T}u=0$ 且 $m^{T}v=0$

即 $4a-2b-c=0$, $-4a+3c=0$

以各選項代入, 只有(b)的 $a=-6, b=-8, c=-8$ 合條件.

題型09B: 正投影

０９Ｂ**０１**【 交大80資料[5] 】

Let V be a vector space with an inner product. Let V_0 be a subspace of V, and let $S=\{v_1, v_2, \ldots, v_q\}$ be an orthogonal basis of V_0 .

(A) Write out the orthogonal projection $P_0(v)$ of a vector $v\in V$ onto V_0 (with respect to the orthogonal basis S).

(B) Suppose that $T=\{u_1, u_2, \ldots, u_q\}$ is another orthogonal basis for V_0 . With respect to basis T we may define another orthogonal projection $P_0'(v)$ similarly as in part (A). Prove that $P_0(v)=P_0'(v)$ for all $v\in V$.

(C) Take $V=\mathbb{R}^3$ with usual inner product, and

$$V_0=\text{Column space of }\begin{bmatrix}1&0&2&3\\0&1&1&2\\1&1&3&5\end{bmatrix}$$

Find the orthogonal projection $P_0(v)$ of $v\in\mathbb{R}^3$ onto V_0 .

(D) In part (C), find the matrix representation (with respect to usual basis e_1 , e_2 , and e_3) of the orthogonal projection.

【解】(A) $P_0(v)=\sum_{i=1}^{q}\dfrac{(v, v_i)}{(v_i , v_i)}v_i$

則$P_0(v)\in V_0$ 且$\forall w\in V_0$, $(v-P_0(v), w)=0$ (綜線CH9定理12)

(B) 由(A)部分可知, 對任意$v\in V$, 必有

$P_0(v)\in V_0$, $P_0'(v)\in V_0$, 且

$\forall w\in V_0$, $(v-P_0(v), w)=0$, $(v-P_0'(v), w)=0$

$\therefore$ $\forall w\in V_0$,

$(P_0(v)-P_0'(v), w)=(P_0(v)-v+v-P_0'(v), w)=(P_0(v)-v, w)+(v-P_0'(v), w)=0$

取 $w=P_0(v)-P_0'(v)$, 可得

$(P_0(v)-P_0'(v), P_0(v)-P_0'(v))=0$

$\therefore\ P_0(v)-P_0'(v)=0$ (內積的正定性)

$\therefore\ P_0(v)=P_0'(v)$

(C) 對矩陣做column operation:

$$\begin{bmatrix}1&0&2&3\\0&1&1&2\\1&1&3&5\end{bmatrix}\overset{C}{\sim}\begin{bmatrix}1&0&0&0\\0&1&1&2\\1&1&1&2\end{bmatrix}\overset{C}{\sim}\begin{bmatrix}1&0&0&0\\0&1&0&0\\1&1&0&0\end{bmatrix}$$

$\therefore\ u_1=\begin{bmatrix}1\\0\\1\end{bmatrix}, u_2=\begin{bmatrix}0\\1\\1\end{bmatrix}$ 為 V_0 之basis

對u_1 , u_2 做Gram–Schmidt process:

$u_1'=u_1$

$$u_2'=u_2-\frac{(u_2, u_1')}{(u_1', u_1')}u_1'=\begin{bmatrix}0\\1\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}-1\\2\\1\end{bmatrix}$$

$\therefore\ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}$ 為V_0 之orthogonal basis

利用(A)部分之公式可得

$$P_0(v)=P_0\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\frac{x_1+x_3}{2}\begin{bmatrix}1\\0\\1\end{bmatrix}+\frac{-x_1+2x_2+x_3}{6}\begin{bmatrix}-1\\2\\1\end{bmatrix}$$

$$=(1/3)\begin{bmatrix} 2x_1-x_2+x_3 \\ -x_1+2x_2+x_3 \\ x_1+x_2+2x_3 \end{bmatrix}$$

(D)

$$P_0\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right)=\begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

∴ P_0之矩陣表示為 $\begin{bmatrix} 2/3 & -1/3 & 1/3 \\ -1/3 & 2/3 & 1/3 \\ 1/3 & 1/3 & 2/3 \end{bmatrix}$ (綜線CH7定理15)

０９B**02** 【 交大81資工[4](d) 】

True or false, two points for each.

(d) Let S be a subspace of $\mathbb{R}^n$, and $\{u_1, u_2, ..., u_q\}$ be an orthonormal basis for S. For the orthogonal projection P defined as:

$$P(u)=\sum_{i=1}^{q}(u_i, u)u_i$$

for all $u\in\mathbb{R}^n$, the projection P can be represented as

$$P(u)=UU^Tu$$

where $U=[\,u_1, u_2, ..., u_q\,]$.

【解】(d) True. 證明如下:

$$UU^Tu=\begin{bmatrix} u_1, ..., u_q \end{bmatrix}\begin{bmatrix} u_1^T \\ . \\ u_q^T \end{bmatrix} u$$

$$= \begin{bmatrix} & & \\ u_1, ..., u_q \\ & & \end{bmatrix} \begin{bmatrix} u_1^T u \\ \vdots \\ u_q^T u \end{bmatrix}$$ (左橫切：CH2定理7)

$$=(u_1^T u)u_1 + \ .... \ + (u_q^T u)u_q$$ (左直切：CH2定理6)

$$= \sum_{i=1}^{q} (u_i, u)u_i$$ (綜線CH9定義1)

０９B03 【 台大84資工[3] 】

Let V be a finite dimentional inner product space (with $\langle\ ,\ \rangle$ as the inner product on V), and W a subspace of V. Let $\{x_1, x_2, ..., x_k\}$ be an orthonormal basis of W. For $y \in V$, define $y_1 = \sum \langle y, x_i \rangle x_i$. Prove that $\| y-y_1 \| \leq \| y-u \|$ for any $u \in W$.

【解】請參閱綜線CH9定理12及定理13. 稍加改寫即得.

０９B04 【 台大80資工[8] 】

[True or False Problem]

Let W be a finite dimensional subspace of an inner product space V. If $\{x_1, x_2, ..., x_k\}$ is an orthogonal basis for W and $y \in V$, then

$\| y-u \| \geq \| y-y_1 \|$, $\forall u \in W$, where $y_1 = \sum_{i=1}^{k} (y, x_i)\, x_i$

【解】False, 反例如下:

令$V = \mathbb{R}^{3\times1}$

$$W = \left\{ \begin{bmatrix} r \\ s \\ 0 \end{bmatrix} \middle| \ r, s, 0 \in \mathbb{R} \right\}, x_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix},$$

則 $\{x_1, x_2\}$爲W的正交基底

對 $y=\begin{bmatrix}1\\1\\1\end{bmatrix}$,

$$y_1=\sum_{i=1}^{2}(y,x_i)=(y,x_1)x_1+(y,x_2)x_2=2\begin{bmatrix}2\\0\\0\end{bmatrix}+2\begin{bmatrix}0\\2\\0\end{bmatrix}=\begin{bmatrix}4\\4\\0\end{bmatrix}$$

不合"$\|y-u\|\geq\|y-y_1\|, \forall u\in W$"的要求.

例如, 以$u=\begin{bmatrix}1\\1\\0\end{bmatrix}\in W$而言, $\|y-u\|=1<\|y-y_1\|$

【討論】若將題目中的orthogonal改爲orthonormal, 或將y_1 的定義改爲

$$y_1=\sum_{i=1}^{k}\frac{(y,x_i)}{(x_i,x_i)}x_i,$$

則本題爲True. (綜線CH9定理12, 定理13.)

０９B**05**【 台大85資工[8] 】

[複選題]

Let V be an inner product space. Which of the following are true.

(1) Let W be a finite–dimensional subspace of V. If $\{x_1, ..., x_k\}$ is an orthonormal basis of W and $y\in V$, then $y=\sum_{i=1}^{k}\langle y,x_i\rangle x_i+z$ for some $z\in W$.

(2) Let $y_1=\sum_{i=1}^{k}\langle y,x_i\rangle x_i$ as in (1) above. If $u\in W$, then

$\|y-u\|\geq\|y-y_1\|$, where $\|u\|=\sqrt{<u,u>}$.

(3) Every orthonormal subset of V must be linearly independent.

(4) Let T be a linear operator on V. If $\| T(x) \| = \| x \|$, then T is onto.
(5) If V is finite–dimensional, and W_1 and W_2 are subspaces of V,
then $(W_1+W_2)^{\perp}=W_1^{\perp}\cap W_2^{\perp}$ and $(W_1\cap W_2)^{\perp}=W_1^{\perp}+W_2^{\perp}$

【解】選(2)(3)(5)

【討論】(1) False.

應該是 for some $z\in W^{\perp}$ 才對. (綜線CH9定義11,定理12)

反例隨便舉就有.

(2) True. 此爲定理. (綜線CH9定理13)

(3) True. 此爲定理. (綜線CH9定理15)

(4) False. 反例如下:

$V=\mathbb{R}[x]$,

對$p(x), q(x)\in V$, 設$k=\max\{ \deg f(x), \deg g(x) \}$,

可令 $p(x)=a_0+a_1x+a_2x^2+\ldots+a_kx^k$,

$q(x)=b_0+b_1x+b_2x^2+\ldots+b_kx^k$. (若缺項可補0爲係數)

定義 $\langle p(x), q(x)\rangle=a_0b_0+a_1b_1+\ldots+a_kb_k$.

顯然此合於內積的條件(綜線CH9定義2).

考慮 $T: V\longrightarrow V$, $T(p(x))=x\cdot p(x)$.

則 $\forall p(x)\in V$, $\| T(p(x)) \| = \| p(x) \|$

但T並非onto. (常數多項式不屬於T的值域)

若加上“V爲有限維”的條件, 則本小題成立. 證明如下:

若 $T(x)=o$, 則 $\| x \| = \| T(x) \| = \| o \| =0$

$\therefore$ $x=o$.

以上就是說 $\mathrm{Ker}T=\{o\}$.

$\therefore$ T 爲一對一映射. (綜線CH8定理7)

$\therefore$ T 爲可逆映射. (綜線CH8定理11)

$\therefore$ T 爲映成映射. (綜線CH8定理11)

(5) True. 此爲定理. (綜線CH11定理18)

０９Ｂ**06** 【 交大85資工[3] 】

Given the vector space $C[0,1]$ with inner product $\langle f,g\rangle=\int_0^1 f(x)g(x)dx$ and norm $\|f\|=(\langle f,f\rangle)^{1/2}$, find the best least squares approximation to e^x on the interval [0, 1] by a linear function of the form $a+bx$.

【解】 令 $f(x)=e^x$, $W=\{a+bx \mid a,b\in\mathbb{R}\}$

欲使 $\|f(x)-(a+bx)\|$ 儘小,

就是要在內積 $\langle f,g\rangle$ 之下求 f 對 W 的正投影. (綜線CH9定理13)

先對W的基底 $\{1,x\}$ 施行Gram–Schmidt process:

$f_1(x)=1$

$\langle x,1\rangle=\int_0^1 x\,dx = 1/2$,

$f_2(x)=x-(1/2)f_1=x-1/2$

得到W的正交基底 $\{f_1,f_2\}$.

