題型12A: 特徵值與特徵向量

１２A**01** 【 淡江85資工[3] 】

Let A be a 2×2 matrix. Show that A and A^t have the same set of eigenvalues. (25%)

【解】 $\det(A^t-xI)=\det((A-xI)^t)$ (綜線CH2定理23)

$=\det(A-xI)$ (綜線CH4定理5)

∴ A與A^t 的特徵多項式相同.

∴ A與A^t的eigenvalues相同. (綜線CH12定義8要訣3)

【討論】本題與A的階數無關.

１２A**02** 【 交大80資工[3](ab) 】

Let A be a real $n\times n$ matrix. Show that

(a) $Ax=x$ has a nonzero solution x if and only if $\det(A-I)=0$. (3%)

(b) The set $\{ x \mid Ax=x \}$ is a subspace of $\mathbb{R}^n$. (3%)

【分析】本題考綜線CH7定理7的證明的特殊情形.

【解】 (a) 存在$x\neq o$使$Ax=x$

$\Longleftrightarrow$ 存在$x\neq o$使$(A-I)x=o$

$\Longleftrightarrow$ $NS(A-I)\neq\{o\}$ (綜線CH5定義19)

$\Longleftrightarrow$ $A-I$ 不可逆 (綜線CH8定理17)

$\Longleftrightarrow$ $\det(A-I)=0$ (綜線CH4定理17)

(b) 由(a)已證得$\{x \mid Ax=x\}=NS(A-I)$

∴ 此集合爲$\mathbb{R}^n$ 的subspace. (綜線CH5定理20)

(若臨場時間充足，可就此部份展開.)

１２A**03** 【 交大82工工[6](c) 】

Label the following statements as being true or false.

(c). If a real matrix has one eigenvector, then it has an infinite number of eigenvectors.

【解】(c) True.

若v是實數矩陣A的eigenvector, 即 $\exists\lambda\in\mathbb{R}$, 使得 $Av=\lambda v$

對任意$k\in\mathbb{R}$, $A(kv)=k(Av)=k(\lambda v)=\lambda(kv)$

$\therefore$ kv也是A的eigenvector. 這種kv有無限多個。

１２A**04** 【 交大82工工[6](e) 】

[是非題]

(e). The sum of two eigenvectors of an operator T is always an eigenvector of T.

【解】(e) False. (綜線CH12定理3)

考慮 $T:\mathbb{R}^2\longrightarrow\mathbb{R}^2$, $T(x, y)=(2x, 3y)$

對 $u=(1, 0)$, $v=(0, 1)$, $u+v=(1,1)$.

$T(u)=2u$, $T(v)=3v$, $\therefore$ u, v都是T的eigenvector.

但對任意$k\in\mathbb{R}$, $T(u+v)=(2, 3)\neq k(1, 1)$

$\therefore$ $u+v$不是T的eigenvector.

１２A**05** 【 交大84資科[7](a) 】

Let A be a $p\times p$-dimensional matrix.

(a) Let λ be an eigenvalue of A. Show that if both A and λ are real then there is an associated real eigenvector x for λ. (5%)

【解】(a) $\because$ λ be an eigenvalue of A.

$\therefore$ $\det(A-\lambda I)=0$ (綜線CH12定理7)

$\therefore$ $A-\lambda I$ 爲不可逆的實數矩陣. (綜線CH4定理17)

$\therefore$ $\ker(A-\lambda I)\neq\{o\}$ (綜線CH8定理17)

令 $x\in\ker(A-\lambda I)\setminus\{o\}$

則 x爲非零的實數向量, 且$Ax=\lambda x$.

１２A**06** 【 交大80資工[1](a)(ⅲ) 】

True (T) or False (F): (1 for each)

(a) Suppose A is row equivalent to B, i.e., $A\overset{R}{\sim}B$.

(ⅲ). A and B have the same eigenvalues.

