題型11A: 空間分解與投影映射

11A01 【 台大80資工[5] 】

[True or False Problem]

If $V=\sum_{i=1}^{k} W_i$, and $W_i \cap W_j=\{o\}$ for $i\neq j$, then $V=W_1\oplus W_2\oplus \dots \oplus W_k$.

【解】False, 反例如下:

$V=\mathbb{R}^2=\{(x, y) \mid x, y\in \mathbb{R}\}$

$W_1=\{(x,0) \mid x\in \mathbb{R}\}$, $W_2=\{(0,y) \mid y\in \mathbb{R}\}$, $W_3=\{(t,t) \mid t\in \mathbb{R}\}$.

則 $W_i \cap W_j=\{o\}$ for $i\neq j$, 且 $W_1+W_2+W_3=\mathbb{R}^2$

但 $W_1\cap(W_2+W_3)\neq\{o\}$, $\therefore$ $W_1+W_2+W_3$ 並非direct sum.

11A02 【 大同84資工[1](b) 】

(b) A matrix is said to be skew-symmetric if $A^T=-A$. Show that every square matrix can be uniquely expressed as the sum of a symmetric and skew-symmetric matrix.

【解】(b) 對任意矩陣X,

若A爲symmetric, B爲skew-symmetric,且

$X=A+B$. .....(甲)

兩邊取轉置得

$X^T=A-B$ .....(乙)

甲乙兩式相加再除以2, 得 $A=(X+X^T)/2$

甲乙兩式相減再除以2, 得 $B=(X-X^T)/2$

$\therefore$ 得證唯一性.

$(X+X^T)/2+(X-X^T)/2=X$

$((X+X^T)/2)^T=(X+X^T)/2$, $((X-X^T)/2)^T=(X^T-X)/2=-(X-X^T)/2$

$\therefore$ 得證存在性.

１１A**03**【 元智83電資[3] 】

$\mathbb{R}^3 = L((1,1,0)) \oplus L((1,0,0)), (0,1,1))$, where $L((1,1,0))$ is the space generated by (1,1,0) and $L((1,0,0), (0,1,1))$ is the space generated by (1,0,0) and (0,1,1). Find the projection of (1,2,3) on $L((1,1,0))$ along $L((1,0,0),(0,1,1))$. (10%)

【解】令 $(1,2,3) = w_1 + w_2,\quad w_1 = a(1,1,0),\quad w_2 = b(1,0,0) + c(0,1,1)$,

$\therefore\quad (1,2,3) = a(1,1,0) + b(1,0,0) + c(0,1,1)$,

展開得 $\begin{cases} a+b = 1 \\ a + c = 2 \\ c = 3 \end{cases}$

解出 $a = -1,\quad b = 2,\quad c = 3.$

$\therefore\ w_1 = (-1)(1,1,0) = (-1,-1,0),\quad w_2 = 2(1,0,0) + 3(0,1,1) = (2,3,3)$

$\because\ (1,2,3) = w_1 + w_2,\quad w_1 \in L((1,1,0)),\quad w_2 \in L((1,0,0), (0,1,1))$

$\therefore$ 所求爲(–1,–1,0). (綜線CH11定理12要訣3)

【另解】設所求爲 $a(1,1,0) \in L((1,1,0))$,

則$(1,2,3) - a(1,1,0) \in L((1,0,0),(0,1,1))$.

$$\therefore \begin{vmatrix} 1-a & 2-a & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0$$ (綜線CH6定理14)

可解得$a = -1$,

而求得投影$(-1)(1,1,0) = (-1,-1,0)$

【討論】第一種解法較一般化. 即使對高維度的W_1, W_2 仍可適用. 第二種解法是利用 $\mathbb{R}^3$ 的幾何性質靈活求解. 本題若要求算(1,2,3)沿$L(1,1,0)$對$L((1,0,0),(0,1,1))$的投影, 只需取依第一種解法算出的w_2即可.

１１A**04**【 元智83工工[8] 】

[是非題]

If P is the projection matrix onto a vector subspace of $\mathbb{R}^n$, then $I-P$ may not be a projection matrix.

【解】×, 剛好講反了. 解說如下:

$P^2=P \Longrightarrow (I-P)^2=I-2P+P^2=I-2P+P=I-P$

１１A**05** 【 雲技84電資Ｙ[3](b) 】

(b) An $n\times n$ matrix B is said to be idempotent if $B^2=B$. Find the eigenvalues of λ for an idempotent matrix ?

