題型10A: 形式與二次式

１０Ａ**０１** 【 成大81資工丙[1] 】

(1) Please find a real symmetric matrix C such that $Q = X^T CX$, where Q equals

(a) $6x_1^2 - 4x_1x_2 + 2x_2^2$

(b) $14x_1x_2 - x_2^2$

(c) $(x_1 + x_2 + x_3)^2$ (4% each, 12% total)

(2) Please verify whether the above Q and its corresponding C are positive definite in (a), (b) and (c). (2% each, 6% total)

【特別解說】考慮對稱矩陣 $C = [c_{ij}]$, 及 $X = [x_1, \ldots, x_n]^t$,

$$X^T CX = \sum_i c_{ii} x_i^2 + \sum_{i<j} 2c_{ij} x_i x_j$$ (綜線CH10範例2a)

所以寫出C的方法如下:

1° 將c_{ii}設爲x_i^2的係數

2° 將c_{ij}及c_{ji}設爲x_ix_j的係數的一半

【解】(1)(a)

$$6x_1^2 - 4x_1x_2 + 2x_2^2 = [x_1 \quad x_2] \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\therefore \text{ 取 } C = \begin{bmatrix} 6 & -2 \\ -2 & 2 \end{bmatrix}$$

(1)(b)

$$14x_1x_2-x_2^2=[x_1 \quad x_2]\begin{bmatrix}0 & 7\\7 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$\therefore$ 取 $C=\begin{bmatrix}0 & 7\\7 & -1\end{bmatrix}$

(1)(c)

$$(x_1+x_2+x_3)^2=x_1^2+x_2^2+x_3^2+2x_1x_2+2x_2x_3+2x_1x_3$$

$$=[x_1 \quad x_2 \quad x_3]\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$\therefore$ 取 $C=\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix}$

(2)(a) 計算C的principle minors:

$$\Delta_1(C)=6,\quad \Delta_2(C)=\begin{vmatrix}6 & -2\\-2 & 2\end{vmatrix}=8$$

$\because$ principle monors 全都大於零

$\therefore$ C正定 (綜線CH10定理28)

(2)(a) [另解]

$$C=\begin{bmatrix}6 & -2\\-2 & 2\end{bmatrix}\begin{matrix}(1/3)\\ \hookleftarrow\end{matrix} \sim \begin{bmatrix}6 & -2\\0 & 4/3\end{bmatrix}$$

$$\therefore\quad C=\begin{bmatrix}1 & 0\\ -1/3 & 1\end{bmatrix}\begin{bmatrix}6 & -2\\ 0 & 4/3\end{bmatrix}$$ (綜線CH3定理7)

$$=\begin{bmatrix}1 & 0\\ -1/3 & 1\end{bmatrix}\begin{bmatrix}6 & 0\\ 0 & 4/3\end{bmatrix}\begin{bmatrix}1 & -1/3\\ 0 & 1\end{bmatrix}$$ (綜線CH3範例4b)

$$=\begin{bmatrix}1 & 0\\ -1/3 & 1\end{bmatrix}\begin{bmatrix}\sqrt{6} & 0\\ 0 & \sqrt{4/3}\end{bmatrix}^2\begin{bmatrix}1 & -1/3\\ 0 & 1\end{bmatrix}$$

$$=W^{T}W\quad ,$$

其中 $W=\begin{bmatrix}\sqrt{6} & 0\\ 0 & \sqrt{4/3}\end{bmatrix}\begin{bmatrix}1 & -1/3\\ 0 & 1\end{bmatrix}$ 爲可逆矩陣

(綜線CH4定理17,定理6)

$\therefore$ Q, C 爲 positive definite. (綜線CH13定理18 ⅱ)

(2)(b) 計算C的principle minors:

$\Delta_1(C)=0,\quad ....$

$\because$ principle monors 不全是正數

$\therefore$ C不正定 (綜線CH10定理28)