【解】(a) (ⅲ) False. 反例如下:

取$A=\begin{bmatrix}2 & 0\\0 & 3\end{bmatrix}$, $B=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

A之eigenvalues爲2,3. B之eigenvalues爲1,1.

A, B列等價, 但eigenvalue不同.

１２A07 【 大同80資工[3](l) 】

True or False:

(l) Exchanging the rows of a 2 by 2 matrix reverses the signs of its eigenvalues.

【解】(l) False, 反例如下:

取$A=\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}$, A經列對調後爲$A'=\begin{bmatrix}2 & 0\\1 & 0\end{bmatrix}$

A的eigenvalue爲1, 0. A'的eigenvalue爲2, 0. (綜線CH12定理7)

１２A08 【 台大86資工[1](e) 】

[是非題]

(e) 設P是permutation matrix $\in \mathbb{R}_{n\times n}$, A與PA有相同的eigenvalues.

【解】False.

【說明】例如 $A=\begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}$, $P=\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$, $PA=\begin{bmatrix}0 & 2\\2 & 0\end{bmatrix}$

A的特徵值爲 2, 2

$\det(PA-xI)=x^2-2=(x-2)(x+2)$, PA的特徵值爲2, –2.

１２A09 【 清大82工工[1] 】

(a) Consider an eigensystem of $Ax=\lambda x$ with $A=[\ a_{ij}\ ]_{n\times n}$

If $a_{ii}=1$, $\forall i$. Prove that $\sum_{i=1}^{n}\lambda_i=n$ (10%)

(b) Use the following example to explain it. (5%)

$$A=\begin{bmatrix}1&3&5\\10&1&8\\4&9&1\end{bmatrix}$$

【解】(a) 在複數系將A三角化, 取得可逆矩陣P, 使得 $P^{-1}AP=\Gamma$,

其中Γ爲下三角矩陣. 且對角線爲A的特徵值. (綜線CH13定理8)

$$\sum_{i=1}^{n}\lambda_i=\mathrm{tr}(\Gamma)=\mathrm{tr}A=\sum_{i=1}^{n}a_{ii}=n$$ (綜線CH2定理28)

(b)

$$\det(A-xI)=\det\begin{bmatrix}1-x&3&5\\10&1-x&8\\4&9&1-x\end{bmatrix}=\ldots=-x^3+3x^2+119x+425$$

$$=-(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)=-x^3+(\lambda_1+\lambda_2+\lambda_3)x^2-(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)x+\lambda_1\lambda_2\lambda_3$$

比較兩次項係數即得 $3=\lambda_1+\lambda_2+\lambda_3$.

１２A**１０**【 雲技85工工[10] 】

已知線性變換$AX=\lambda X$, 其中$A=[\,a_{ij}\,]_{n\times n}$, X爲非零向量.

假定 $a_{ij}=1$ $\forall i$, 試證明 $\sum_{i=1}^{n}\lambda_i=n$.

【分析】題目很混亂. 不過還看得出$\lambda_1, \ldots, \lambda_n$ 是A的所有特徵值.

【解】

$$\sum_{i=1}^{n}\lambda_i=\mathrm{tr}A=\sum_{i=1}^{n}a_{ii}=n$$ (綜線CH12定理13要訣1)

題型12B: 可對角化理論

１２B０１ 【 中正80資工[2] 】
Prove that an $n\times n$ matrix A is diagonalizable if it has n linearly independent eigenvectors.

【參考章節】CH12定義15,定理16

【解】1° 設$n\times 1$矩陣$v_1, v_2, \ldots, v_n$爲linear independent, 且 $Av_i=\lambda_i v_i$, $i=1, 2, \ldots, n$.

令$n\times n$矩陣P的第j行($j=1, 2, \ldots, n$)爲v_j.

2° 先證P可逆:

∵ $v_1, v_2, \ldots, v_n$ 爲n維行矩陣空間中的獨立集.