【解】設λ為B的特徵值. 取$v\neq o$, 使 $Bv=\lambda v$. (綜線CH12定義1)

$B^2v=B(Bv)=B(\lambda v)=\lambda Bv=\lambda^2 v$

而$B^2v=Bv=\lambda v$

$\therefore\ \lambda^2 v=\lambda v$ $\therefore\ (\lambda^2-\lambda)v=o$

$\therefore\ \lambda^2-\lambda=0,$ $\therefore\ \lambda\in\{0, 1\}$.

１１A**06** 【 台大80資工[10] 】

[True or False Problem]

Every projection operator on a finite dimensional vector space V is diagonalizable.

【解】 True, 證明請參閱綜線CH11定理14.

１１A**07** 【 清大77資科[6](b) 】

Prove or disprove the following statements.

(b) Let A and B be two $n\times n$ matrix over $\mathbb{R}$. If $A^2=A$, $B^2=B$ and trace(A)=trace(B) then A is similar to B.

【解】 (b) [prove]

由 $A^2=A$ 可知A相似於 $\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}$ (綜線CH11定理14)

由 $B^2=B$ 可知B相似於 $\begin{bmatrix} I_s & O \\ O & O \end{bmatrix}$ (綜線CH11定理14)

而 trace(A)$=r$, trace(B)$=s$, 由題意可知 $r=s$.

$$\therefore\ A\text{相似於}\begin{bmatrix} I_r & O \\ O & O \end{bmatrix}\quad \begin{bmatrix} I_r & O \\ O & O \end{bmatrix}\text{相似於}B,$$

$\therefore\ A$相似於B (綜線CH7定義21要訣2)

１１A**08** 【 元智84電資[4] 】

Prove that if A is a real symmetric $n\times n$ matrix and $A^2=\alpha A$, where $\alpha\in\mathbb{R}$, then $\text{tr}(A)=\alpha\cdot\text{rank}(A)$.

【解】 設λ爲A的特徵值, 取$v\neq o$使$Av=\lambda v$. 則

$A^2v=A(\lambda v)=\lambda Av=\lambda^2 v$, 而 $\alpha Av=\alpha\lambda v$

$\therefore\ \lambda^2 v=\alpha\lambda v$, $\therefore\ (\lambda^2-\alpha\lambda)v=o$

由$v\neq o$ 得知 $\lambda^2-\alpha\lambda=0$, $\therefore\ \lambda\in\{\alpha, 0\}$

$\because\ A$爲real symmetric,

$\therefore\ \exists P$, 使得$A=PDP^{-1}$, D爲對角矩陣, 且對角線都是α或0. (綜線CH13定理15)

可調整P的行, 使得$D=\text{diag}(\alpha, \alpha, \ldots, \alpha, 0, 0, \ldots, 0)$.

若$\alpha=0$, 則$D=O$, $A=O$. 求證式顯然成立.

若$\alpha\neq 0$, 則 $\text{tr}A=\text{tr}D=\alpha\cdot[D\text{中}\alpha\text{的個數}]=\alpha\cdot\text{rank}D=\alpha\cdot\text{rank}A$.

(綜線CH2定理28,CH8定理16)

【討論】此題其實不必用到real symmetric的條件. 另證如下:

若$\alpha=0$, 則$A^2=O$, $\therefore\ A$的eigenvalue全是零, (綜線CH14定理17)

$\therefore\ \text{tr}A=0=0\cdot\text{rank}A$ (綜線CH13定理8)

若$\alpha\neq 0$, 則$(A/\alpha)^2=(A/\alpha)$

$\therefore\ \exists P$使得$A/\alpha=PDP^{-1}$, 且$D=\text{diag}(1,1,\ldots,1.0,0,\ldots,0)$. (綜線CH11定理14)

$\therefore\ \text{tr}(A/\alpha)=\text{tr}D=D$中1的個數$=\text{rank}D=\text{rank}(A/\alpha)$

(綜線CH2定理28,CH8定理16)

$\therefore\ \text{tr}A/\alpha=\text{rank}(A)$. (綜線CH2定理28,CH8定理16)

再移項即得證.

１１A**09** 【 台大80資工[4] 】

[True or False Problem]

Let V be a vector space of finite-dimension. If T is a linear operator satisfying $T^2 = T$, then $R(T) \cap N(T) = \varnothing$.

【解】False. 這個敘述對任意T都錯.