(2)(b) [另解]

取 $X=\begin{bmatrix}0\\ 1\end{bmatrix}\neq o\quad ,$ 此時 $Q=X^{T}CX=-1 < 0$

$\therefore$ Q, C並非 positive definite. (綜線CH10定義19)

(2)(c) 計算C的principle minors:

$\Delta_1(C)=1\quad ,\quad \Delta_2(C)=\begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix}=0,\quad ....$

$\because$ principle monors 不全是正數

∴ C不正定 (綜線CH10定理28)

(2)(c) [另解]

取 $X=\begin{bmatrix}1\\-1\\0\end{bmatrix}\neq o$， 此時 $Q=(1-1+0)^2=0^2=0$

∴ Q, C並非 positive definite. (綜線CH10定義19)

【加強演練】

設 $Q=2x_1^2+4x_2^2+10x_3^2+2x_1x_2+6x_1x_3+10x_2x_3$

試求矩陣C使 $Q=X^TCX$，並判定其正定性。

Ans: $C=\begin{bmatrix}2&1&3\\1&4&5\\3&5&10\end{bmatrix}$，正定.

１０A**02** 【 大同82資工[13] 】

Let $u=(x_1, x_2)$ and $v=(y_1, y_2)$. Determine which of the following is not bilinear;

(a) $f(u,v)=2x_1y_2-3x_2y_1$; (b) $f(u,v)=x_1+y_1$;

(c) $f(u,v)=0$; (d) $f(u,v)=3x_2y_2$.

【解】 選(b)

【說明】bilinear在固定u時, v應該是linear; 而在固定v時, u應該是linear. (綜線CH10定義1)

以 $v=(1, 2)$爲例, (b)變成 $f((x_1, x_2), (1, 2))=x_1+1$

這不是 (x_1, x_2) 的線性映射.

１０A**03** 【 元智82電資[12] 】

Show that if A is an indefinite symmetric matrix, there is a nonzero vector x such that $x^TAx=0$.

【分析】題中雖未明說, 但由 x^TAx (而非x^HAx) 可判知命題者所指的矩陣及向量都是在實數系考慮。

【解】令 $A\in\mathbb{R}^{n\times n}$， 則$n\geq 2$ (註1)

已知A爲indefinite, 即

"$\exists v_1\in\mathbb{R}^{n\times 1}$使得$v_1^TAv_1>0$, 且 $\exists v_2\in\mathbb{R}^{n\times 1}$ 使得 $v_2^TAv_2<0$."

可證明 v_1 , v_2線性獨立 ：

假如v_1 , v_2 爲線性相關,

則不妨假設$v_1=kv_2$,

由$v_2^TAv_2=k^2v_1Av_1>0$, 造成矛盾.

定義 $f:[0,1]\longrightarrow\mathbb{R}$ 如下:

$$f(t)=((1-t)v_1+tv_2)^TA((1-t)v_1+tv_2)\ .$$

則 $f(t)=(v_1+t(v_2-v_1))^TA(v_1+t(v_2-v_1))$

$$=v_1^TAv_1+2(v_1^TA(v_2-v_1))t+((v_2-v_1)^TA(v_2-v_1))t^2\ ,$$

∵ 上式爲t的二次多項式, ∴ f 爲連續函數.

而 $f(0)=v_1^TAv_1>0$, $f(1)=v_2Av_2<0$

由微積分的中間值定理(intermediate value theorem, 或稱勘根定理),

$\exists s$ 使得 $0<s<1, f(s)=0$

令 $x=(1-s)v_1+sv_2$, 即 $x^TAx=0$

∵ v_1 , v_2 線性獨立, 而$s\neq 0$, $1-s\neq 0$,

∴ $x\neq o$ (綜線CH6定義9)

[註 1]若$n=1$, 則$A\in\mathbb{R}$.