$\langle f,f_1\rangle=\int_0^1 e^x\,dx=e-1$

$\langle f,f_2\rangle=\int_0^1 e^x(x-(1/2))dx=\int_0^1 (x-(1/2))de^x= 3/2-e/2$

$\langle f_1,f_1\rangle=\int_0^1 1\,dx=1$, $\langle f_2,f_2\rangle=\int_0^1 (x-1/2)^2dx=1/12$

所求爲 $\dfrac{\langle f,f_1\rangle}{\langle f_1,f_1\rangle}f_1+\dfrac{\langle f,f_2\rangle}{\langle f_2,f_2\rangle}f_2=(e-1)\cdot 1+6(3-e)(x-1/2)=(18-6e)x+(4e-10)$

【另解】

$F(a,b)=\int_0^1 (e^x-(ax+b))^2dx = ...$

$=(1/3)a^2+ab+b^2-2a-2(e-1)b+(1/2)(e^2-1)$

$\dfrac{\partial F}{\partial a}=\dfrac{2}{3}a+b-2$, $\dfrac{\partial F}{\partial b}=a+2b-2(e-1)$

欲使 $F(a,b)$ 取極小值, 令 $\dfrac{\partial F}{\partial a}=0$, $\dfrac{\partial F}{\partial b}=0$

可解得 $a=18-6e$, $b=4e-10$

$\therefore$ $(18-6e)x+(4e-10)$ 爲所求

０９Ｂ**07** 【 交大82工工[9] 】 Find the best least squares approximation to e on the interval [0,1] by a linear function.

【解】同上題.

０９Ｂ**08** 【 雲技84電資Ｙ[2] 】 Let the continuous real–valued functions $C[a,b]$ with inner product defined by $\langle f,g\rangle=\int_a^b f(x)g(x)dx$ and norm defined by $\Vert f\Vert=\sqrt{\langle f,f\rangle}$ Find the best least squares approximation to e^x on the internal [0, 2] by a linear function.

【解】 令 $f(x)=e^x$, $W=\{\,a+bx \mid a, b\in\mathbb{R}\,\}$

在所設內積之下求f對W的正投影:

先對W的基底 $\{1, x\}$ 施行Gram–Schmidt process: (綜線CH9定理16)

$f_1(x)=1$

$\langle f_1, f_1\rangle=\int_0^2 1\,dx=2$, $\langle x, 1\rangle=\int_0^2 x\,dx=2$,

$f_2(x)=x-(2/2)f_1=x-1$.

得到W的正交基底 $\{f_1, f_2\}$.

$\langle f_2,f_2\rangle=\int_0^2(x-1)^2dx=2/3$, $\langle f,f_1\rangle=\int_0^2 e^x dx=e^2-1$.

$\langle f,f_2\rangle=\int_0^2 e^x(x-1)dx=\int_0^2(x-1)de^x=(x-1)e^x\Big|_0^2-\int_0^2 e^x d(x-1)=\ldots=2$

所求為 $\dfrac{\langle f, f_1\rangle}{\langle f_1, f_1\rangle} f_1 + \dfrac{\langle f, f_2\rangle}{\langle f_2, f_2\rangle} f_2$ (綜線CH9定理12)

$= \dfrac{(e^2-1)}{2}\cdot 1 + \dfrac{2}{2/3}(x-1) = 3x+(e^2-7)/2$ #

０９B**09**【交大83工工[8]】

Let $C[a,b]$ denote the set of all real–valued functions that are defined and continuous on the closed interval $[a,b]$. Consider the inner product defined by

$$\langle f,g\rangle = \int_0^1 f(x)g(x)dx$$

Let S be the subspace spanned by the vectors 1 and $2x-1$.

(a) Show that 1 and $2x-1$ are orthogonal. (3%)

(b) Determine $\|1\|$ and $\|2x-1\|$. (2%)

(c) Find the best least square aproximation to $\sqrt{x}$ by a function from the subspace S. (8%)

【解】

(a) $\langle 1, 2x-1\rangle = \int_0^1 (2x-1)dx = (x^2-x)\Big|_0^1 = 0$

∴ 1 與 $2x-1$ 正交.

(b) $\|1\|^2 = \langle 1, 1\rangle = \int_0^1 1\, dx = 1$

$\|2x-1\|^2 = \int_0^1 (2x-1)^2 dx = \ldots = 1/3$.

∴ $\|1\| = 1$, $\|2x-1\| = \sqrt{1/3}$. (綜線CH9定義4)

(c) 所求為$\sqrt{x}$ 對S的正投影. (綜線CH9定理13)

$\langle \sqrt{x}, 1\rangle = \int_0^1 \sqrt{x}\, dx = 2/3$

$\langle \sqrt{x}, 2x-1\rangle = \int_0^1 \sqrt{x}\cdot(2x-1)dx = \ldots = 2/15$.

$\therefore$ 所求爲 $\dfrac{2/3}{1}\cdot 1+\dfrac{2/15}{1/3}(2x-1)$ (綜線CH9定理12)

$=(4/5)x+4/15$.

09B10 【清大78資料[5]】

Let A be an m by n matrix.

(a) Show that if $(A^tA)x=0$ then $Ax=0$, $x\in\mathbb{R}^n$.

(b) Show that rank$(A^tA)=$rank(A).

【分析】本題必須是實數矩陣, 複數矩陣時要將轉置改爲共軛轉置. (綜線CH9定理20)

【解】 (a) 若$A^tAx=0$, 左乘x^t , 得 $x^tA^tAx=0$

$\therefore$ $(Ax)^t(Ax)=0$

由實數行向量內積的正定性可知

$Ax=0$ (綜線CH9定義1①, CH1定理5)

(b) 由(a) 可知

$Ax=o \Longleftrightarrow A^tAx=o$

$\therefore$ Ker$A=$Ker(A^tA)

A爲$m\times n$矩陣, A^tA爲$n\times n$矩陣

$\therefore$ rank$A=n-$dim Ker$A$ (綜線CH8定理8)

$=n-$dim Ker$(A^tA)=$rank(A^tA) (綜線CH8定理8)

09B11 【交大83資工[6]】

Let Q be an $m\times n$ matrix with orthogonal columns, B an $n\times r$ matrix, and $C=Q^TQB$. Show that the rank of matrix B is equal to the rank of matrix C.

【勘誤】本題錯誤. 反例如下:

$$Q=\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix},\quad B=\begin{bmatrix}1&0&0&0\\0&1&0&0\end{bmatrix}$$

則Q的兩個column彼此正交.

$$C = Q^{\mathrm{T}}QB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\operatorname{rank}C = 1 \neq \operatorname{rank}B = 2$.

本題應更正為:

"Let Q be an $m\times n$ matrix with nonzero orthogonal columns, ..." 或

"Let Q be an $m\times n$ matrix with orthonormal columns, ..." .

而且所討論的矩陣只能是實數矩陣.

(若討論複數矩陣, 還必須將各個轉置 $^{\mathrm{T}}$ 修正為共軛轉置 $^{\mathrm{H}}$ 才行.)

【解】Q的行是非零正交集, 所以是線性獨立集,

$\therefore$ $\operatorname{rank}(Q) = n$ (綜線CH8定理17)

$\therefore$ $\operatorname{rank}(Q^{\mathrm{T}}Q) = n$ (綜線CH9定理20)

而$Q^{\mathrm{T}}Q$為$n\times n$矩陣,

$\therefore$ $Q^{\mathrm{T}}Q$為可逆矩陣. (綜線CH8定理17)

$\therefore$ $\operatorname{rank}(C) = \operatorname{rank}(Q^{\mathrm{T}}QB) = \operatorname{rank}(B)$.

０９B**１２**【 交大84資工[3] 】

Let A be an $m\times n$ matrix of rank n and b be a vector in $\mathbb{R}^m$. Find the projection of b onto $R(A) = \{b \in \mathbb{R}^m \mid b = Ax \text{ for some } x \in \mathbb{R}^n\}$. (you should derive your solution.)

【參考章節】綜線CH9定理22

【解】所求為 $A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b$. 以下加以證明:

$\because$ $\operatorname{rank}(A^{\mathrm{T}}A) = \operatorname{rank}A = n$, (綜線CH9定理20)

$\therefore$ $A^{\mathrm{T}}A$可逆. (綜線CH8定理17)

令$p = A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b$,

顯然 $p \in R(A)$. (定義中所需的x取$(A^{\mathrm{T}}A)^{\mathrm{T}}A^{\mathrm{T}}b$即可)

對任意 $Az \in R(A)$,

$(Az)^{\mathrm{T}}(b-p) = (Az)^{\mathrm{T}}(b-A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b) = z^{\mathrm{T}}A^{\mathrm{T}}b - z^{\mathrm{T}}A^{\mathrm{T}}A(A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}b$

$= z^{\mathrm{T}}A^{\mathrm{T}}b - z^{\mathrm{T}}A^{\mathrm{T}}b = 0$

$\therefore$ $b-p$垂直於$R(A)$上的每個向量.

０９B**13**【 大同83資工[2] 】

Let $P=A(A^TA)^{-1}A^T$, where A is a $m\times n$ matrix of rank n.

(a) Show that $P^k=P$ for k=1,2,...(3%)

(b) Show that P is symmetric. (2%)

【分析】本題與清大71年計管所第五題同. 本題背景請參閱綜線CH11定理20.

【解】 (a) 以數學歸納法證明. $k=1$時顯然成立.

假設$k=r$時成立, 即 $P^r=P$

當$k=r+1$時,

$$P^k=P^{r+1}=P^r\cdot P=P\cdot P=A(A^TA)^{-1}A^T\cdot A(A^TA)^{-1}A^T$$
$$=A(A^TA)^{-1}(A^TA)(A^TA)^{-1}A^T=A(A^TA)^{-1}A^T=P$$

(b) $P^T=(A(A^TA)^{-1}A^T)^T=(A^T)^T((A^TA)^{-1})^TA^T=A((A^TA)^T)^{-1}A^T=A(A^TA)^{-1}A^T=P$

０９B**14**【 元智85工工乙[1] 】

Let $A\in\mathbb{R}^{m\times n}$. Prove the following assertions.

(1) A^TA has the same nullspace as A. (10%)

(2) If A has linearly independent columns, then A^TA is invertible. (15%)

(3) Find the projection of b onto the column space of A, where

$$A=\begin{bmatrix}1&1\\2&-1\\-2&0\\1&1\end{bmatrix}\quad b=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\quad (25\%)$$

【解】(1) 本小題考定理證明, 請參閱綜線CH9定理20.

(2) 本小題考定理證明, 請參閱綜線CH9定理21.

(3) $A^{T}A = \begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 3 \end{bmatrix}$

所求為 $A(A^{T}A)^{-1}A^{T}b$ (綜線CH9定理22)

$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/10 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1/10 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \dots = \begin{bmatrix} 13/10 \\ -2/5 \\ -3/5 \\ 13/10 \end{bmatrix}$$

#

【討論】第3小題因為A的兩個行彼此正交，所以也可直接用綜線CH9定理12求解，但計算的難度完全一樣.