$\because$ $R(T), N(T)$都是V的子空間. (綜線CH5定理17,定理20)

$\therefore$ $o \in R(T) \cap N(T)$ (綜線CH5定理11要訣2)

$\therefore$ $R(T) \cap N(T) \neq \varnothing$

【討論】本題若改爲$R(T) \cap N(T) = \{ o \}$, 則爲True. (綜線CH11定理12)

１１A**１０** 【 台大83資工[4] 】

[複選題]

Let V be a finite-dimension vector space, and there is a linear operator $T: V \longrightarrow V$ satisfying $T^2 = T$. Then,

(1) $R(T) \cap N(T) = \varnothing$,

(2) $R(T) + N(T) = V$,

(3) $R(T) = \{x \in v : T(x) = x\}$,

(4) $\text{rank}(T) \geq \text{nullity}(T)$.

【分析】第(3)小題須熟悉下述性質: (綜線CH11定義10要訣2)

$$T^2 = T \iff \forall w \in R(T),\ T(w) = w$$

【解】 選(2)(3).

【討論】(1) 應是$R(T) \cap N(T) = \{o\}$ (綜線CH11定理12)

(2) 此爲定理. (綜線CH11定理12)

(3) 若$x = T(x)$, 當然$x \in R(T)$.

反之, 若$x \in R(T)$, 則存在z, 使得$x = T(z)$

$\therefore$ $T(x) = T(T(z)) = (T^2)(z) = T(z) = x$

(4) 反例如下:

設$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, 定義如$T(x, y, z) = (x, 0, 0)$.

則$T^2 = T$, $\text{rank}(T) = 1$, $\text{nullity}(T) = 2$.

１１A**１１** 【 台大85資工[5] 】

[複選題]

Let T be a linear transformation on a finite-dimentional vector space V satisfying $T^2 = T$. Which of the following are true.

(1) range(T)$\cap$nullspace(T)$=\varnothing$.

(2) range(T)$+$nullspace(T)$=V$.

(3) T has only two distinct eigenvalues, namely, $\lambda = 0$ and $\lambda = 1$.

(4) There exists an ordered basis β such that $\det([T]_\beta) \neq 0$.

【解】選(2)(3)

【討論】

(1) False. 應該是 range(T)$\cap$nullspace(T)$=\{o\}$. (綜線CH11定理12)

(2) True. 此爲定理. (綜線CH11定理12)

(3) True.

idempotent 的 eigenvalue 只能是0或1. (綜線CH11定理14)

本題題意有點模糊. 若解釋成"恰有兩個相異特徵値0,1", 就變成不成立. 例如零映射也是映兩次等於映一次, 但只有一個eigenvalue.

(4) False. 例如

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

不論用什麼基底, 矩陣表示的行列式都是0. (綜線CH7定理19,CH4定理6)

１１A１２【 元智83電資[4] 】

Let φ be an idempotent linear operator on V, that is; $\varphi^2 = \varphi$. Prove that $V = \operatorname{Im}\varphi \oplus \ker\varphi$, where $\operatorname{Im}\varphi$ is the range of φ and $\ker\varphi$ is the kernel of φ.

【解】此題考定理證明(並不難!). 請參閱綜線CH11定理12②.

１１A**13** 【 交大78資工[4](b) 】

(b) Is the matrix $\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}$ involutory ? Explain.

【要訣】 $A^2 = I$ 時稱A爲involutory.

【解】

$$\left(\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}\right)^2 = \frac{1}{9}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ -2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Yes. (綜線CH11定義10④)

題型11B:　正交分解

１１B**01** 【 師大84資教[10] 】

Let V and W be two vector spaces. Which ones are orthogonal.

(a) V=X軸, W=Y軸,　　(b) V=Z軸, W=XY平面,　　(c) V=XY平面, W=XZ平面

【解】(a)(b)

【說明】XY平面與XZ平面並不正交!

例如$(1,1,0)\cdot(0,1,1)\neq 0$.　(綜線CH11定義15)

１１B**02** 【 師大84資教[11] 】

If X and Y are subspaces of $\mathbb{R}^n$ and are mutually orthogonal, what is $X\cap Y$?

【解】 $\{o\}$ (綜線CH11定理16)

１１B**03** 【 師大84資教[12] 】

Let X＝span(e_1) and Y＝span(e_2), where e_1 and e_2 are the standard axes of $\mathbb{R}^3$. Are X and Y orthogonal complements ?

【解】No. (綜線CH11定義1,定義15)

１１B**04** 【 交大78資工[1] 】

(a) Give an example of n-dimensional vector space, $n\leq 10$.

(b) In (a), give two subspaces that have no intersection if possible.

(c) In (a), give two subspaces that are orthogonal to each other.