$A>0$時, 對非零數v, $v^TAv=Av^2>0$, ∴ A爲正定.

$A<0$時, 對非零數v, $v^TAv=Av^2<0$, ∴ A爲負定.

$A=0$時, 對非零數v, $v^TAv=0$,

以上三情形都不可能indefinite.

題型10B:　Hermitian矩陣

本題型涉及Hermitian矩陣(及對稱矩陣), 所用到的定理分佈在第十章及第十三章.

１０B**01** 【 元智81工工[3] 】

若A爲一complex Hermitian matrix, 則 $\det(A)$ 有無可能爲$1+i$?

【解】不可能. 理由如下:

$\because$ $A^H = A$,　　(綜線CH10定義12)

$\therefore$ $\det(A^H) = \det(A)$

$\therefore$ $\overline{\det(A)} = \det(A)$　　(綜線CH4定理5)

$\therefore$ $\det(A) \in \mathbb{R}$

【加強演練】

True or False? “$A^T = -A$, 則 $\det A = 0$”

Ans:　False.　　(綜線CH10定理13)

１０B**02** 【 中正79資工[2] 】

Let A be a Hermitian matrix. Show that

(a) any eigenvalue of A is real;　(5%)

(b) any two eigenvectors of A corresponding to two distinct eigenvalues must orthogonal. (5%)

【解】請參閱綜線CH13定理14.

１０B**03** 【 淡江82資工[2] 】

Let A be a symmetric matrix and suppose λ_i and λ_j are two eigenvalues of A with corresponding eigenvectors u_i and u_j . Show that if $\lambda_i \neq \lambda_j$, then the inner product $u_i^T \cdot u_j = 0$

【解】　$u_i^T A u_j = u_i^T \lambda_j u_j = \lambda_j u_i^T u_j$

$u_i^T A u_j = u_i^T A^T u_j = (Au_i)^T u_j = (\lambda_i u_i)^T u_j = \lambda_i u_i^T u_j$

$\therefore\ \lambda_i u_i^T u_j = \lambda_j u_i^T u_j$ $\therefore\ (\lambda_i - \lambda_j) u_i^T u_j = 0$

但λ_i與λ_j相異，

$\therefore\ u_i^T u_j = 0$ （即 u_i 與 u_j 垂直.）

１０B**04** 【台大75資工[11]】

Let $A = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$ where a, b, c real and $c \neq 0$

(a) Show that A has different eigenvalues.

(b) Let $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigenvector of A, $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ an another eigenvector of A.

Show that if their corresponding eigenvalues are different then they are orthogonal to each other.

【解】

(1) $\det(A - xI) = \begin{vmatrix} a-x & c \\ c & b-x \end{vmatrix} = x^2 - (a+b)x + ab - c^2$

$\Delta = (a+b)^2 - 4(ab - c^2)$ (判別式)

$= (a-b)^2 + 4c^2 > 0$ $(\because\ c \neq 0)$

$\therefore\ \det(A - xI)$ has different roots.

$\therefore$ has different eigenvalues.

(2) $\begin{bmatrix} x_1 & x_2 \end{bmatrix} A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \lambda_2 \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

$\begin{bmatrix} x_1 & x_2 \end{bmatrix} A \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} A^T \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ ($\because A$對稱)

$$= \left(A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)^{\mathrm{T}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \left(\lambda_1 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)^{\mathrm{T}} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \lambda_1 \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\therefore\ \lambda_1 \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \lambda_2 \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$\therefore\ (\lambda_1 - \lambda_2) \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0$$

$$\therefore\ \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = 0 \qquad (\because \lambda_1 - \lambda_2 \neq 0)$$

i.e. $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ are orthogonal to each other.

１０B**05** 【 台大85資工[9] 】

[複選題]

Which of the following are true.

(1) The eigenvalues of a self-adjoint operator must be real.

(2) Every normal operator is diagonalizable.