０９B**１５**【 中正85資工[5] 】

Let T be a projection on a finite–dimensional inner product space V. Prove that if $\| T(x) \| \leq \| x \|$ for all $x \in V$, then T is an orthogonal projection.

【分析】本題較難. 須先有幾何直覺，再拼湊證明.

【解】T為projection即$T^2 = T$

$\therefore\ V = \mathrm{Ker}T \oplus \mathrm{Im}T$ (綜線CH11定理12)

對$p \in \mathrm{Im}T$，令$p = T(v)$， 則$T(p) = T(T(v)) = (T^2)v = Tv = p$

欲證 Ker$T$與ImT正交，即：

$\forall p \in \mathrm{Im}T, \forall q \in \mathrm{Ker}T$，必$\langle p,q \rangle = 0$. (綜線CH11定義15)

假設$\exists p\in \mathrm{Im}T, \exists q\in \mathrm{Ker}T$ 使得 $\langle p,q\rangle\neq 0$，欲導出矛盾:

$T(tp+q)=T(tp)+T(q)=tT(p)+o=tp$

$\| tp \|^2=\| T(tp+q)\|^2$

$\le \| tp+q\|^2=\| tp\|^2+\| q\|^2+2\mathrm{Re}\langle tp,q\rangle$ (綜線CH9定理8)

$\therefore\ 0\le \| q\|^2+2\mathrm{Re}(t\langle p,q\rangle)$

令 $t=-\| q\|^2/\langle p,q\rangle$， 代入上式得 $0\le -\| q\|^2$

$\therefore\ \| q\|^2=0$ ， $\therefore\ q=o$

$\therefore\ \langle p,q\rangle=0$，此爲矛盾.

０９B**１６**【 大同80資工[3](i) 】

True or False:

(i) If the vectors x and y are orthogonal, and P is a projection matrix, then Px and Py are orthogonal.

【解】(i) False, 反例如下:

取 $P=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, x=\begin{bmatrix}1\\ 1\end{bmatrix}, y=\begin{bmatrix}-1\\ 1\end{bmatrix}$,

則 $P^2=P,\quad P^{\mathrm{T}}=P$ (綜線CH11定理20)

且 x, y正交 (內積爲0) (綜線CH1定義3)

但 $Px=\begin{bmatrix}1\\ 0\end{bmatrix}, Py=\begin{bmatrix}-1\\ 0\end{bmatrix}$

Px與Py並未正交.

０９B**１７**【 交大86資工[6](c) 】

[是非倒扣題]

Every projection matrix is symmetric.

【解】True.

[解說] 一般的投影矩陣的條件是 $A^2=A$. (綜線CH11定義10)

而實數正投影矩陣的條件是 $A^2 = A, A^T = A$. (綜線CH11定理20)

這題本來應該是False.

但有些書只討論正投影, 而將正投影矩陣稱爲projection matrix.

由本試卷第8題的用詞可判定命題者的意思應是正投影, 而且還是實數矩陣.

０９B**１８**【 元智83工工[13] 】

What is a basis of the plane $2X_1 + 2X_2 + X_3 = 0$? Find the projection matrix that projects any point $X \in \mathbb{R}^3$ onto the plane.

【分析】所謂 "the plane $2X_1 + 2X_2 + X_3 = 0$", 其實該是

" the plane $\{(X_1, X_2, X_3)^T \mid 2X_1 + 2X_2 + X_3 = 0\}$ "

習慣上常將方程式與方程式的軌跡混爲一談.

【解】$\{(X_1, X_2, X_3)^T \mid 2X_1 + 2X_2 + X_3 = 0\} = \{(s, t, -2s-2t) \mid s, t \in \mathbb{R}\}$

$= \{s(1, 0, -2) + t(0, 1, -2) \mid s, t \in \mathbb{R}\}$

∴ 可取 $\{(1, 0, -2), (0, 1, -2)\}$ 爲基底.

令 $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & -2 \end{bmatrix}$,

所求爲 $A(A^TA)^{-1}A^T = \ \dots\ = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$ (綜線CH9定理22)

【另解】

設E代表平面$2X_1 + 2X_2 + X_3 = 0$,

∵ $n = (2, 2, 1)^T$ 爲E的法向量, ∴ $E^{\perp} =$ span($(2,2,1)^T$)

對$v \in \mathbb{R}^3$,

v對$E^{\perp}$的正投影爲 $(n^Tn)^{-1}(n^Tv)n$ (綜線CH9定理12)

$$= (n^{\mathrm{T}}n)^{-1}\, nn^{\mathrm{T}}v = \frac{1}{9}\begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} v$$

∴ v對E的正投影爲 $v-(n^{\mathrm{T}}n)^{-1}nn^{\mathrm{T}}v$

$$= (I-(n^{\mathrm{T}}n)^{-1}nn^{\mathrm{T}})\, v = \frac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix} v$$

∴ 所求矩陣爲$\frac{1}{9}\begin{bmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{bmatrix}$

題型09C: Gram–Schmidt程序

０９Ｃ０１【 中正82資工[2] 】

(a) Let V be a finite dimensional vector space with a positive definite scalar product. Suppose we are given an arbitrary basis $\{v_1, v_2, \ldots, v_n\}$ of V. How to obtain an orthonormal basis $\{u_1, u_2, \ldots, u_n\}$ for V? (the Gram–Schmidt process).

(b) Apply the Gram–Schmidt process to the following three vectors:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

【解】(a) 令$\langle x, y \rangle$代表其positive definite scalar product.

再由下述依序求出orthogonal set $\{w_1, \ldots, w_n\}$:

$$w_1 = v_1,$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1,$$

....

$$w_i = v_i - \sum_{j=1}^{i-1} \frac{\langle v_i, w_j \rangle}{\langle w_j, w_j \rangle} w_j,$$

....

$$w_n = v_n - \sum_{j=1}^{n-1} \frac{\langle v_n, w \rangle}{\langle w_j, w_j \rangle} w_j \quad .$$

(以上爲Gram–Schmidt process)

接下來令$u_i = w_i / \| w_i \|$, 則$\{u_1, u_2, \ldots, u_n\}$爲orthonormal basis.

(b) $w_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$,

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - \frac{-1}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 7/3 \\ 5/3 \end{bmatrix}$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{26/3}{78/9}\begin{bmatrix} -2/3 \\ 7/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

０９C**02** 【 中正80資工[3] 】

(a) Describe the Gram–Schmidt orthonormalization process to construct an orthonormal basis $\{u_1, u_2, \ldots, u_n\}$ in $\mathbb{R}^n$ starting with an arbitary basis $\{v_1, v_2, \ldots, v_n\}$ in $\mathbb{R}^n$. (5%)

(b) Construct an orthonormal basis in $\mathbb{R}^3$ starting with the basis

$$\{v_1, v_2, v_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\} \quad (5\%)$$

【參考章節】CH9定理16, 範例17

【解】(a) 同上題.

(b) $w_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$,

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1/2}{3/2}\begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1/\sqrt{6} \\ 1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}, \quad u_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

０９C**03** 【 清大85工工[1] 】

(a) Try to find an orthonormal basis for the subspace V_0 of $\mathbb{R}^4$ spanned by

$v_1 = [1 \quad 1 \quad 1 \quad -1]^T$, $v_2 = [2 \quad -1 \quad -1 \quad 1]^T$,

$v_3 = [0 \quad 3 \quad 3 \quad -3]^T$, $v_4 = [-1 \quad 2 \quad 2 \quad 1]^T$. (10%)

(b) Find an orthonormal projection P onto V_0 . (10%)

【註】 原考題將$\mathbb{R}$打成IR .

【解】(a) 經Gram–Schmidt process (細步過程讀者自解) (綜線CH9定理12)

可將v_1 , v_2 , v_3 , v_4 化成正交集如下:

$v_1' = [\,1 \quad 1 \quad 1 \quad -1\,]^T$, $v_2' = (3/4)\,[\,3 \quad -1 \quad -1 \quad 1\,]^T$,

$v_3' = [\,0 \quad 0 \quad 0 \quad 0\,]^T$, $v_4' = [\,0 \quad 1 \quad 1 \quad 2\,]^T$.

再對非零向量做單位化即得出V_0的orthonormal basis如下:

$\{ \quad u_1 = (1/\sqrt{4})[\,1 \quad 1 \quad 1 \quad -1\,]^T,$

$u_2 = (1/\sqrt{12})[\,3 \quad -1 \quad -1 \quad 1\,]^T,$

$u_4 = (1/\sqrt{6})[\,0 \quad 1 \quad 1 \quad 2\,]^T \quad \}$.

(b)

$$u_1u_1^T = \frac{1}{4}\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & -1\end{bmatrix} = \frac{1}{4}\begin{bmatrix}1 & 1 & 1 & -1\\1 & 1 & 1 & -1\\1 & 1 & 1 & -1\\-1 & -1 & -1 & 1\end{bmatrix}$$

$$u_2u_2^T = \frac{1}{12}\begin{bmatrix}3\\-1\\-1\\1\end{bmatrix}\begin{bmatrix}3 & -1 & -1 & 1\end{bmatrix} = \frac{1}{12}\begin{bmatrix}9 & -3 & -3 & 3\\-3 & 1 & 1 & -1\\-3 & 1 & 1 & -1\\3 & -1 & -1 & 1\end{bmatrix}$$

$$u_4u_4^T = \frac{1}{6}\begin{bmatrix}0\\1\\1\\2\end{bmatrix}\begin{bmatrix}0 & 1 & 1 & 2\end{bmatrix} = \frac{1}{6}\begin{bmatrix}0 & 0 & 0 & 0\\0 & 1 & 1 & 2\\0 & 1 & 1 & 2\\0 & 2 & 2 & 4\end{bmatrix}$$

令$v = [\,x \quad y \quad z \quad w\,]^T$,

$P(v) = (u_1^Tv)u_1 + (u_2^Tv)u_2 + (u_4^Tv)u_4$ (綜線CH9定理12)

$= u_1u_1^Tv + u_2u_2^Tv + u_4u_4^Tv$ (係數積改寫成矩陣乘法)

$= (\,u_1u_1^T + u_2u_2^T + u_4u_4^T\,)\,v$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ (y+z)/2 \\ (y+z)/2 \\ w \end{bmatrix}$$

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09C**04** 【 清大83工工[6] 】

Given three vectors

$[\,1 \quad 1 \quad -1\,]^T$, $[\,-1 \quad 2 \quad 2\,]^T$, and $[\,1 \quad 4 \quad 0\,]^T$.

(a) Are they linear dependent or independent ? why ? (5%)

(b) Apply Gram–Schmidt process to find an orthonormal basis for $\mathbb{R}^3$ spanned by them.(10%)

(c) Find the eigensystem of the matrix formed by the basis obtained from (b) and discuss its properties. (15%)

【分析】矩陣的特徵值(eigenvalues)及各特徵值所結合的特徵向量(eigenvectors)稱爲這矩陣的特徵系統(eigen–system). 但必須是方陣才能定義特徵值.