【解】(a) 令$V=\{(x, y, z)\mid x, y, z\in\mathbb{R}\}$, 以$\mathbb{R}$爲scalar field,

向量加法定義爲 $(x_1, y_1, z_1)+(x_2, y_2, z_2)=(x_1+x_2, y_1+y_2, z_1+z_2)$

純量積定義爲 $c\cdot(x, y, z)=(cx, cy, cz)$

則V爲vector space over $\mathbb{R}$. (綜線CH5定義3, 範例4)

可取$B=\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ 爲V之基底,

所以 $\dim V=3\leq 10$ (綜線CH6定理19)

(b) 任二subspace都有零向量在內, 不可能 "have no intersection"

(綜線CH5定理11要訣(2))

(c) 令$W_1=\{(x,0,0)\mid x\in\mathbb{R}\}$, $W_2=\{(0,y,0)\mid y\in\mathbb{R}\}$

$\forall\ (x,0,0)\in W_1$, $(0,y,0)\in W_2$, $(x,0,0)\cdot(0,y,0)=0$

$\therefore$ W_1, W_2 are orthogonal to each other. (綜線CH11定義15)

１１B**05**【 交大82工工[6](b) 】

Label the following statements as being true or false.

(b). The orthogonal complement of any set is a subspace. (T. F).

【解】(b) True. (綜線CH11定理17)

１１B**06**【 中央85資工[2](g) 】

[是非論證題]

(g) $V\cap V^{\perp}$ is always non-empty, where $V^{\perp}$ is the orthogonal complement of V.

【解】(g) T,

依習慣, V代表某向量空間的子空間, 於是V與$V^{\perp}$ 都擁有零向量. 所以交集非空. (綜線CH5定理11, CH11定理17)

【討論】若V只是某向量空間的子集, 則本題變成False. 例如在$\mathbb{R}^3$,

$\{(1,0,0)\}\cap\{(1,0,0)\}^{\perp}=\{(1,0,0)\}\cap yz$平面$=\varnothing$

１１B**07**【 中央84資工[1](ij) 】

True or False. (一定要有說明、證明或反例。每小題5分)

(i) If V is orthogonal to W, then $V^{\perp}$ is orthogonal to $W^{\perp}$, where $V^{\perp}$ is the orthogonal complement of V.

(j) $V\cap V^{\perp}$ may be an empty set.

【解】(i) False. 例如

在$\mathbb{R}^3$ 中取$V=\{(x,0,0)\mid x\in\mathbb{R}\}$, $W=\{(0,y,0)\mid y\in\mathbb{R}\}$,

則 V is orthogonal to W,

$V^{\perp}=\{(0,y,z)\mid y,z\in\mathbb{R}\}$, $W^{\perp}=\{(x,0,z)\mid x,z\in\mathbb{R}\}$.

$V^{\perp}$與$W^{\perp}$並不orthogonal. (綜線CH11定義15)

(j) False.

$V\cap V^{\perp}=\{o\}\neq\varnothing$ (綜線CH11定理17)

１１B**08** 【 中正84資工[5] 】

Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{w_1, w_2,...,w_p\}$ and let $\{v_1, v_2, ..., v_q\}$ be an orthogonal basis for $W^{\perp}$ (the orthogonal complement of W).

(a) Explain why $\{w_1,w_2,...,w_p, v_1, v_2,..., v_q\}$ is an orthogonal set.

(b) Explain why the set in part (a) spans $\mathbb{R}^n$.

(c) Show that $\dim(W)+\dim(W^{\perp})=n$.

【分析】若套用定理$V=W\oplus W^{\perp}$，則本題將變成極爲trivial. (綜線CH11定理17)

這題應將證明過程展出.

【解】(a) 對任意$i=1,2,..., p, j=1,2,...,q$,

$\because\ v_j\in W^{\perp}$, $\therefore\ \langle w_i , v_j\rangle=0$ (綜線CH9定義4)

已知$\{w_1 , ..., w_p\}$ 兩兩正交， $\{v_1 , ..., v_q\}$ 兩兩正交.

$\therefore\ \{w_1 , ..., w_p , v_1 , ..., v_q\}$ 兩兩正交.

(b) 對$v\in\mathbb{R}^n$,

令 $a_i=\dfrac{\langle v,w_i\rangle}{\langle w_i ,w_i\rangle}$, $i=1, 2, ... , p$

則v對W的正投影向量爲 $x=\sum a_i w_i$ (綜線CH9定理12)

$\therefore\ v-x\in W^{\perp}$ (綜線CH9定義11)

$\therefore\ \exists b_j$ 使得 $v=\sum b_j v_j$

$\therefore\ v=\sum a_i w_i+\sum b_j v_j$.

(c) 由(a)已知$\{w_1, ... , w_p, v_1, ... , v_q\}$ 爲正交集，且其中皆非零.