∴ $v_1, v_2, \ldots, v_n$ 形成n維行矩陣空間的基底. (綜線CH6定理22)

∴ $\text{rank}P=\dim(\text{CSP}P)$ (綜線CH8定義12)

$=n$ (綜線CH6定理19)

∴ P可逆 (綜線CH8定理17)

3°再證$P^{-1}AP$爲對角矩陣:

$$Av_1=\lambda_1 v_1=P\begin{bmatrix}\lambda_1\\0\\\vdots\\\vdots\\0\end{bmatrix}$$ (綜線CH2定理6②)

$$Av_2=\lambda_2 v_2=P\begin{bmatrix}0\\\lambda_2\\0\\\vdots\\0\end{bmatrix}$$ (綜線CH2定理6②)

$\cdots\cdots$

$$Av_n = \lambda_n v_n = P\begin{bmatrix} 0 \\ 0 \\ \vdots \\ \lambda_n \end{bmatrix}$$ (綜線CH2定理6②)

$$\therefore\ AP = P\begin{bmatrix} \lambda_1 & & & \bigcirc \\ & \lambda_2 & & \\ & & \ddots & \\ \bigcirc & & & \lambda_n \end{bmatrix}$$ (綜線CH2定理6①)

$$\therefore P^{-1}AP = \begin{bmatrix} \lambda_1 & & & \bigcirc \\ & \lambda_2 & & \\ & & \ddots & \\ \bigcirc & & & \lambda_n \end{bmatrix}$$ (綜線CH2定理12a)

１２B**02** 【 中原85工工[3] 】

Let A be a $n\times n$ matrix and λ be one of its eigenvalues and v be the corresponding eigenvectors of λ.

(a) Can λ be 0 ? Can v be a zero vector ? (5%)

(b) Write down the definition that A is diagonalizable. (5%)

(c) Show that if A has n linearly independent eigenvectors then A is diagonalizable. (10%)

【解】(a) eigenvalue可以是0.

通常eigenvector不能是零向量. 但也有某些書(Hoffman)將零向量也算是eigenvector. (綜線CH12定義1)

(b) 若存在可逆矩陣P, 使$P^{-1}AP$爲對角矩陣, 就稱A爲diagonalizable.

(綜線CH12定義15)

(c) (本小題可參閱綜線CH12定理16)

設 $v_1, v_2, \ldots, v_n$ 線性獨立.

將行向量排成矩陣P, 則$\ker P=\{o\}$ (綜線CH6定理15)

∴ P爲可逆矩陣. (綜線CH8定理17)

若 $Av_i=\lambda_i v_i$, $i=1,2,\ldots,n$.

$$\text{則 } Av_1=P\begin{bmatrix}\lambda_1\\0\\\vdots\\0\end{bmatrix}, Av_2=P\begin{bmatrix}0\\\lambda_2\\\vdots\\0\end{bmatrix}, \ldots, Av_n=P\begin{bmatrix}0\\0\\\vdots\\\lambda_n\end{bmatrix}$$

(左直切:綜線CH2定理6②)

$$\therefore\ AP=P\begin{bmatrix}\lambda_1&0&\cdots&0\\0&\lambda_2&\cdots&0\\\vdots&\vdots&\cdots&\vdots\\0&0&\cdots&\lambda_n\end{bmatrix}$$

(右直切:綜線CH2定理6①)

∴ $P^{-1}AP$爲對角線矩陣.

12B03 【 中央86資工[1](i) 】

[是非論證題]

(i) If $n\times n$ matrix A has n linearly independent eigenvectors, then A is invertible.

【解】(i) False.

例如 $A=\begin{bmatrix}2&0\\0&0\end{bmatrix}$ 有2個線性獨立的特徵向量 $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\end{bmatrix}$. 但$A$不可逆.

１２B**04** 【 中央86資工[5] 】

Show that if A is diagonalizable, then A^T and A^{-1} are also diagonalizable.