【勘誤】本題(b)應修正如下:

" Apply Gram–Schmidt process to find an orthonormal basis for the subspace of $\mathbb{R}^3$ spanned by them. "

【解】(a) (本小題屬於題型06A)

$$\begin{vmatrix} 1 & -1 & 1 \\ 1 & 2 & 4 \\ -1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & 3 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0.$$

∴ 此三向量爲linear dependent. (綜線CH6定理14)

(b) 令$v_1 = [\,1 \quad 1 \quad -1\,]^T$, $v_2 = [\,-1 \quad 2 \quad 2\,]^T$, $v_3 = [\,1 \quad 4 \quad 0\,]^T$.

對v_1 , v_2 , v_3 進行Gram–Schmidt process:

$u_1 = v_1 = [\,1 \quad 1 \quad -1\,]^T$,

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} - \frac{-1}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 7/3 \\ 5/3 \end{bmatrix}$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u_1／\| u_1 \| = \begin{bmatrix} 1／\sqrt{3} \\ 1／\sqrt{3} \\ -1／\sqrt{3} \end{bmatrix}, \quad u_2／\| u_2 \| = \begin{bmatrix} -2\sqrt{78} \\ 7\sqrt{78} \\ 5\sqrt{78} \end{bmatrix}.$$

$$\therefore \left\{ \begin{bmatrix} 1／\sqrt{3} \\ 1／\sqrt{3} \\ -1／\sqrt{3} \end{bmatrix}, \begin{bmatrix} -2\sqrt{78} \\ 7\sqrt{78} \\ 5\sqrt{78} \end{bmatrix} \right\}$$ 爲所求之正交單位基底.

(c) 所拼成的爲3×2矩陣，並無eigensystem.

０９C**05** 【 交大79工工[25] 】

Construct an orthonormal set from the set S_1={ w_1=(1, 1, 1), w_2=(1, 2, 1), w_3=(0, 1, 1) }

【解】1° 先造orthogonal set:

$$w_1' = w_1 = (1, 1, 1)$$

$$w_2' = w_2 - \frac{\langle w_2, w_1' \rangle}{\langle w_1', w_1' \rangle} w_1' = (1, 2, 1) - \frac{4}{3}(1, 1, 1) = \frac{1}{3}(-1, 2, -1)$$

$$w_3' = w_3 - \frac{\langle w_3, w_1' \rangle}{\langle w_1', w_1' \rangle} w_1' - \frac{\langle w_3, w_2' \rangle}{\langle w_2', w_2' \rangle} w_2'$$

$$=(0,1,1)-\frac{2}{3}(1,1,1)-\frac{1/3}{2/3}(1/3)(-1,2,-1)=(1/2)(-1,0,1)$$

2° 再造orthonormal set:

$$w_1''=w_1'/\|w_1'\|=(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})$$

$$w_2''=w_2'/\|w_2'\|=(-1/\sqrt{6},2/\sqrt{6},1/\sqrt{6})$$

$$w_3''=w_3'/\|w_3'\|=(-1/\sqrt{2},0,1/\sqrt{2})$$

０９Ｃ**06** 【 交大82資工[2](b) 】

Given matrix A with row vectors $r_1=(1, 2, 4)$, $r_2=(2, 1, 3)$,

(b) construct an orthonormal basis for $\mathbb{R}^3$ starting with r_1, r_2, and $r=(2, 5, -3)$.

(Use Gram–Schmidt process)

【解】(b) 先對r_1, r_2, r 執行正交化: (綜線CH9定理16)

$$r_1'=r_1=[1\quad 2\quad 4]$$

$$r_2'=r_2-\frac{r_2\cdot r_1'}{r_1'\cdot r_1'}r_1'$$

$$=[2\quad 1\quad 3]-\frac{16}{21}[1\quad 2\quad 4]=\frac{1}{21}[26\quad -11\quad -1]$$

$$r'=r-\frac{r\cdot r_1'}{r_1'\cdot r_1'}r_1'-\frac{r\cdot r_2'}{r_2'\cdot r_2'}r_2'$$

$$=[2\quad 5\quad -3]-0\,r_1'-0\,r_2'=[2\quad 5\quad -3]$$

再對r_1', r_2', r' 執行單位化:

$$r_1''=r_1'/\|r_1'\|=\frac{1}{\sqrt{21}}[1\quad 2\quad 4]$$

$$r_2''=r_2'/\|r_2'\|=\frac{1}{\sqrt{798}}[26\quad -11\quad -1]$$

$$r_3'' = r_3' / \| r_3' \| = \frac{1}{\sqrt{38}} [\,2 \quad 5 \quad -3\,]$$

０９Ｃ**07**【 台大79資工[5] 】

Let V = {real coefficient polynomial $f(x)$ | degree $f(x) \le 2$ } and let the inner product on $V \times V$ is defined as

$$(f(x), g(x)) = \int_0^1 f(x)g(x)dx\ ,$$

find an orthonormal basis for V.

【解】 $\{1, x, x^2\}$爲V的一組基底, 爲求orthonormal basis, 執行Gram–Schmidt process如下:

令$v_1 = 1$, $v_2 = x$, $v_3 = x^2$

$v_1' = v_1 = 1$,

$$(v_1', v_1') = \int_0^1 1dx = 1, \quad (v_2, v_1') = \int_0^1 xdx = 1/2$$

$$v_2' = v_2 - \frac{(v_2, v_1')}{(v_1', v_1')} v_1' = x - 1/2$$

$$(v_2', v_2') = \int_0^1 (x-1/2)^2 dx = 1/12\ , \quad (v_3, v_1') = \int_0^1 x^2 dx = 1/3\ .$$

$$(v_3, v_2') = \int_0^1 x^2(x-1/2)dx = \int_0^1 (x^3 - (1/2)x^2)dx = 1/12$$

$$v_3' = v_3 - \frac{(v_3, v_1')}{(v_1', v_1')} v_1' - \frac{(v_3, v_2')}{(v_2', v_2')} v_2' = x^2 - x + 1/6$$

$$(v_3', v_3') = \int_0^1 (x^2 - x + 1/6)^2 dx = 1/180$$

$v_1'' = v_1' / \| v_1' \| = 1$

$v_2'' = v_2' / \| v_2' \| = \sqrt{12}\,(x - 1/2)$

$v_3''=v_3'/\|v_3'\|=\sqrt{180}\,(x^2-x+1/6)$

$\therefore\ \{1,\ \sqrt{12}(x-1/2),\ \sqrt{180}\,(x^2-x+1/6)\}$ 爲所求

０９Ｃ**08** 【 交大78資料[1] 】

Construct orthonormal vectors which are linear combinations of

(a) (1, 1, 0), (1, 0, 1), (0, 1, 1). (9%)

(b) 1, t using $\langle f,g\rangle=\int_0^1 f(t)g(t)dt.$ (9%)

【解】 (a) 令$v_1=(1, 1, 0)$, $v_2=(1, 0, 1)$, $v_3=(0, 1, 1)$

由Gram–Schmidt process, (綜線CH9定理16)

$v_1'=v_1=(1, 1, 0)$

$$v_2'=v_2-\frac{\langle v_2, v_1'\rangle}{\langle v_1', v_1'\rangle}v_1'=(1/2, -1/2, 1)$$

$$v_3'=v_3-\frac{\langle v_3, v_1'\rangle}{\langle v_1', v_1'\rangle}v_1'-\frac{\langle v_3, v_2'\rangle}{\langle v_2', v_2'\rangle}v_2'=(-2/3, 2/3, 2/3)$$

$$v_1''=\frac{v_1'}{\|v_1'\|}=(1/\sqrt{2})\,(1, 1, 0)$$

$$v_2''=\frac{v_2'}{\|v_2'\|}=\sqrt{2/3}\,(1/2, -1/2, 1)$$

$$v_3''=\frac{v_3'}{\|v_3''\|}=(1/\sqrt{3})\,(-1, 1, 1)$$

v_1'', v_2'', v_3'' 爲所求之orthonormal vectors

(b) $v_1=1, v_2=t$ ， 依同法可得 $v_1''=1$, $v_2''=\sqrt{3}\,(t-1/2)$

詳情請參閱上題(台大79資工[5]).

０９Ｃ**09** 【 大同85資工[3] 】

Starting with the basis $\{1, x, x+x^2\}$, find an orthonormal basis for P_3 if the inner product on P_3 is defined by

$$\langle p, q \rangle = \sum_{i=1}^{3} p(x_i)q(x_i)$$

where $x_1=0$, $x_2=1$ and $x_3=2$, and P_n also denotes the set of all polynomials of degree less than n. (10%)

【解】令 $u_1=1$, $u_2=x$, $u_3=1+x^2$.

依Gram–Schmidt process 進行正交化: (綜線CH9定理16)

$v_1=u_1=1$

$\langle v_1, v_1 \rangle = \langle 1, 1 \rangle = 1\cdot 1+1\cdot 1+1\cdot 1=3$

$\langle u_2, v_1 \rangle = \langle x, 1 \rangle = 0\cdot 1+1\cdot 1+2\cdot 1=3$

$v_2= u_2-(3/3)v_1= x-1$

$\langle v_2, v_2 \rangle = \langle x-1, x-1 \rangle = (-1)\cdot(-1)+0\cdot 0+1\cdot 1=2$

$\langle u_3, v_1 \rangle = \langle x+x^2, 1 \rangle = 0\cdot 1+2\cdot 1+6\cdot 1=8$

$\langle u_3, v_2 \rangle = \langle x+x^2, x-1 \rangle = 0\cdot(-1)+2\cdot 0+6\cdot 1=6$

$v_3= u_3-(8/3)v_1-(6/2)v_2=(x+x^2)-(8/3)\cdot 1-3\cdot(x-1) = x_2-2x+1/3$

$\langle v_3, v_3 \rangle= (1/3)(1/3) +(-2/3)(-2/3)+(1/3)(1/3)=2/3$

再對v_1 , v_2 , v_3 單位化即得出orthonormal basis如下: (綜線CH9定義4)

$\{ (1/\sqrt{3}), (1/\sqrt{2})(x-1), \sqrt{3/2}(x^2-2x+1/3) \}$ #

０９C**１０**【 交大82工工[6](a) 】

Label the following statements as being true or false.

(a). The Gram–Schmidt orthogonalization process allows us to construct an orthonormal set from an arbitrary set of vectors. (T, F).

【解】(a) False.

Gram–Schmidt process 施行的對象是有限集合, 頂多可推廣到可數集合(countable set)。對不可數(uncountable)的集合無從下手。 但假如把arbitrary解釋成“不管是否獨立的”有限集合, 則本題答案爲True.