$\therefore$ 此集合線性獨立. (綜線CH9定理15)

由(b)知此集合生成$\mathbb{R}^n$.

$\therefore$此集合爲$\mathbb{R}^n$的基底.

$\therefore\ n=\dim\mathbb{R}^n=p+q=\dim W+\dim W^{\perp}$.

１１B**09** 【 交大81資科[5] 】

Let V be a finite dimensional real vector space with inner product, W be a subspace of V, and

$W^{\perp}$ be the set of all vectors of V which are perpendicular to W.

(a) Prove that $W^{\perp}$ is a subspace of V.

(b) Prove that $\dim W + \dim W^{\perp} = \dim V$.

(c) For any given vector $v \in V$, prove that v can be expressed uniquely as a sum $w_1 + w_2$ with $w_1 \in W$ and $w_2 \in W^{\perp}$.

(d) Suppose $V = \mathbb{R}^2$, $W = \text{span}\left\{ \begin{bmatrix} 1 \\ m \end{bmatrix} \right\}$ and $v = \begin{bmatrix} a \\ b \end{bmatrix}$. Find w_1 and w_2 as stated in part (c)

【分析】本題(a)(b)(c)考定理證明，請參閱綜線CH11定理17.
其中(c)相當於是在證明$V = W \oplus W^{\perp}$, (綜線CH11定理3a)
而(b)是(c)的推論. 所以應先證明(c)再證明(b).

【解】(d)

$$\langle v, \begin{bmatrix} 1 \\ m \end{bmatrix} \rangle = a + bm, \quad \langle \begin{bmatrix} 1 \\ m \end{bmatrix}, \begin{bmatrix} 1 \\ m \end{bmatrix} \rangle = 1 + m^2$$

$$w_1 = \frac{a+bm}{1+m^2} \begin{bmatrix} 1 \\ m \end{bmatrix}$$

$$w_2 = v - w_1 = \begin{bmatrix} a \\ b \end{bmatrix} - \frac{a+bm}{1+m^2} \begin{bmatrix} 1 \\ m \end{bmatrix} = \frac{b-am}{1+m^2} \begin{bmatrix} -m \\ 1 \end{bmatrix}$$

１１B**１０**【 交大80資工[4] 】

Let $\{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ be an orthonormal basis for a vector space V with an inner product $(\cdot, \cdot)$. Let S_1 be the subspace of V spanned by $v_1, \ldots, v_k$ and let S_2 be the subspace spanned by $v_{k+1}, \ldots, v_n$.

(a) Show that $S_1 \perp S_2$, i.e. $(u, v) = 0$, $\forall u \in S_1$ and $\forall v \in S_2$. (5%)

(b) Let x be an element of V and let p_1 and p_2 be the orthogonal projections of x onto S_1 and S_2, respectively. Show that $x = p_1 + p_2$. (5%)

【分析】$\{v_1, \ldots, v_n\}$爲orthonormal set $\Longleftrightarrow (v_i, v_j) = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$

【解】(a) $\forall u \in S_1$, $Av \in S_2$,

令 $u = a_1v_1 + a_2v_2 + \ldots + a_kv_k$, $v = a_{k+1}v_{k+1} + a_{k+2}v_{k+2} + \ldots + a_nv_n$

則 $(u, v) = (\sum_{i=1}^{k} a_iv_i, v) = \sum_{i=1}^{k} a_i(v_i, v)$ (左線性條件)

$= \sum_{i=1}^{k} a_i(v_i, \sum_{j=k+1}^{n} a_jv_j) = \sum_{i=1}^{k} a_i \sum_{j=k+1}^{n} \overline{a_j}(v_i, v_j)$ (右半線性條件)

$= \sum_{i=1}^{k} a_i \sum_{j=k+1}^{n} \overline{a_j} \cdot 0 = 0$

(b) 令$x = a_1v_1 + a_2v_2 + \ldots + a_nv_n$

則$(x, v_i) = (a_1v_1 + a_2v_2 + \ldots + a_nv_n, v_i)$

$= a_1(v_1, v_i) + a_2(v_2, v_i) + \ldots + a_n(v_n, v_i) = a$

$\therefore p_1 = \sum_{i=1}^{k} \dfrac{(x, v_i)}{(v_i, v_i)} v_i = \sum_{i=1}^{k} a_iv_i$ (綜線CH9定理12)

$p_2 = \sum_{i=k+1}^{n} \dfrac{(x, v_i)}{(v_i, v_i)} v_i = \sum_{i=k+1}^{n} a_iv_i$ (綜線CH9定理12)