(3) Every self-adjoint operator is a linear combination of orthogonal projections.

(4) Let W be a finite-dimensional subspace of an inner product space V. If T is an orthogonal projection on W, then $(I–T)$ is the orthogonal projection on $W^{\perp}$.

【解】選(1)(2)(3)(4)

【討論】

(1) True. 此爲定理. (綜線CH13定理14)

(2) True. 此爲定理. (綜線CH13定理15)

(3) True.

取正交單位基底的矩陣表示來觀察. self-adjoint operator的矩陣表示就是Hermitian矩陣. (綜線CH10定義31,CH10定理29a)

將矩陣單式對角化: $A=UDU^{-1}=UDU^{H}$.

再如光譜分解, 則A化爲如$\sum\lambda_i P_i$之型, 其中各P_i皆形如USU^{H}, S爲對角線矩陣, 且對角線元素爲0或1. (綜線CH12範例25)

顯然 $P^{H}=P$, 且 $P^2=P$,

所以 P爲正投影映射之矩陣表示. (綜線CH11定理20)

(4) True. 證明如下:

∵ T爲對W的正投影

∴ $T=T^{*}$, $T^2=T$, (綜線CH11定理20)

且Im$T=W$, Ker$T=W^{\perp}$. (綜線CH11定理21)

$$\begin{cases} (I-T)^{*}=I^{*}-T^{*}=I-T \\ (I-T)\circ(I-T)=I-2T+T\circ T=I-2T+T=I-T \\ \mathrm{Im}(I-T)=\mathrm{Ker}T=W^{\perp}, \\ \mathrm{Ker}(I-T)=\mathrm{Im}T=W=W^{\perp\perp} \end{cases}$$

(綜線CH11定理12,定理17)

∴ $I-T$爲對$W^{\perp}$的正投影.

１０Ｂ06 【 大同84資工[4](a) 】

(a) Assume that A is a symmetric $n\times n$ matrix. Also x and y are vectors in $\mathbb{R}^n$ with the property that $Ax=3x$ and $Ay=5y$. Prove that x and y are orthogonal.

【解】(a) $y^{T}Ax=y^{T}(3x)=3y^{T}x$

$y^{T}Ax=y^{T}A^{T}x=(Ay)^{T}x=(5y)^{T}x=5y^{T}x$

∴ $3y^{T}x=5y^{T}x$ ∴ $2y^{T}x=0$

∴ $y^{T}x=0$

１０B**０７**【 大同86資工[5](ab) 】

(a) Show that the eigenvalues of a Hermittian matrix are all real. Furthermore, eigenvectors belonging to distinct eigenvalues are orthogonal. (7%)

(b) Given $A=\begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$, find an orthogonal matrix U that diagonalizes .(7%)

【解】(a) 請參閱綜線CH13定理14, 此處不再重覆.

(b) $\det(A-xI)=-x^3+3x^2+9x+5=-(x+1)^2(x-5)$

$$A+I=\begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ -1 & -2 & 1 \end{bmatrix} \sim \underset{\cdots\cdots}{\text{列運算}} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(A+I)=\left\{ \begin{bmatrix} -2s+t \\ s \\ t \end{bmatrix} \middle| \, s,t\in\mathbb{R} \right\}=\text{CSP}\begin{bmatrix} -2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

令 $v_1=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $v_2=\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$,

對 v_1 , v_2 做Gram-Schmidt正交化:

$v_1'=v_1$, $\|v_1'\|=\sqrt{2}$

$$v_2' = v_2 - ((v_2 \cdot v_1')/(v_1' \cdot v_1'))v_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\| v_2' \| = \sqrt{3}$$

令 $u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ ， $u_2 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$ ，

則 u_1, u_2 爲orthonormal的特徵向量.