【證】 ∵ A可對角化

∴ ∃可逆矩陣P使得 $A = PDP^{-1}$ (綜線CH12定義15)

∴ $A^T = (P^{-1})^T D^T P^T = (P^T)^{-1} D (P^T)$ (綜線CH2定理23)

∴ A^T有n個線性獨立的特徵向量.

依題目的寫法應是已假設A可逆. (否則不能寫A^{-1})

$A^{-1} = (P^{-1})^{-1} D^{-1} P^{-1}$ (綜線CH2定理12)

∴ A^{-1}有n個線性獨立的特徵向量. (綜線CH12定理16)

１２B**05** 【 中央85資工[2](hij) 】

[是非論證題]

(h) If $n \times n$ matrix A has n linear-independent eigenvectors, then so do both A^T and A^{-1}.

(i) If A is row equivelent to the identity matrix I, then A is diagonalizable.

(j) If A is diagonalizable, then the columns of A are linearly independent.

【分析】(h) 有n個線性獨立的特徵向量 $\Longleftrightarrow$ 可對角化. (綜線CH12定理16)

(j) 方陣的行線性獨立表示可逆，也就是行列式不為零.

(綜線CH4定理17,CH8定理17)

對可三角化的矩陣來說，可逆的充要條件是特徵值都不為零.

(綜線CH14定理17)

【解】(h) T,

∵ A有n個線性獨立的特徵向量

∴ ∃可逆矩陣P使得 $A = PDP^{-1}$ (綜線CH12定理16)

∴ $A^T = (P^{-1})^T D^T P^T = (P^T)^{-1} D (P^T)$ (綜線CH2定理23)

∴ A^T有n個線性獨立的特徵向量. (綜線CH12定理16)

依題目的寫法應是已假設A可逆. (否則不能寫A^{-1})

$A^{-1} = (P^{-1})^{-1} D^{-1} P^{-1}$ (綜線CH2定理12)

∴ A^{-1}有n個線性獨立的特徵向量. (綜線CH12定理16)

(i) F,

對 $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$, 顯然 A列等價於I.

但A不可對角化. (因A已是Jordan form. 也可用CH12定理21判定)

(j) F.

對$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, 顯然 $\det(A-xI)=x(x-1)$

∵ A有完全相異之eigenvalues,

∴ A可對角化. (綜線CH12定理23)

但A的兩個行並非線性獨立.

１２B**06** 【 中央84資工[1](g) 】

(g) If matrix $A_{n\times n}$ has n independent eigenvectors, then A has n distinct eigenvalues.

【解】(g) False.

例如 $A=\begin{bmatrix}3 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$有independent的eigenvector $\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\1\end{bmatrix}$.

但只有兩個相異的eigenvalue 3, 2.

１２B**07** 【 台大83資工[7] 】

[複選題]

Let V be a finite-dimension vector space, and T is a linear operator on that space. Then,

(1) T can not be diagonalizable if T has fewer than n distinct eigenvalues,

(2) T can not be diagonalizable if the multiplicity of each eigenvalue of λ equals the dimension of E_λ, the eigenspace corresponding to the eigenvalue λ.

(3) T is diagonalizable if there exists an ordered basis β for V such that $[T]_\beta$ is diagonal.

(4) T can not be diagonalizable if T is a projection map.

【勘誤】 第(1)小題漏列$n=\dim V$的條件.
本題的“... can not be diagonalizable ...”應修改爲“... can not be diagonalized ...”或“... is not diagonalizable ...”.

【解】 選(3).

【討論】(1) 要可對角化必須有n個(重根需重計)特徵值(綜線CH12定理21). 不足n個相異特徵值仍可能是可對角化. 例如

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

(2) 所述爲可對角化的條件的一部份. (綜線CH12定理21)

(3) 此爲定義. (綜線CH12定義15)

(4) 投影映射都可對角化. (綜線CH11定理14)

１２B**08** 【 中央83資工[3](defg) 】

True or False (Give a reason if true, and give a counterexample if false).