題型09D: QR分解

9D01 【中央86資工[6]】

Find a QR factorization of matrix $A=\begin{bmatrix}1 & -2 & -1\\ 2 & 0 & 1\\ 2 & -4 & 2\\ 4 & 0 & 0\end{bmatrix}$

【解】

令$a=\begin{bmatrix}1\\2\\2\\4\end{bmatrix}$, $b=\begin{bmatrix}-2\\0\\-4\\0\end{bmatrix}$, $c=\begin{bmatrix}-1\\1\\2\\0\end{bmatrix}$

對a, b, c作Gram–Schmidt process: (綜線CH9定理16)

$a'=a=\begin{bmatrix}1\\2\\2\\4\end{bmatrix}$. $\|a'\|=\sqrt{25}=5$

$$b'=b-\frac{b\cdot a'}{a'\cdot a'}a'=b-\frac{-10}{25}a'=b+\frac{2}{5}a'=\begin{bmatrix}-8/5\\4/5\\-16/5\\8/5\end{bmatrix}$$

$\| b' \| = 4$

$$c' = c - \frac{c \cdot a'}{a' \cdot a'}a' - \frac{c \cdot b'}{b' \cdot b'}b' = c - \frac{5}{25}a' - \frac{-4}{16}b' = \begin{bmatrix} -8/5 \\ 4/5 \\ 4/5 \\ -2/5 \end{bmatrix}$$

$\| c' \| = 2$

$$\begin{cases} a = a' = 5a'' \\ b = (-2/5)a' + b' = -2a'' + 4b'' \\ c = (1/5)a' - (1/4)b' + c' = a'' - b'' + 2c'' \end{cases}$$

$$\therefore A = \begin{bmatrix} a'' & b'' & c'' \end{bmatrix} \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$ (綜線CH2定理6)

$$取\ Q = \begin{bmatrix} a'' & b'' & c'' \end{bmatrix} = \begin{bmatrix} 1/5 & -2/5 & -4/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & -4/5 & 2/5 \\ 4/5 & 2/5 & -1/5 \end{bmatrix}$$

$$R = \begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

#

０９D**02** 【 大同86資工[4] 】

Let

$$A=\begin{bmatrix}1 & -2 & -1\\ 2 & 0 & 1\\ 2 & -4 & 2\\ 4 & 0 & 0\end{bmatrix}$$

factor A into $A=QR$ where Q is a 4×3 matrix with orthonormal columns and R is a 3×3 upper triangular matrix. (6%)

【解】本題與上題(中央86資工[6])的數據完全相同.

０９D**03** 【 中央84資工[6] 】

Find a QR factorization of matrix $\begin{bmatrix}1 & 2 & 5\\ -1 & 1 & -4\\ -1 & 4 & -3\\ 1 & -4 & 7\\ 1 & 2 & 1\end{bmatrix}$

【解】

令$a=\begin{bmatrix}1\\ -1\\ -1\\ 1\\ 1\end{bmatrix}$, $b=\begin{bmatrix}2\\ 1\\ 4\\ -4\\ 2\end{bmatrix}$, $c=\begin{bmatrix}5\\ -4\\ -3\\ 7\\ 1\end{bmatrix}$

對a, b, c, d作Gram–Schmidt process: (綜線CH9定理16)

$a'=a=[1, -1, -1, 1, 1]^T$

$\| a' \| = \sqrt{5}$

$$b' = b - \frac{b \cdot a'}{a' \cdot a'}\, a' = b - \frac{-5}{5} a' = b + a' = [\,3, 0, 3, -3, 3\,]^T$$

$$\| b' \| = 6$$

$$c' = c - \frac{c \cdot a'}{a' \cdot a'}\, a' - \frac{c \cdot b'}{b' \cdot b'}\, b' = c - 4a' + (1/3)b' = [\,2, 0, 2, 2, -2\,]^T$$

$$\| c' \| = 4 .$$

令$a'' = \dfrac{a'}{\| a' \|},\ b'' = \dfrac{b'}{\| b' \|},\ c'' = \dfrac{c'}{\| c' \|}$,

則有

$$\begin{cases} a = a' & = \sqrt{5}\, a'' \\ b = -a' + b' & = -\sqrt{5}\, a'' + 6b'' \\ c = 4a' - (1/3)b' + c' & = 4\sqrt{5}\, a'' - 2b'' + 4c'' \end{cases}$$

$$\therefore\ A = \begin{bmatrix} a'' & b'' & c'' \end{bmatrix} \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\text{取 } Q = \begin{bmatrix} a'' & b'' & c'' \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & 1/2 & 1/2 \\ -1/\sqrt{5} & 0 & 0 \\ -1/\sqrt{5} & 1/2 & 1/2 \\ 1/\sqrt{5} & -1/2 & 1/2 \\ 1/\sqrt{5} & 1/2 & -1/2 \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

０９D**04** 【 元智84工工X[3] 】

令$A=\begin{bmatrix}1&1&2\\1&2&3\\1&1&1\end{bmatrix}$，試問$A$可否被QR分解(QR–factorization), 若可,

求Q與R. (即$A\overset{?}{=}QR$, 其中Q是實直交矩陣(real orthogonal matrix) R是一矩陣其下三角形部份皆為0).

【解】可, 求算如下:

經Gram–Schmidt process 得 (讀者參考綜線CH9定理19自解)

$$A=\begin{bmatrix}1&-1/3&1/2\\1&2/3&0\\1&-1/3&-1/2\end{bmatrix}\begin{bmatrix}1&4/3&2\\0&1&3/2\\0&0&1\end{bmatrix}$$

左因子各行的長度為$\sqrt{3}$, $\sqrt{2/3}$, $\sqrt{1/2}$.

$$A=\begin{bmatrix}1/\sqrt{3}&-1/\sqrt{6}&1/\sqrt{2}\\1/\sqrt{3}&2/\sqrt{6}&0\\1/\sqrt{3}&-1/\sqrt{6}&-1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{3}&0&0\\0&\sqrt{2/3}&0\\0&0&\sqrt{1/2}\end{bmatrix}\begin{bmatrix}1&4/3&2\\0&1&3/2\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}1/\sqrt{3}&-1/\sqrt{6}&1/\sqrt{2}\\1/\sqrt{3}&2/\sqrt{6}&0\\1/\sqrt{3}&-1/\sqrt{6}&-1/\sqrt{2}\end{bmatrix}\begin{bmatrix}\sqrt{3}&4/\sqrt{3}&2\sqrt{3}\\0&\sqrt{2/3}&\sqrt{3/2}\\0&0&\sqrt{1/2}\end{bmatrix}$$

０９D**05** 【 元智84工工Y[3] 】

Apply the Gram–Schmidt process to

$$a=\begin{bmatrix}0\\0\\1\end{bmatrix},\quad b=\begin{bmatrix}0\\1\\1\end{bmatrix},\quad c=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

and write the result in the form $A=QR$.

【解】(細節略, 計算過程請參閱下題)

$$Q=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix},\qquad R=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}.$$

０９D**06** 【 元智82工工[5] 】

Let $A=\begin{bmatrix}1&1&1\\0&1&-1\\1&1&0\\0&0&1\end{bmatrix}$. Find the QR–decomposition of A, that is,

find a 4×3 matrix Q with orthonormal column vectors and an upper triangular 3×3 matrix R such that $A=QR$.

【解】1° 令

$$a_1=\begin{bmatrix}1\\0\\1\\0\end{bmatrix},\ a_2=\begin{bmatrix}1\\1\\1\\0\end{bmatrix},\ a_3=\begin{bmatrix}1\\-1\\0\\1\end{bmatrix},$$

對 a_1 , a_2 , a_3 進行Gram–Schmidt process: (綜線CH9定理16)

$a_1'=a_1$, ...(甲)

$\langle a_1', a_1'\rangle = 2$, $\langle a_2, a_1'\rangle = 2$

$a_2' = a_2 - (2/2)a_1'$ ....(乙)

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$\langle a_2', a_2'\rangle = 1$, $\langle a_3, a_1'\rangle = 1$, $\langle a_3, a_2'\rangle = -1$

$a_3' = a_3 - (1/2)a_1' - (-1/1)a_2'$ ....(丙)

$$= \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - (1/2)\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 1 \end{bmatrix}$$

此時{ a_1', a_2', a_3' }已是正交集, 接下來再做單位化:

$\langle a_3', a_3'\rangle = 3/2$

$$a_1'' = a_1'/\|a_1'\| = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$a_2'' = a_2'/\|a_2'\| = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$a_3'' = a_3' / \| a_3' \| = \begin{bmatrix} 1/\sqrt{6} \\ 0 \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$$

$\{ a_1'', a_2'', a_3'' \}$ 即爲所求.

2° 由以上(甲)(乙)(丙)三式移項得:

$$\begin{cases} a_1 = a_1' \\ a_2 = a_1' + a_2' \\ a_3 = (1/2)a_1' - a_2' + a_3' \end{cases}$$

$$\therefore \quad a_1 = \begin{bmatrix} a_1' & a_2' & a_3' \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$ (左直切,CH2定理6)

$$a_2 = \begin{bmatrix} a_1' & a_2' & a_3' \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$ (左直切,CH2定理6)

$$a_3 = \begin{bmatrix} a_1' & a_2' & a_3' \end{bmatrix} \begin{bmatrix} 1/2 \\ -1 \\ 1 \end{bmatrix}$$ (左直切,CH2定理6)

$$\therefore \quad \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} = \begin{bmatrix} a_1' & a_2' & a_3' \end{bmatrix} \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(右直切,CH2定理6)

$$= \begin{bmatrix} a_1'' & a_2'' & a_3'' \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{3/2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(綜線CH3定理24)

$$= \begin{bmatrix} a_1'' & a_2'' & a_3'' \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2}/2 \\ 0 & 1 & -1 \\ 0 & 0 & \sqrt{3/2} \end{bmatrix}$$ (綜線CH3定理15)

$$\therefore \quad Q = \begin{bmatrix} a_1'' & a_2'' & a_3'' \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{6} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{6} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$R = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2/2} \\ 0 & 1 & -1 \\ 0 & 0 & \sqrt{3/2} \end{bmatrix}$$

#

０９D07 【 交大84資料[6] 】

Let $v_1 = [1\ 2\ 2\ 1]^T$, $v_2 = [1\ 0\ 2\ 0]^T$, and $v_3 = [2\ 0\ 4\ -3]^T$.

(a) Find an orthonormal basis for the vector space spanned by the vector v_1 , v_2 , and v_3 .(5%)

(b)Find the transformation matrix from the coordinate system $B = \{v_1 , v_2 , v_3\}$ to the coordinate system formed by the orthonormal basis in (a). (5%)

【分析】本題(a)屬於題型09C. (b)屬於題型06D. 整個是QR分解的變形.

【解】(a)

令 $a=\begin{bmatrix}1\\2\\2\\1\end{bmatrix}$, $b=\begin{bmatrix}1\\0\\2\\0\end{bmatrix}$, $c=\begin{bmatrix}2\\0\\4\\-3\end{bmatrix}$

對a, b, c作Gram–Schmidt process: (綜線CH9定理16)

$$a'=a=\begin{bmatrix}1\\2\\2\\1\end{bmatrix}$$

$$\|a'\|=\sqrt{10}$$

$$b'=b-\frac{b\cdot a'}{a'\cdot a'}a'=b-\frac{1}{2}a'=\begin{bmatrix}1\\2\\2\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}1\\0\\2\\0\end{bmatrix}=\begin{bmatrix}1/2\\-1\\1\\-1/2\end{bmatrix}$$

$$\|b'\|=\sqrt{5/2}$$

$$c'=c-\frac{c\cdot a'}{a'\cdot a'}a'-\frac{c\cdot b'}{b'\cdot b'}b'=c-\frac{7}{10}a'-\frac{13}{5}b'$$

$$=\begin{bmatrix}2\\0\\4\\-3\end{bmatrix}-\frac{7}{10}\begin{bmatrix}1\\2\\2\\1\end{bmatrix}-\frac{13}{5}\begin{bmatrix}1/2\\-1\\1\\-1/2\end{bmatrix}=\begin{bmatrix}0\\6/5\\0\\-12/5\end{bmatrix}$$

$\| c' \| = \sqrt{36/5}$

令$a''=\dfrac{a'}{\| a' \|}$, $b''=\dfrac{b'}{\| b' \|}$, $c''=\dfrac{c'}{\| c' \|}$.