$$A-5I = \begin{bmatrix} -5 & 2 & -1 \\ 2 & -2 & -2 \\ -1 & -2 & -5 \end{bmatrix} \sim \overset{\text{列運算}}{\cdots\cdots\cdots} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\ker(A+I) = \left\{ \begin{bmatrix} -t \\ -2t \\ t \end{bmatrix} \middle| \ t \in \mathbb{R} \right\} = \text{CSP}\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

令 $v_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ ，則 $\| v_3 \| = \sqrt{6}$

令 $u_3 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$ ，

則 u_3 為單位長的特徵向量.

$$令\ U = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \end{bmatrix},$$

則 $U^{-1}AU = \text{diag}(-1,-1,5)$，且 $U^{T}U = UU^{T} = I$ #

題型10C: 正定矩陣

本題型涉及正定矩陣，所用到的定理分佈在第十章及第十三章.

１０C**01** 【 交大81資工[7](b) 】

(b) (4%) A real symmetric matrix A is said to be positive definite if $x^T A x > 0$ for all nonzero x in $\mathbb{R}^n$. Let B be an $m\times n$ matrix of rank n. Show that $B^T B$ is positive definite.

【解】(b) $\dim \ker B = \dim \mathbb{R}^n - \text{rank}B = n - n = 0$ (綜線CH8定理8)

$\therefore\ \ker B = \{ o \}$

對非零向量x, 由上式得知 $Bx \neq o$, (綜線CH5定義19)

$\therefore\ (Bx)^T(Bx) > 0$

即 $x^T B^T B x > 0$

１０C**02** 【 交大84資工[4] 】

Let A be a real symmetric positive definite $n\times n$ matrix. (A symmetric $n\times n$ matrix is positive definite if $x^T Ax > 0$ for all nonzero vectors x in $\mathbb{R}^n$.) Prove that

(a) All eigenvalues of A are positive.

(b) The leading principal submatrices $A_1, A_2, ..., A_n$ of A are all positive definite. (A leading principal submatrix A_r is formed by deleting the last n-r rows and columns of A).

【分析】由 $x \in \mathbb{R}^n$, 及 $x^T Ax$ 的式子可判定題目的$\mathbb{R}^n$是指$\mathbb{R}^{n\times 1}$.

【參考章節】本題(b)請參閱綜線第9版CH10定理26a

【解】(a) 設λ爲A的eigenvalue, 取(非零的)eigenvector x, 則 $x^T Ax = x^T(\lambda x) = \lambda x^T x$

由A的正定性得知 $x^T Ax > 0$, 又因$x \neq o$ 得知$x^T x \neq 0$,

$\therefore\ \lambda > 0$.

(b) 令$B = A_r$, (由A刪除末$(n-r)$列及末$(n-r)$行所得的矩陣.)

再令 $A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$. 欲證$B$爲正定:

顯然B仍為對稱的實數矩陣.

對任意非零(行)向量$u\in\mathbb{R}^r$, (欲證 $u^TBu>0$)

令 $x=\begin{bmatrix} u \\ o \end{bmatrix}\in\mathbb{R}^n$, (o為$\mathbb{R}^{n-r}$的零向量)

則x為非零向量. 由A的正定性得知 $x^TAx > 0$.

$$x^TAx=\begin{bmatrix} u^T & o \end{bmatrix}\begin{bmatrix} B & C \\ D & E \end{bmatrix}\begin{bmatrix} u \\ o \end{bmatrix}$$

$$=\begin{bmatrix} u^T & o \end{bmatrix}\begin{bmatrix} Bu \\ Du \end{bmatrix}$$ (塊狀乘法: 綜線CH2定理8)

$$= u^TBu$$ (塊狀乘法: 綜線CH2定理8)

$\therefore\ u^TBu>0$.

１０C**03** 【 大同84資工[6] 】

Which of the following matrices are positive definite? Negative definite? Indefinite?