$\therefore p_1 + p_2 = x$

題型11C: 四子空間的互補性質

１１C**01** 【 交大86資工[8] 】

Let $A=\begin{bmatrix}2&1\\1&1\\2&1\end{bmatrix}$ and $b=\begin{bmatrix}12\\6\\18\end{bmatrix}$

(a) Determine the projection matrix Q that projects vectors in $\mathbb{R}^3$ onto the nullspace of A^T. (5%)

(b) Solve the least square problem $Ax=b$ through computing the QR factorization of matrix A. (10%)

【解】依題目(b)小題之要求，先對A做Gram-Schmidt process及QR分解:

$$v_1'=v_1=\begin{bmatrix}2\\1\\2\end{bmatrix},\quad v_2'=v_2-\frac{5}{9}v_1'=\begin{bmatrix}-1/9\\4/9\\-1/9\end{bmatrix}$$

再做單位化: $\|v_1'\|=\sqrt{9}=3$, $\|v_2'\|=\sqrt{18}/9=\sqrt{2}/3$

$$v_1''=v_1'/\|v_1'\|=\begin{bmatrix}2/3\\1/3\\2/3\end{bmatrix},\quad v_2''=v_2'/\|v_2'\|=\begin{bmatrix}-1/\sqrt{18}\\4/\sqrt{18}\\-1/\sqrt{18}\end{bmatrix}$$

移項得 $$\begin{cases}v_1=v_1' & =3v_1''.\\ v_2=(5/9)v_1'+v_2'=(5/3)v_1''+(\sqrt{2}/3)v_2''\end{cases}$$

$$\therefore\quad A=\begin{bmatrix} 2/3 & -1/\sqrt{18} \\ 1/3 & 4/\sqrt{18} \\ 2/3 & -1/\sqrt{18} \end{bmatrix}\begin{bmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{bmatrix}$$

<--- A的QR分解

(綜線CH定理6)

(a) (請參閱題型09B)

對$v\in\mathbb{R}^{3\times 1}$，令$v$對[$A^{\mathrm{T}}$的null space]的正投影向量爲$q$.

$\because$ A^{T}的null space是A的column space的正交補空間. (綜線CH11定理23)

$\therefore$ $v-q$爲v對[A的column space]的正投影向量.

由前述計算已取得A的column space的orthonormal basis $\{v_1'', v_2''\}$.

$\therefore$ $v-q=(v_1''^{\mathrm{T}}v)v_1''+(v_2''^{\mathrm{T}}v)v_2''=(v_1''v_1''^{\mathrm{T}})v+(v_2''v_2''^{\mathrm{T}})v$

$\therefore$ $q=v-(v_1''v_1''^{\mathrm{T}})v-(v_2''v_2''^{\mathrm{T}})v=(I-(v_1''v_1''^{\mathrm{T}})-(v_2''v_2''^{\mathrm{T}}))v$

$$\therefore\ \text{所求矩陣}Q=I-(v_1''v_1''^{\mathrm{T}})-(v_2''v_2''^{\mathrm{T}})=\begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

(b) (請參閱題型09E)

QR分解後的least-square problem應解: (綜線CH9定理21a)

$$\begin{bmatrix} 3 & 5/3 \\ 0 & \sqrt{2}/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -1/\sqrt{18} & 4/\sqrt{18} & -1/\sqrt{18} \end{bmatrix}\begin{bmatrix} 12 \\ 6 \\ 18 \end{bmatrix}$$

即 $3x_1+(5/3)x_2=22$， $(\sqrt{2}/3)x_2=-\sqrt{2}$

解得 $x_2=-3$, $x_1=9$

所求 $x=\begin{bmatrix} 9 \\ -3 \end{bmatrix}$.

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１１C**02** 【 中央83資工[5] 】

Let S be the subspace of $\mathbb{R}^4$ containing all vectors $(x_1, x_2, x_3, x_4)^T$ with $x_1+x_2+x_3+x_4=0$ and $x_1+2x_2+3x_3+4x_4=0$.

(a) Find two bases for the space S and the space $S^\perp$ (the space containing all vectors orthogonal to S) respectively. (10%)

(b) Find the projection of $(1, 2, 7)^T$ onto the space $S^\perp$. (10%)

【勘誤】本題(b)的向量 $(1, 2, 7)^T$ 並不在$\mathbb{R}^4$ 內, 無法對S作投影.
應修改爲 $(1, 2, 7, 0)^T$.