Let A and B be two different $n\times n$ nonsingular matrices.

(d) If A has n different eigenvectors, then A has n independent eigenvectors. (3%)

(e) If A has repeated eigenvalues, then A may be diagonalizable and invertible. (3%)

(f) If A has zero eigenvalues, then A may be diagonalizable and invertible. (3%)

(g) Triangular factor $A=LDU$, where L and U have 1's on the diagonal, and D is a diagonal matrix. {eigenvalue of A} = { eigenvalues of D }. (3%)

【分析】(甲) 若存在可逆矩陣P, 使$P^{-1}AP$爲對角線矩陣, 通常說: “A is a diagonalizable matrix. (A是可對角化矩陣.)” ,或“A is diagonalizable. (A可對角化.)”, 或說“A can be diagonalized. (A能被對角化.)” .前兩種說法的diagonalizable是形容詞, 第三種說法的be diagonalized是動詞diagonalize使用被動語態.

(乙) 本題中(e)的“may”應解釋成“可能”. 其題意爲:
“There exists a matrix A which has repeated eigenvalues, and is diagonalizable and invertible.” 本題(f)的意義及修飾法同上.

在(e),(f)陳述“可能性”的情形下，反而是True時要舉例說明可能, False 時要證明不可能.

(丙) 本題(g)的敘述也很含糊. 就習慣用法判斷，本題漏講“L是下三角矩陣”和“U是上三角矩陣”的條件. 而它的原意是說 “若將A分解為LDU....的格式, 則A與D的eigenvalues相同”.

【解】 (d) False. 反例如下:

設 $A=\begin{bmatrix}0&0\\1&0\end{bmatrix}$, 則 $\det(A-xI)=x^2$,

A的特徵值為0. 可解得A的特徵向量為 $\begin{bmatrix}0\\t\end{bmatrix}, t\neq 0$.

A有無限多個特徵向量, 但並沒有兩個彼此獨立的特徵向量.

(與此題類似的正確敘述是: (綜線CH12定理23)
If A has n different eigenvalues, then A has n independent eigenvectors.)

(e) True. 理由如下:

A有重複的特徵值並不禁止可對角化或可逆. 例如

$A=\begin{bmatrix}2&0\\0&2\end{bmatrix}$ 的特徵值是2, 2. 而仍可對角化，可逆.

(f) False. 證明如下:

A有eigenvalue 0時, $\det(A-0I)=0$, 即$\det A=0$, $\therefore$ A不可逆. (綜線CH4定理17)

(g) False. 反例如下:

設 $L=\begin{bmatrix}1&0\\1&1\end{bmatrix}, D=\begin{bmatrix}2&0\\0&3\end{bmatrix}, U=\begin{bmatrix}1&1\\0&1\end{bmatrix}$.

$A=LDU=\begin{bmatrix}2&2\\2&5\end{bmatrix}$

$\det(A-xI)=x^2-7x+6=(x-6)(x-1)$.

A的特徵值爲 6, 1, 而D的特徵值爲2, 3.

１２B**09**【 台大86資工[3] 】

$A\in\mathbb{R}_{n\times n}$, $\lambda_1, \ldots, \lambda_k$ 爲A的不同的eigenvalues, $x_1, \ldots, x_k$ 是$\lambda_1, \ldots, \lambda_k$ 對應的eigenvectors, 試證 $\{x_1, x_2, \ldots, x_k\}$ linearly independent.

【解】請參閱綜合線性代數CH12定理20, 此處不再重覆.

１２B**10**【 交大84資科[7](b) 】

Let A be a $p\times p$-dimensional matrix.

(b) Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of A and let x_i be an eigenvector associated with λ_i, $1\le i\le k$. show that $\{x_1, x_2, \ldots, x_k\}$ is linearly independent. (5%)

【解】(b) 同上題.