(a) $\begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & 4 \\ 4 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -2 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

【解說】對$n\times n$實數對稱方陣A,

“ $\forall x\in\mathbb{R}^{n\times1}\backslash\{o\}, x^TAx>0,$ ” 稱為正定(positive definite).

“ $\forall x\in\mathbb{R}^{n\times1}\backslash\{o\}, x^TAx<0,$ ” 稱為負定(negative definite).

“ $\forall x\in\mathbb{R}^{n\times1}, x^TAx\geq0,$ ” 稱為正半定(positive semi-definite).

“ $\forall x\in \mathbb{R}^{n\times 1}, x^{T}Ax\leq 0,$ ” 稱爲負半定(negative semi-definite).

不正半定且不負半定稱爲不定(indefinite), 也就是

“ $\exists x\in \mathbb{R}^{n\times 1}$ 使得 $x^{T}Ax>0$ 且 $\exists y\in \mathbb{R}^{n\times 1}$ 使得 $y^{T}Ay<0.$ ”

關於這些矩陣, 有下列性質:

◎ A正定 $\Longleftrightarrow$ A的所有特徵值都是正數.

◎ A正半定 $\Longleftrightarrow$ A的所有特徵值都是“正數或零”.

◎ A負定 $\Longleftrightarrow$ $-A$正定 $\Longleftrightarrow$ A的所有特徵值都是負數.

◎ A負半定 $\Longleftrightarrow$ $-A$正半定 $\Longleftrightarrow$ A的所有特徵值都是“負數或零”.

◎ A不定 $\Longleftrightarrow$ A的特徵值中有正數也有負數.

【解】 各矩陣都是實數對稱矩陣, 所以都可(正交)對角化. 計算特徵多項式以判定其特徵值的正負 :

(a) $\begin{vmatrix} 3-x & 2 \\ 2 & 2-x \end{vmatrix} = x^2-5x+2=(x-\lambda_1)(x-\lambda_2)$

顯然 λ_1, λ_2 同爲正數. ∴ 正定.

(b) $\begin{vmatrix} 3-x & 4 \\ 4 & 1-x \end{vmatrix} = x^2-4x-13=(x-\lambda_1)(x-\lambda_2)$

顯然 λ_1, λ_2一正一負. ∴ 不定.

(c) $\begin{vmatrix} -2-x & 0 & 1 \\ 0 & -1-x & 0 \\ 1 & 0 & -2-x \end{vmatrix} = (-1-x)\begin{vmatrix} -2-x & 1 \\ 1 & -2-x \end{vmatrix}$

$=(-1-x)(x^2+4x+3)=-(x+1)^2(x+3)$

特徵值爲–1, –1, –3 ∴ 負定.

(d) $\begin{vmatrix} 1-x & 2 & 1 \\ 2 & 1-x & 1 \\ 1 & 1 & 2-x \end{vmatrix} = \ldots. = -(x^3-4x^2-x+4) = -(x-4)(x-1)(x+1)$

特徵值爲4, 1, –1 ∴ 不定.

【另解】計算左上角行列式:

(a) $\Delta_1 = 3 > 0$, $\Delta_2 = \begin{vmatrix} 3 & 2 \\ 2 & 2 \end{vmatrix} = 2 > 0$

∴ 正定. (綜線CH10定理28)

(b) $\Delta_1 = 3 > 0$, $\Delta_2 = \begin{vmatrix} 3 & 4 \\ 4 & 1 \end{vmatrix} = -1 < 0$

∴ 不定.

(c) $\Delta_1 = -2 < 0$, $\Delta_2 = \begin{vmatrix} -2 & 0 \\ 0 & -1 \end{vmatrix} = 2 > 0$

$$\Delta_3 = \begin{vmatrix} -2 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -2 \end{vmatrix} = (-1)\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} = (-1)\cdot 3 < 0$$

將矩陣變號後 $\Delta_1 > 0$, $\Delta_2 > 0$, $\Delta_3 > 0$. (∵ $\det(-A) = (-1)^n \det A$)

∴ 變號後的矩陣爲正定. ∴ 原矩陣爲負定.