則所求爲 $\{ a'', b'', c'' \}$

$$=\left\{ \frac{1}{\sqrt{10}}\begin{bmatrix}1\\2\\2\\1\end{bmatrix}, \sqrt{\frac{2}{5}}\begin{bmatrix}1/2\\-1\\1\\-1/2\end{bmatrix}, \sqrt{\frac{5}{36}}\begin{bmatrix}0\\6/5\\0\\-12/5\end{bmatrix} \right\}$$

(b) 由(a):

$$\begin{cases} a=a'=\sqrt{10}a'' \\ b=\dfrac{1}{2}a'+b'=\dfrac{\sqrt{10}}{2}a''+\dfrac{\sqrt{10}}{2}b'' \\ c=\dfrac{7}{10}a'+\dfrac{13}{5}b'+c'=\dfrac{7\sqrt{10}}{10}a''+\dfrac{13\sqrt{10}}{10}b''+\dfrac{6\sqrt{5}}{5}c'' \end{cases}$$

$$\therefore \text{所求爲}\begin{bmatrix} \sqrt{10} & \sqrt{10}/2 & 7\sqrt{10}/10 \\ 0 & \sqrt{10}/2 & 13\sqrt{10}/10 \\ 0 & 0 & 6\sqrt{5}/5 \end{bmatrix}$$

#

０９D**08** 【 台大78資工[4] 】

Let

$$D=\begin{bmatrix}1&2&1\\2&1&1\\1&0&0\\0&1&0\\0&0&2\end{bmatrix}$$

Find a 3×3 matrix E such that the matrix $(DE)^T(DE)=I$. Where I be 3×3 identity matrix and A^T be the transpose of A.

【分析】(1) 對實數矩陣Q, $Q^TQ=I$表示Q的行形成正交單位集. (綜線CH13定理3)

(2) 須了解Gram–Schmidt process的矩陣形 (綜線CH9定理19)

(3) 主要核心觀念爲直切原理 (綜線CH2定理6)

【解】 令$Q=DE$, 由$Q^TQ=I$ 可知要求Q的各行爲orthonormal set.

現對D的各行施行Gram–Schmidt process: (綜線CH9定理16)

$$令d_1=\begin{bmatrix}1\\2\\1\\0\\0\end{bmatrix}, d_2=\begin{bmatrix}2\\1\\0\\1\\0\end{bmatrix}, d_3=\begin{bmatrix}1\\1\\0\\0\\2\end{bmatrix}$$

$$d_1'=d_1=\begin{bmatrix}1\\2\\1\\0\\0\end{bmatrix}$$

$$d_2' = d_2 - \frac{d_2 \cdot d_1'}{d_1' \cdot d_1'} d_1' = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/3 \\ -1/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix} \quad (= \frac{1}{3} \begin{bmatrix} 4 \\ -1 \\ -2 \\ 3 \\ 0 \end{bmatrix})$$

$$d_3' = d_3 - \frac{d_3 \cdot d_1'}{d_1' \cdot d_1'} d_1' - \frac{d_3 \cdot d_2'}{d_2' \cdot d_2'} d_2'$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{10} \begin{bmatrix} 4/3 \\ -1/3 \\ -2/3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/10 \\ 1/10 \\ -3/10 \\ -3/10 \\ 2 \end{bmatrix} (= \frac{1}{10} \begin{bmatrix} 1 \\ 1 \\ -3 \\ -3 \\ 20 \end{bmatrix})$$

$$\| d_1' \| = \sqrt{6}, \quad \| d_2' \| = \frac{1}{3}\sqrt{30}, \| d_3' \| = \frac{1}{10}\sqrt{420}$$

$$令 d_1'' = \frac{d_1}{\| d_1' \|} = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \\ 0 \end{bmatrix}$$

$$令 d_2'' = \frac{d_2}{\| d_2' \|} = \begin{bmatrix} 4/\sqrt{30} \\ -1/\sqrt{30} \\ -2/\sqrt{30} \\ 3/\sqrt{30} \\ 0 \end{bmatrix}$$

$$令 d_3'' = \frac{d_3}{\| d_3' \|} = \begin{bmatrix} 1/\sqrt{420} \\ 1/\sqrt{420} \\ -3/\sqrt{420} \\ -3/\sqrt{420} \\ 20/\sqrt{420} \end{bmatrix}$$

則 $\{ d_1'' , d_2'' , d_3'' \}$形成orthonormal set

$$令 Q = \begin{bmatrix} d_1'' , d_2'' , d_3'' \end{bmatrix} ,\ 則 Q^T Q = I .$$ (綜線CH13定理3)

$$d_1'' = \frac{1}{\| d_1' \|} d_1 = \frac{1}{\sqrt{6}} D \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = D \begin{bmatrix} 1/\sqrt{6} \\ 0 \\ 0 \end{bmatrix}$$ (左直切原理)

$$d_2'' = \frac{1}{\| d_2' \|} d_2' = \frac{3}{\sqrt{30}} \left(d_2 - \frac{2}{3} d_1' \right) = \frac{3}{\sqrt{30}} \left(-\frac{2}{3} d_1 + d_2 \right)$$

$$=\frac{3}{\sqrt{30}}D\begin{bmatrix}-2/3\\1\\0\end{bmatrix}=D\begin{bmatrix}-2/\sqrt{30}\\3/\sqrt{30}\\0\end{bmatrix}$$ (左直切原理)

$$d_3''=\frac{1}{\|d_3'\|}d_3'=\frac{10}{\sqrt{420}}\left(d_3-\frac{1}{2}d_1'-\frac{3}{10}d_2'\right)$$

$$=\frac{10}{\sqrt{420}}\left(d_3-\frac{1}{2}d_1-\frac{3}{10}(d_2-\frac{2}{3}d_1)\right)$$

$$=\frac{10}{\sqrt{420}}\left(\frac{-3}{10}d_1+\frac{-3}{10}d_2+d_3\right)$$

$$=\frac{10}{\sqrt{420}}D\begin{bmatrix}-3/10\\-3/10\\1\end{bmatrix}=D\begin{bmatrix}-3/\sqrt{420}\\-3/\sqrt{420}\\10/\sqrt{420}\end{bmatrix}$$ (左直切原理)

$$\therefore\ Q=D\begin{bmatrix}1/\sqrt{6}&-2/\sqrt{30}&-3/\sqrt{420}\\0&3/\sqrt{30}&-3/\sqrt{420}\\0&0&10/\sqrt{420}\end{bmatrix}$$ (右直切原理)

令$E=\begin{bmatrix}1/\sqrt{6}&-2/\sqrt{30}&-3/\sqrt{420}\\0&3/\sqrt{30}&-3/\sqrt{420}\\0&0&10/\sqrt{420}\end{bmatrix}$ 即可.

０９D**09**【 交大80資工[3](c) 】

Let A be a real $n\times n$ matrix. Show that

(c) (4%) Suppose A has rank $=n$, and that $A=QR$,where Q is $n\times n$ and has orthonormal columns, and R is a $n\times n$ upper-triangular matrix. How do you solve $Ax=b$ using $A=QR$? Show explicitly your steps.

【解】 (c) ∵ Q具有orthonormal columns

∴ $Q^TQ=I$. (綜線CH13定理3)

$Ax=b$ 即 $QRx=b$, 等式兩邊左乘Q^H，即得

即$Rx=Q^Hb$

$$設R=\begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ & r_{22} & \cdots & r_{2n} \\ \bigcirc & & \ddots & \\ & & & r_{nn} \end{bmatrix}\quad x=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},\ Q^Hb=\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$則\ \begin{cases} r_{11}x_1+r_{12}x_2+\cdots+r_{1n}x_n=c_1 \\ r_{22}x_2+\cdots+r_{2n}x_n=c_2 \\ \cdots\cdots \\ r_{nn}x_n=c_n \end{cases}$$

於是可由最後一式解出x_n，再代入倒數第二式解出x_{n-1}，再將x_n,x_{n-1}代入倒數第二式解出x_{n-2}，依此法一步步解出x的各個分量.

【討論】本題其實是最小平方問題. (綜線CH9定理21a)

由於本題假設rank$A=n$，於是 CSP$A=\mathbb{R}^{n\times 1}$. 所以$Ax=b$ 一定有解.

０９D**１０**【 清大78工工[7] 】

Suppose that A is real and $p\times q$ with rank k, and that $A=QR$,where Q is $p\times k$ and has orthonormal columns while R is $k\times q$, upper triangular, and of rank k. Prove that the columns of Q form an orthonormal basis for the column space of A and that $P=QQ^T$ represent orthogonal projection onto the column space of A.

【解】 以下對任意矩陣X，以CSPX表示X的column space，並以X_j 表示X的第j行.

1°∵ Q的各行Q_1, Q_2, ..., Q_k 是正交單位集, (已知)

$\therefore$ $Q_1, Q_2, \ldots, Q_k$ 線性獨立. (綜線CH9定理15)

2° $Q_1, Q_2, \ldots, Q_k$ 生成CSPQ (綜線CH5定義16, CH6定義1)

再由1°得知$Q_1, Q_2, \ldots, Q_k$爲CSPQ的orthonormal basis

3° $\because$ $A = QR$, $\therefore$ CSP$A \subseteq$ CSPQ . (綜線CH5定理21a)

又 $\dim(\text{CSP}A) = \text{rank}A = k$ (已知)

$= \dim(\text{CSP}Q)$ (由2°)

$\therefore$ CSPA = CSPQ (綜線CH6定理22a)

$Q_1, Q_2, \ldots, Q_k$ 形成CSPA的orthonormal basis.

4° $\forall v \in \mathbb{R}^{p\times 1}$,

$$Pv = \sum_{i=1}^{k} (Q_i^T v)\, Q_i$$ (綜線CH9定理12)

$$= \sum_{i=1}^{k} Q_i (Q_i^T v)$$ ($\because Q^T v$是1×1矩陣)

$$= \sum_{i=1}^{k} (Q_i Q_i^T) v = \left(\sum_{i=1}^{k} Q_i Q_i^T \right) v$$ (矩陣乘法結合律及分配律)

$$= QQ^T v$$ (綜線CH2定理8)

$\therefore$ $P = QQ^T$

題型09E: 最小平方問題

０９E**01** 【 清大83資料[4] 】

Let $A=\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}$, $b=\begin{bmatrix}1\\1\\0\end{bmatrix}$

What is the least squares solution of $Ax=b$? (5%)

【解】

$$A^TA=\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1&0\\0&1\\1&1\end{bmatrix}=\begin{bmatrix}2&1\\1&2\end{bmatrix}$$

$$A^Tb=\begin{bmatrix}1&0&1\\0&1&1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

解 $A^TAx=A^Tb$， 得 $x=\begin{bmatrix}1/3\\1/3\end{bmatrix}$

０９E**02** 【 交大85資料[5] 】

[複選題]

In the following, choose each one which is a solution of the least–squares problem $Ax\approx Y$, where

$$A=\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix} \text{ and } Y=\begin{bmatrix}2\\ 8\end{bmatrix}$$

(a) $X=\begin{bmatrix}2\\ 2\end{bmatrix}$, (b) $X=\begin{bmatrix}3.5\\ 0.2\end{bmatrix}$, (c) $X=\begin{bmatrix}2\\ 1.5\end{bmatrix}$,

(d) none of (a), (b) and (c).