【解】(a) 解方程式 $\begin{cases} x_1+x_2+x_3+x_4=0 \\ x_1+2x_2+3x_3+4x_4=0 \end{cases}$

對係數矩陣做列運算:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

$$\therefore\ S=\left\{ \begin{bmatrix} t+2u \\ -2t-3u \\ t \\ u \end{bmatrix} \middle|\ t, u\in\mathbb{R} \right\}.$$ (綜線CH3範例7,CH5範例21)

$$\therefore\ \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$$ 爲S的基底. (∵線性獨立且生成S)

S是矩陣 $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ 的null space,

$\therefore$ $S^{\perp}$ 是它的row space取transpose. (綜線CH11定理23,範例24)

$\therefore$ $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$是$S^{\perp}$的基底. ($\because$線性獨立且生成$S^{\perp}$)

(b) (請參閱題型09B)

令$B=\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$, 則所求投影向量為:

$$B(B^{T}B)^{-1}B^{T}\begin{bmatrix} 1 \\ 2 \\ 7 \\ 0 \end{bmatrix}=\ldots=\frac{1}{5}\begin{bmatrix} 11 \\ 12 \\ 13 \\ 14 \end{bmatrix}$$ (綜線CH9定理22)

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１１C**03** 【 交大83工工[7] 】

Prove: If A is an $m\times n$ matrix and $x\in\mathbb{R}^{n}$, then either $Ax=o$ or there exists $y\in\mathcal{R}(A^{T})$ such that $x^{T}y\neq 0$ [$\mathcal{R}(A^{T})$ is the column space of A^{T}].

【分析】CSP(A^{T})就是(RSPA)^{T}，又是ker$A$的正交補空間.

證明“α 或 β” 通常是採用“若($\neg\alpha$)則β” 的格式.

但這題其實不只是“α或β”, 而且還是 “α XOR β”

證明 “α XOR β” 是採用 “ $\neg\alpha \Longleftrightarrow \beta$” 的格式.

【解】 $\neg(\exists\, y\in\mathcal{R}(A^{\mathrm{T}})$ such that $x^{\mathrm{T}}y\neq 0)$

$\Longleftrightarrow$ $\forall\, y\in\mathcal{R}(A^{\mathrm{T}})$, $x^{\mathrm{T}}y=0$

$\Longleftrightarrow$ $x\in (\mathcal{R}(A^{\mathrm{T}}))^{\perp}$

$\Longleftrightarrow$ $x\in \ker A$ (綜線CH11定理23)

$\Longleftrightarrow$ $Ax=o$.

１１C04 【 元智83工工[12] 】

Explain why exactly one of the following two systems has a solution.

(1) $Ax=b$ (2) $A^{\mathrm{T}}y=o$, $y^{\mathrm{T}}b\neq 0$.

A is a m by n matrix, $x\in\mathbb{R}^{n}$, $y\in\mathbb{R}^{m}$.

【分析】ker(A^{T})就是(lker$A$)$^{\mathrm{T}}$，又是CSPA的正交補空間.

本題與上題是天生一對的絕配.

【解】 “$A^{\mathrm{T}}y=o$, $y^{\mathrm{T}}b\neq 0$. ”無解 $\Longleftrightarrow$ $\forall y$, $\neg(A^{\mathrm{T}}y=o$ 且 $y^{\mathrm{T}}b\neq 0)$

$\Longleftrightarrow$ $\forall y$, ($A^{\mathrm{T}}y\neq o$ 或 $y^{\mathrm{T}}b=0$)

$\Longleftrightarrow$ $\forall y$, (若$A^{\mathrm{T}}y=o$ 則 $y^{\mathrm{T}}b=0$) $\Longleftrightarrow$ 若 $y\in\ker(A^{\mathrm{T}})$ 則 $b\perp y$)

$\Longleftrightarrow$ $b\in(\ker(A^{\mathrm{T}}))^{\perp}$ $\Longleftrightarrow$ $b\in$CSP(A) (綜線CH11定理23)

$\Longleftrightarrow$ “$Ax=b$” 有解. (綜線CH8定理18)

１１C05 【 中央82資電[6] 】

(a) What is the definition of “orthogonal complement” of vector space. (6%)

(b) What is the orthogonal complement of column space. (3%)

(c) If V is orthogonal to W. Is $V^{\perp}$ orthogonal to $W^{\perp}$? (3%)

(d) If V is orthogonal to W and W is orthogonal to Z. Is V orthogonal to Z ? (3%)

【解】(a) 設V是內積空間U的子空間, V的orthogonal complement 定義爲

$\{\, u\in U \mid \forall v\in V, \langle u,v\rangle=0 \,\}$. (綜線CH9定義4)

(b) 當A爲複數矩陣時, A的column space的orthogonal complement爲

“ A^{H} 的null space ” (綜線CH11定理23)

當A爲實數矩陣，且取實數系爲scalar field時A的column space的

orthogonal complement爲 “A^{T}的null space”．有的書(G.Strang)在討論正交

時專討論實數系，且在正文中只承認行矩陣是向量. 它將A的left null space 定義成A^T的null space. 若依此書的定義, A的column space的orthogonal complement就正是A的left null space.