(d) $\Delta_1 = 1 > 0$, $\Delta_2 = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} < 0$

∴ 不定.

１０C**04** 【 台大79資工[4](v) 】

(Yes or No question and explain the reason:)

(v) If a matrix A is positive definite, then the eigenvalues of A are positive.

【參考章節】CH13定理17c

【解】 Yes

設 λ 爲A的eigenvalue, x爲所對應的(非零)eigenvector,

則 $x^H A x > 0$ 而 $x^H A x = x^H(\lambda x) = \lambda x^H x = \lambda \|x\|^2$

∴ $\lambda \|x\|^2 > 0$

∴ $\lambda > 0$

題型10D: 伴隨映射

１０D01 【 交大86資料[1] 】

Let the inner product be $\langle x, y \rangle = x^{T}y$ for $x, y \in \mathbb{R}^{n}$. Show that for any $n\times n$ matrix A, $\langle Ax, y \rangle = \langle x, A^{T}y \rangle$.

【分析】由內積之形狀可判定這裡的$\mathbb{R}^{n}$ 是指$\mathbb{R}^{n\times 1}$. 本題背景請參閱綜線CH10定義29.

【解】 $\langle Ax, y \rangle = (Ax)^{T}y = x^{T}A^{T}y = \langle x, A^{T}y \rangle$.

１０D02 【 台大81資工[2] 】

Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then there exists a unique, linear function $T^{*}: V \longrightarrow V$ (called the adjoint of T) such that

$$\langle T(X), Y \rangle = \langle X, T^{*}(Y) \rangle \text{ for all } X, Y \in V,$$

where $\langle \cdot , \cdot \rangle$ is the inner product function defined on V.

【解】本題考定理證明, 請參閱綜線CH10定理29.

【加強演練】

設$T: \mathbb{C}^{3} \longrightarrow \mathbb{C}^{2}$ 定義如

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2ix-9y+6iy+4z \\ 4x+5ix-6y-9z+iz \end{bmatrix}$$

試求 $S: \mathbb{C}^{2} \longrightarrow \mathbb{C}^{3}$, 使

$\forall v \in \mathbb{C}^{3}, \forall w \in \mathbb{C}^{2}$, $\langle T(v), w \rangle = \langle v, S(w) \rangle$,

其中$\langle\ ,\ \rangle$代表 $\mathbb{C}^{k}$ 中的標準內積。

【解】

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)=\begin{bmatrix} 2i & -9+6i & 4 \\ 4+5i & -6 & -9+i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

令 $S:\mathbb{C}^2\longrightarrow\mathbb{C}^3$ 定義如: $S\left(\begin{bmatrix} u \\ v \end{bmatrix}\right)=\begin{bmatrix} -2i & 4-5i \\ -9-6i & -6 \\ 4 & -9-i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

則 $\left\langle T\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right\rangle=\overline{\begin{bmatrix} u\ v \end{bmatrix}}\begin{bmatrix} 2i & -9+6i & 4 \\ 4+5i & -6 & -9+i \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$=\overline{\begin{bmatrix} -2i & 4-5i \\ -9-6i & -6 \\ 4 & -9-i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\overline{S\begin{bmatrix} u \\ v \end{bmatrix}}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\left\langle\begin{bmatrix} x \\ y \\ z \end{bmatrix}, S\begin{bmatrix} u \\ v \end{bmatrix}\right\rangle$$

１０D**03** 【 元智84電資[3] 】

Let $\varphi:\mathbb{C}^2\longrightarrow\mathbb{C}^2$ be the linear operator on $\mathbb{C}^2$ defined by: $\varphi(z_1,z_2)=(2iz_1+3z_2, z_1-z_2)$ $\forall(z_1, z_2)\in\mathbb{C}^2$. Let φ^* be the adjoint operator of φ. Find $\varphi^*(w_1, w_2)$