【解】選(d).

【說明】

$$A^{T}A=\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}\begin{bmatrix}1 & 1\\ 2 & 2\end{bmatrix}=\begin{bmatrix}5 & 5\\ 5 & 5\end{bmatrix},\quad A^{T}y=\begin{bmatrix}1 & 2\\ 1 & 2\end{bmatrix}\begin{bmatrix}2\\ 8\end{bmatrix}=\begin{bmatrix}18\\ 18\end{bmatrix}$$

解 $A^{T}Ax=A^{T}y$:

得 $x=\begin{bmatrix}18/5-t\\ t\end{bmatrix}$ #

(a),(b),(c)皆不合.

０９E**03**【 交大86資工[6](d) 】

[是非倒扣題]

A linear system has a least–squares solution only if the number of equations is greater than the number of unknowns.

【解】False. (綜線CH9定理21a)

０９E**04**【 交大81資工[4](c) 】

True or false, two points for each.

(c) For a general $m\times n$ matrix A with $m>n$, the least square problem $Ax=b$ has a unique solution.

【分析】$Ax=b$ 的最小平方問題是解 $A^{H}Ax=A^{H}b$, (綜線CH9定理21a)

(實數情形變成解 $A^{T}Ax=A^{T}b$)

這個方程式必有解, (綜線CH9定理21a要訣4)

而解唯一的充份必要條件是 $\ker(A^{T}A)=\{o\}$ (綜線CH3定理2)

亦即 $\ker(A)=\{o\}$ (綜線CH9定理20)

【解】(c) False. 反例如下:

令 $A=\begin{bmatrix}1&1\\2&2\\3&3\end{bmatrix}$, $b=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$A^{T}A=\begin{bmatrix}1&2&3\\1&2&3\end{bmatrix}\begin{bmatrix}1&1\\2&2\\3&3\end{bmatrix}=\begin{bmatrix}14&14\\14&14\end{bmatrix}$$

解方程式 $A^{T}Ax=A^{T}b$ (綜線CH9定理21a)

通解爲 $x=\begin{bmatrix}t\\-t\end{bmatrix}$, t爲任意常數

∴ 解並不唯一.

０９E**05** 【 大同82資工[14] 】

Let $Ax=b$ represent a system of linear equations. Which of the following methods is used to find its least square solution $\hat{x}$:

(a) $\hat{x}=A^{-1}b$; (b) $\hat{x}=A^{T}(AA^{T})^{-1}b$;

(c) $\hat{x}=(A^{T}A)^{-1}b$; (d) $\hat{x}=(A^{T}A)^{-1}A^{T}b$.

【解】 選(d) (綜線CH9定理22)

０９E**06** 【 清大85工工[2] 】

(a) Let $A = QR$ be a normalized QR–decomposition of the $p \times q$ matrix A. Show that all solutions to the least–square problem of finding x to min $\| Ax-y \|_2$ can be obtained by applying back–substitution to solve $Rx = Q^T y$. (10%)

(b) Using the result of (a) to solve the least–square problem:

$$Ax \approx y = [1 \quad -1 \quad 2 \quad 1]^T$$

when $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 1 & -1 & 3 & 2 \\ 1 & -1 & 3 & 2 \\ -1 & 1 & -3 & 1 \end{bmatrix}$ (10%)

【解】(a) 本題考定理證明, 請參閱綜線CH9定理21a, 此處不再重複.

(b) 由題意可知必須做QR分解.

依Gram–Schmidt process可得Q_0R_0 分解如下(細節讀者自解)

$$A = \begin{bmatrix} 1 & 9/4 & 0 & 0 \\ 1 & -3/4 & 0 & 1 \\ 1 & -3/4 & 0 & 1 \\ -1 & 3/4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1/4 & 9/4 & 1/2 \\ 0 & 1 & -1 & -2/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(刪除Q_0 的第三行及R_0 的第三列得)

$$= \begin{bmatrix} 1 & 9/4 & 0 \\ 1 & -3/4 & 1 \\ 1 & -3/4 & 1 \\ -1 & 3/4 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1/4 & 9/4 & 1/2 \\ 0 & 1 & -1 & -2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(再做normalization)

$$= \begin{bmatrix} 1/2 & 3/(2\sqrt{3}) & 0 \\ 1/2 & -1/(2\sqrt{3}) & 1/\sqrt{6} \\ 1/2 & -1/(2\sqrt{3}) & 1/\sqrt{6} \\ -1/2 & 1/(2\sqrt{3}) & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3\sqrt{3}/2 & 0 \\ 0 & 0 & \sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & -1/4 & 9/4 & 1/2 \\ 0 & 1 & -1 & -2/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & 3/(2\sqrt{3}) & 0 \\ 1/2 & -1/(2\sqrt{3}) & 1/\sqrt{6} \\ 1/2 & -1/(2\sqrt{3}) & 1/\sqrt{6} \\ -1/2 & 1/(2\sqrt{3}) & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & -1/2 & 9/2 & 1 \\ 0 & 3\sqrt{3}/2 & -3\sqrt{3}/2 & -\sqrt{3} \\ 0 & 0 & 0 & \sqrt{6} \end{bmatrix}$$

#

$Rx = Q^T y$ 乘開, 即

$$\begin{cases} 2x_1 - (1/2)x_2 + (9/2)x_3 + x_4 = 1/2 \\ (3\sqrt{3}/2)x_2 - (3\sqrt{3}/2)x_3 - \sqrt{3}\,x_4 = \sqrt{3}/2 \\ \sqrt{6}\,x_4 = 3/\sqrt{6} \end{cases}$$

依back–substitution: 先解得$x_4 = 1/2$.

代回解得 $x_3 = t$, $x_2 = (2/3) + t$.

再代回解得 $x_1 = (1/6) - 2t$.

∴ $x = [\ (1/6)-2t,\quad (2/3)+t,\quad t,\quad 1/2\ ]^T$,

此爲最小平方問題的的通解. #

$\|x\|^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2 = (1/4) + t^2 + (2/3+t)^2 + (1/6-2t)^2$

由 $0 = \dfrac{d\|x\|^2}{dt} = 12t + 2/3$, 解得 $t = -1/18$

∴ 最佳解爲 $x = [\ 5/18 \quad 11/18 \quad -1/18 \quad 1/2\ ]^T$.

０９E**07** 【 交大84工工[2] 】

(a) Let $A = \begin{bmatrix} 1 & -2 & -1 \\ 2 & 0 & 1 \\ 2 & -4 & 2 \\ 4 & 0 & 0 \end{bmatrix}$, factor A into a product QR, where Q

has an orthonormal set of column vectors and R is upper triangular. (10%)

(b) Use the results from part (a) to find the least squares solution

to $Ax = b$, where $b = \begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}$. (10%)

【解】(a) (請參閱題型09D)

對A的行做Gram–Schmidt process，可得出A的Q_0R_0分解:

$$A = \begin{bmatrix} 1 & -8/5 & -8/5 \\ 2 & 4/5 & 4/5 \\ 2 & -16/5 & 4/5 \\ 4 & 8/5 & -2/5 \end{bmatrix} \begin{bmatrix} 1 & -2/5 & 1/5 \\ 0 & 1 & -1/4 \\ 0 & 0 & 1 \end{bmatrix}$$ (綜線CH9定理19)

上式Q_0 的三個行的長度分別是 5, 4, 2，調整如下:

$$A = \begin{bmatrix} 1/5 & -2/5 & -4/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & -4/5 & 2/5 \\ 4/5 & 2/5 & -1/5 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2/5 & 1/5 \\ 0 & 1 & -1/4 \\ 0 & 0 & 1 \end{bmatrix}$$ (綜線CH3範例23a)

$$=\begin{bmatrix} 1/5 & -2/5 & -4/5 \\ 2/5 & 1/5 & 2/5 \\ 2/5 & -4/5 & 2/5 \\ 4/5 & 2/5 & -1/5 \end{bmatrix}\begin{bmatrix} 5 & -2 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & 2 \end{bmatrix}=QR$$ (綜線CH3範例14b)

(b) least–square problem在已有QR分解時應解$Rx=Q^{T}b$ (綜線CH9定理21a)

$$Q^{T}b=\begin{bmatrix} 1/2 & 2/5 & 2/5 & 4/5 \\ -2/5 & 1/5 & -4/5 & 2/5 \\ -4/5 & 2/5 & 2/5 & -1/5 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -2 \end{bmatrix}=\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\left[R \mid Q^{T}b\right]\sim\left[\begin{array}{ccc|c} 5 & -2 & 1 & -1 \\ 0 & 4 & -1 & -1 \\ 0 & 0 & 2 & 2 \end{array}\right]\sim\ldots\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2/5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$\therefore$ least square solution 爲 $\begin{bmatrix} -2/5 \\ 0 \\ 1 \end{bmatrix}$ #

０９E**08** 【 交大81資工[5] 】

(RS(A): the column space of A. NS(A) the null space of A.)

Solve the least square problem $Ax=b$ by the following steps, where

$$A=\begin{bmatrix}1&1&2\\1&2&3\\1&2&1\\1&1&6\end{bmatrix},\quad b=\begin{bmatrix}3\\-4\\2\\1\end{bmatrix}$$

(a) (4%) Derive an orthonormal basis for RS(A) using Gram–Schmidt method.

(b) (2%) Factor A into $A=QR$, where Q has an orthonormal set of columns and R is upper triangular.

(c) (2%) Solve explicitly $Ax=b$.

【分析】RS通常是表示row space, 但此考題定義RS(A)為A的column space.

本題(c)的方程式其實是無解(讀者自試!). 這題應是求最小平方解.

【解】(a) (細節略, 計算過程請參閱題型09D)

$$\text{令}\quad a_1=\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\ a_2=\begin{bmatrix}1\\2\\2\\1\end{bmatrix},\ a_3=\begin{bmatrix}2\\3\\1\\6\end{bmatrix},$$

對 a_1, a_2, a_3 進行Gram–Schmidt process, 再做單位化得出orthonormal set:

$$a_1''=\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix},\quad a_2''=\begin{bmatrix}-1/2\\1/2\\1/2\\-1/2\end{bmatrix},\quad a_3''=\begin{bmatrix}-2/\sqrt{10}\\1/\sqrt{10}\\-1/\sqrt{10}\\2/\sqrt{10}\end{bmatrix}$$

$\{a_1'', a_2'', a_3''\}$ 即為所求.