(編註: 依此份考題的符號習慣研判，命題用書應是G.Strang)

(c) No.　反例如下:

在$\mathbb{R}^3$ 中，取 V為x軸, W為y軸，則 $V^\perp$ 為yz平面, $W^\perp$ 為xz平面.

但x軸與y軸為orthogonal，但yz平面與xz平面並不orthogonal! (綜線CH11定義15)

(d) No.　反例如下:

在$\mathbb{R}^3$ 中，取 V為x軸, W為z軸, $Z = \{ (t, t, 0) \mid t \in \mathbb{R}\}$

則 $V \perp W$, $W \perp Z$, 但 V與Z並不為orthogonal .

１１C**06** 【 交大81資工[4](a) 】

(RS(A): the column space of A.　NS(A) the null space of A.)

True or false, two points for each.

(a) Let A be an $m\times n$ matrix repersenting a mapping from the Euclidean vector space $\mathbb{R}^n$ to $\mathbb{R}^m$. Then $\mathbb{R}^n = \text{RS}(A^T) \oplus \text{NS}(A)^\perp$.

【分析】RS常被用來表示row space，但此份考題定義RS(A)為A的column space.

【解】(a) False.

應是$\mathbb{R}^n = \text{RS}(A^T) \oplus \text{NS}(A)$　　(綜線CH11定理23)

(因為敘述上剛好相反，所以反例隨便舉就可適用。)

取$n=2$，並令 $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$,

column space　$\text{RS}(A^T) = x$軸

null space　$\text{NS}(A) = y$軸,　$\text{NS}(A)^\perp = x$軸.

１１C**07** 【 交大82資工[3](e) 】

[是非倒扣題]

(e) Let A be an $m\times n$ matrix with rank r, where $r < m$ and $r < n$. For each vector x in the column

space of A, there is a unique vector y in the row space of A such that $y=Ax$.

【解】(e) False.

【討論】A是$m\times n$矩陣，而x在A的column space內,是$m\times 1$矩陣，句末的乘積Ax 可能因$m\neq n$而變成無意義。 本題原意應該是

"... For each vector y in the column space of A, there is a unique vector x in the row space of A such that $y=Ax$" 但因這題是要倒扣的是非題，而且沒有解說理由的機會，所以最好不要擅自修改題目。

題型11D: 不變子空間

１１D**01**【 台大85資工[7] 】

[複選題]

Let T be a linear operator on a vector space V. Which of the following are true.

(1) It is possible that T does not have T-invariant subspaces.

(2) For any $x\in V$, then the T-cyclic subspace generated by x is the same as that T-cyclic subspace generated by $T(x)$.

(3) Let V be finite-dimensional, and let E_λ, $K_\lambda$denote the eigenspace and the generalized eignespace of T corresponding to the eigenvalue λ of T, respectively. Then, K_λ is a T-invariant subspace of V containing E_λ.

(4) Let W be a T-invariant subspace of V. Then, W is $g(T)$-invariant for any polynomial $g(t)$.

【解】選(3)(4)

【討論】

(1) False. 至少V自己就是T-invariant subspace. (綜線CH11定理30)

(2) False. 例如 $T:\mathbb{R}^3\longrightarrow\mathbb{R}^3$, $T(a,b,c)=(0, a, 0)$

對$x=(1,0,0)$, 求得 $T(x)=(0,1,0)$, $T(T(x))=(0,0,0)$

x生成的cyclic subspace為 $\{ (p, q, 0) \mid p, q\in\mathbb{R} \}$

$T(x)$生成的cyclic subspace為 $\{ (0, q, 0) \mid q\in\mathbb{R} \}$

兩者並不相等.

(3) True. 此為定理. (綜線CH15定義3,定理5)

(4) True. 對$g(t)=a_0+a_1t+a_2t^2+...+a_kt^k$

$g(T)w=(a_0I+a_1T+a_2T^2+...+a_kT^k)w=a_0w+a_1Tw+a_2T^2w+...+a_kT^kw$

$\because$ W為T-invariant,

$\therefore$ $w\in W \Longrightarrow Tw\in W \Longrightarrow T(Tw)\in W \Longrightarrow ... \Longrightarrow T^kw\in W \Longrightarrow ...$

再由W的封閉性即得知 $g(T)w\in W$