【解】

$$\begin{cases} \varphi(1,0)=(2i,1)=2i(1,0)+1(0,1) \\ \varphi(0,1)=(3,-1)=3(1,0)+(-1)(0,1) \end{cases}$$

在標準基底下，φ 的矩陣表示爲 $\begin{bmatrix} 2i & 3 \\ 1 & -1 \end{bmatrix}$ (綜線CH7定義9)

在標準基底下，φ^* 的矩陣表示為 $\begin{bmatrix} 2i & 3 \\ 1 & -1 \end{bmatrix}^{H} = \begin{bmatrix} -2i & 1 \\ 3 & -1 \end{bmatrix}$

(綜線CH10定理29a)

$\therefore \begin{cases} \varphi^*(1,0) = -2i(1,0) + 3(0,1) = (-2i,3) \\ \varphi^*(0,1) = 1(1,0) - 1(0,1) = (1,-1) \end{cases}$ (綜線CH7定義9)

$\therefore\ \varphi^*(w_1, w_2) = w_1\varphi^*(1,0) + w_2\varphi^*(0,1)$ (綜線CH7定義1)

$= (-2iw_1, 3w_1) + (w_2, -w_2)$

$= (-2iw_1 + w_2, 3w_1 - w_2)$

１０D**04** 【 大同82資工[11] 】

A linear operator T on an inner product space V is said to have an adjoint operator T^* if

$\langle T(u),v\rangle = \langle u, T^*(v)\rangle$ for every $u,v \in V$.

Consider the linear operator T on $\mathbb{C}^3$ defined by

$T(x, y, z) = (2x + iy, y - 5iz, x + (1-i)y + 3z)$.

Then $T^* = ?$

(a) $T^*(x, y, z) = (2x - iy, y + 5iz, x + (1+i)y + 3z)$;

(b) $T^*(x, y, z) = (2x + z, ix + y + (1-i)z, -5iy + 3z)$;

(c) $T^*(x, y, z) = (2x + z, -ix + y + (1+i)z, 5iy + 3z)$;

(d) T has no adjoint operator.

【解】 選 (c)

【說明】

$$T(x, y, z) = (x, y, z)\begin{bmatrix} 2 & 0 & 1 \\ i & 1 & 1-i \\ 0 & -5i & 3 \end{bmatrix}$$

令上式的3x3矩陣為A. 在標準基底之下, T的矩陣表示為A^{T} (綜線CH7定理9a)

$\therefore\ T^*$ 的矩陣表示為$(A^{T})^{H} = (A^{H})^{T}$ (綜線CH2定理23)

$\therefore\ T^*(x, y, z) = (x, y, z)A^{H}$, (綜線CH7定理9a)

$$=(x, y, z)\begin{bmatrix} 2 & -i & 0 \\ 0 & 1 & 5i \\ 1 & 1+i & 3 \end{bmatrix}$$

$$=(2x+z, -ix+y+(1+i)z, 5iy+3z)$$

１０D**05** 【 清大76資料[7](b) 】

Prove or disprove the following statements.

(b) Let N be a normal linear transformation on V. If $N(v)=o$ for $v\in V$, then $N^*(v)=o$ where N^* is the adjoint matrix of N. (5%)

【解】(b) (prove)

$\|N^*(v)\|^2$

$=\langle N^*(v), N^*(v)\rangle$ (綜線CH9定義4)

$=\langle v, NN^*(v)\rangle$ (綜線CH10定義29)

$=\langle v, N^*N(v)\rangle$ (綜線CH13定義12)

$=\langle Nv, Nv\rangle$ (綜線CH10定義29)

$=\|Nv\|^2=0$

$\therefore N^*(v)=o$

【討論】由以上證明內容得知:

對 normal operator N, $\|Nv\|=\|N^*v\|$.