(b) (細節略, 計算過程請參閱題型09D)

$$Q=\begin{bmatrix} 1/2 & -1/2 & -2/\sqrt{10} \\ 1/2 & 1/2 & 1/\sqrt{10} \\ 1/2 & 1/2 & -1/\sqrt{10} \\ 1/2 & -1/2 & 2/\sqrt{10} \end{bmatrix},\ R=\begin{bmatrix} 2 & 3 & 6 \\ 0 & 1 & -2 \\ 0 & 0 & \sqrt{10} \end{bmatrix}$$

(c) 以 $Rx=Q^{T}b$ 求解: (綜線CH9定理21a)

$$Q^{T}b=\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -2/\sqrt{10} & 1/\sqrt{10} & -1/\sqrt{10} & 2/\sqrt{10} \end{bmatrix}\begin{bmatrix} 3 \\ -4 \\ 2 \\ 1 \end{bmatrix}=\begin{bmatrix} 1 \\ -3 \\ -\sqrt{10} \end{bmatrix}$$

$$\left[R \mid Q^{T}b\right]=\left[\begin{array}{ccc|c} 2 & 3 & 6 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & \sqrt{10} & -\sqrt{10} \end{array}\right]\sim\ldots\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\therefore\quad x=\begin{bmatrix} 11 \\ -5 \\ -1 \end{bmatrix}$ 為所求.

(c) 另解（若因無(a)(b)小題而未做成QR分解, 本小題用下述解法較易筆算:)
解 $A^{T}Ax=A^{T}b$:

$$A^{T}A=\begin{bmatrix} 4 & 6 & 12 \\ 6 & 10 & 16 \\ 12 & 16 & 50 \end{bmatrix},\quad A^{T}b=\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 6 & 12 & 2 \\ 6 & 10 & 16 & 0 \\ 12 & 16 & 50 & 2 \end{array}\right] \sim \ldots\ldots \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$\therefore\ x = \begin{bmatrix} 11 \\ -5 \\ -1 \end{bmatrix}$ 為所求.

０９E**09** 【 清大80工工[3] 】

Find a normalized QR-decomposition for A and use it to solve the least-squares problem $Ax \approx y$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } y = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$$

【參考章節】CH9定理19, CH9定理21a.

【解】1°對A的行做Gram–Schmidt process, 並求得A的QR分解: (過程略, 見題型09D)

$$Q = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 2/\sqrt{6} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} \end{bmatrix}, \quad R = \begin{bmatrix} \sqrt{6} & \sqrt{6}/2 \\ 0 & 3\sqrt{2}/2 \end{bmatrix}.$$

2° $Ax \approx y$可由$Rx = Q^t b$解出: (綜線CH9定理21a)

即解

$$\begin{bmatrix} \sqrt{6} & \sqrt{6}/2 \\ 0 & 3\sqrt{2}/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{bmatrix} 8/\sqrt{6} \\ \sqrt{2} \end{bmatrix}$$

可解出 $x_2=2/3,\quad x_1=1.$

０９E**１０** 【 大同85資工[4] 】

(a) Derive the least squares solution to an overdetermined $m\times n$ system $Ax=b$ with $m>n$. (5%)

(b) Find the best quadratic least squares fit to the data

x	1	2	3	4
y	3	2	4	4

(5%)

【解】(a) 欲解x使 $\|Ax-b\|$儘小

就是要使Ax成為b在CSPA上的正投影. (綜線CH9定理13)

即 $\forall w\in \text{CSP}A, \langle b-Ax, w\rangle=0$ (綜線CH9定義11)

即 $\forall u, \langle b-Ax, Au\rangle=0$ (綜線CH5定理17①)

即 $\forall u, (Au)^*(b-Ax)=0$ (綜線CH9定義1)

(A為實數矩陣時, A^*為A的轉置; A為複數矩陣時, A^*為A的共軛轉置)

即 $\forall u, u^*A^*(b-Ax)=0$ (綜線CH10定理11)

即 $A^*(b-Ax)=0$ (綜線CH9範例10)

即 $A^*Ax=A^*b$

解此方程式即得出least square solution.

當A的行線性獨立時, A^*A可逆, (綜線CH9定理21)

$\therefore\ x=(A^*A)^{-1}A^*b$

(b) 設所求方程式為 $y=a+bx+cx^2$

由所給數據得:

$$\begin{cases} 3 = a + b + c \\ 2 = a + 2b + 4c \\ 4 = a + 3b + 9c \\ 4 = a + 4b + 16c \end{cases}$$

即
$$\begin{bmatrix}1&1&1\\1&2&4\\1&3&9\\1&4&16\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}3\\2\\4\\4\end{bmatrix}$$

兩邊左乘 $\begin{bmatrix}1&1&1&1\\1&2&3&4\\1&4&9&16\end{bmatrix}$，得 (如(a)小題)

$$\begin{bmatrix}4&10&30\\10&30&100\\30&100&354\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}13\\35\\111\end{bmatrix}$$

經列運算解得a=13/4, b= –3/4, c=1/4.

$\therefore\quad y=(13-3x+x^2)/4$

０９E**１１**【 淡江84資工[3] 】

Find the least squares approximating guadratic $y=a_0+a_1x+a_2x^2$ for the following five data points (–3, 3), (–1, 1), (0, 1), (1, 2), (3, 4).

【解】以所給數據代入，得

$$\begin{cases}3\approx a_0-3a_1+9a_2\\1\approx a_0-a_1+a_2\\1\approx a_0\\2\approx a_0+a_1+a_2\\4\approx a_0+3a_1+9a_2\end{cases}\quad\text{即}\quad\begin{bmatrix}1&-3&9\\1&-1&1\\1&0&0\\1&1&1\\1&3&9\end{bmatrix}\begin{bmatrix}a_0\\a_1\\a_2\end{bmatrix}\approx\begin{bmatrix}3\\1\\1\\2\\4\end{bmatrix}$$

兩邊左乘 $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -3 & -1 & 0 & 1 & 3 \\ 9 & 1 & 0 & 1 & 9 \end{bmatrix}$ 得normal equation: (綜線CH9定理21a)

$$\begin{bmatrix} 5 & 0 & 20 \\ 0 & 20 & 0 \\ 20 & 0 & 164 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \\ 66 \end{bmatrix}$$

經列運算解得a_0=121/105, a_1=1/5, a_2=11/42

∴ 所求為 $y=(121/105)+(1/5)x+(11/42)x^2$

０９E**１２**【 大同84資工[7] 】

Find the line of best fit (in the Least Square sense) for the data (1, 2), (2, 5), (3, 3), (4, 6).

You are required to slove this problem with matrix manipulation.

【解】設所求直線為 $y=ax+b$，以所給數據代入得

$$\begin{cases} 2 \approx a+b \\ 5 \approx 2a+b \\ 3 \approx 3a+b \\ 6 \approx 4a+b \end{cases} \quad \text{即} \quad \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \approx \begin{bmatrix} 2 \\ 5 \\ 3 \\ 6 \end{bmatrix}$$

以 $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$ 左乘兩邊， 得出normal equation: (綜線CH9定理21a)

$$\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 45 \\ 16 \end{bmatrix}$$

解得 $a=1$， $b=3/2$.

∴ 所求直線為 $y=x+3/2$.

０９E**13** 【 大同83資工[4] 】

Find the best least squares fit by a linear function to the data.

x	-1	0	1	2
y	0	1	3	9

【解】設 $y=a+bx$為所求，依上題之法解得一次多項式 $1.8+2.9x$.

０９E**14** 【 交大82資工[5] 】

Given the data

x	-2	-1	1	2
y	6	0	8	17

find the quadratic polynomial

$$y=a+bx+cx^2$$

that best fits these data in least–squares sense.

【解】設所求多項式為 $y=a+bx+cx^2$，依前述方法解得 $y=(3/2)+3x+(5/2)x^2$.

０９E**15** 【 交大80工工[9] 】

已知公式$y=a+bx+cx^2$，但做實驗時，量到的數據卻是

x	-1	0	0	1
y	0	-2	-1	0

試問a, b, c為何值時， $\sum_{i=1}^{4}(y_i-(a+bx_i+cx_i^2))^2$ 值最小.

$(x_1=-1, y_1=0, x_2=0, y_2=-2, x_3=0, y_3=-1, x_4=1, y_4=0)$

【解】 令$S=\sum_{i=1}^{4}(y_i-(a+bx_i+cx_i^2))^2$

令 $A = \begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \\ 1 & x_4 & x_4^2 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$,

$$x = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -1 \\ 0 \end{bmatrix}.$$

則 $S = \| y - Ax \|_2^2$.

使S最小的x可由normal equation $A^{T}Ax = A^{T}y$求得:

$$A^{T}A = \begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \quad A^{T}y = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

經列運算解得 $a = -3/2, b = 0, c = 3/2$.

０９E**１６** 【 大同80資工[1] 】

Let data points $(x_1, y_1), (x_2, y_2), \ldots, (x_5, y_5)$ be given as in the following table. Find the line that is the least square approximation to these data

x	–1	0	1	2	3
y	–1	1	2	4	6

【解】設$y = ax + b$爲所求直線, 則

$$\begin{cases} ax_1+b\approx y_1 \\ ax_2+b\approx y_2 \\ ax_3+b\approx y_3 \\ ax_4+b\approx y_4 \\ ax_5+b\approx y_5 \end{cases} \quad 即 \quad \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \\ x_4 & 1 \\ x_5 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \approx \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

將上個矩陣方程式記爲$Au\approx v$,

欲取u使 $\| Au-v \|$ 爲極小，應解方程式$A^TAu=A^Tv$.　(綜線CH9定理21a)

$$A^TA=\begin{bmatrix} 15 & 5 \\ 5 & 5 \end{bmatrix}, \quad A^Tv=\begin{bmatrix} 29 \\ 12 \end{bmatrix}$$

解 $\begin{bmatrix} 15 & 5 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} 29 \\ 12 \end{bmatrix}$，得 $u=\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} 17/10 \\ 7/10 \end{bmatrix}$

$\therefore\ y=\frac{17}{10}x+\frac{7}{10}$爲所求直線.

#

０９E**17**【清大86工工[7]】

The number of an IC (Integrated Circuit) produced by a company for the last four–month period were as follows. Determine a least square line (called the cost–volume formula) and use it to predict the total cost for the next month (November) if production was planned to rise sharply to 1000 IC's. (15%)

	July	Aug	Sept	Oct
Number of IC produced	200	400	600	800
Total cost (in dollars)	820	1160	1430	1750

【解】設所求之least square line為 $y=a+bx$.

依所給數據代入得

$820 \doteqdot a+200b,\ 1160 \doteqdot a+400b,\ 1430 \doteqdot a+600b,\ 1750 \doteqdot a+800b$

即
$$\begin{bmatrix} 820 \\ 1160 \\ 1430 \\ 1750 \end{bmatrix} \doteqdot \begin{bmatrix} 1 & 200 \\ 1 & 400 \\ 1 & 600 \\ 1 & 800 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

兩邊左乘係數矩陣的轉置(transpose)即得出normal equation:

$$\begin{bmatrix} 5160 \\ 2886000 \end{bmatrix} = \begin{bmatrix} 4 & 2000 \\ 2000 & 1200000 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

解得 $a=525,\quad b=1.53$,

$\therefore$ 所求之least square line為 $y=525+1.53x$.

以$x=1000$代入，可得預測之total cost為 $525+1530=2055$ (in dollars)