１２B**11**【 中正82資工[1] 】

Let A be a $p\times p$ matrix. Show that if the set $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ is a collection of distinct eigenvalues of A and v_i is an eigenvector associated with λ_i for each i, then $v_1, v_2, \ldots, v_n$ are linearly independent. (10%)

【解】同上題.

１２B**12**【 清大81資科[28] 】

Show that the eigenvectors corresponding to different eigenvalues are linearly indepnedent.

【解】同上題.

１２B**13** 【 大同82資工[9] 】

[單選題]

(a) eigenvectors corresponding to the same eigenvalue are always linearly dependent;

(b) if λ_1 and λ_2 are distinct eigenvalues of a linear operator T, then $E_{\lambda_1} \cap E_{\lambda_2} = \{ o \}$;

(c) a linear operator T on a finite-dimensional vector space is diagonalizable if and only if the multiplicity of each eigenvalue λ equals the dimension of E_λ.

(d) every diagonalizable linear operator has at least one eigenvalue.

【解】 選 a

【說明】此題之E_λ 代表λ的eigenspace, 即 Ker($T-\lambda I$) .

(a) 爲false. 因有可能 $\dim(E_\lambda) > 0$, 此時E_λ中可取得線性獨立的兩個向量.
例如對 $T(x, y, z) = (3x, 3y, 2z)$,
(1,0,0) 與 (0,1,0) 線性獨立, 且都是eigenvalue 3 的eigenvector

(b) 爲定理, 證明如下:
設$v \in E_{\lambda_1} \cap E_{\lambda_2}$, 則 $T(v) = \lambda_1 v$, 且 $T(v) = \lambda_2 v$
$\therefore\ \lambda_1 v = \lambda_2 v$. $\therefore\ (\lambda_1 - \lambda_2) v = o$
而 $\lambda_1 - \lambda_2 \neq 0$ $\therefore\ v = o$

(c) T可對角化的充份必要條件爲內含兩條件. (綜線CH12定理21)
本題只述及第二條件, 照理說也是false!
但因本題爲單選題, 比較起來(a)是較明顯的錯誤, 我們認定(c)是命題上的疏忽(也可能命題者的想法與眾不同), 所以還是選答(a) .

１２B**14** 【 台大85資工[6] 】

[複選題]

Let V be an n-dimensional vector space, and let T, U be linear operators on V. Which of the following are true.

(1) If T has fewer that n distinct eigenvalues, then T is not diagonalizable.

(2) Let λ_1 and λ_2 be two distinct eigenvalues of T, and let E_{λ_1} and E_{λ_2} be the two corresponding eigenspaces. For any $x_1 \in E_{\lambda_1}$ and $x_2 \in E_{\lambda_2}$, if $x_1 + x_2 = 0$, then $x_1 = x_2 = 0$.

(3) Let T and U be diagonalizable operators. Then, T and U are simultaneueously diagonalizable if $TU = UT$.

(4) There can not be a polynomial $g(t)$ with degree less than n such that $g(T)=O$. (zero operator).

【解】選(2)(3)

【討論】

(1) False.

若有n個相異的eigenvalue則可對角化. (綜線CH12定理23)

但可對角化未必要有n個相異的eigenvalue:

例如 $T:\mathbb{R}^3\longrightarrow\mathbb{R}^3$,

$$T\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)=\begin{bmatrix} 2&0&0\\ 0&2&0\\ 0&0&3\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix}$$

$n=3$, T只有兩個相異的eigenvalue, 但仍可對角化.

線性獨立的eigenvector不足n個才是不可對角化. (綜線CH12定理16)

(2) True.

$\because$ λ_1, λ_2 為相異的eigenvalue

$\therefore$ E_{λ_1} 與 E_{λ_2} 為獨立子空間. (綜線CH12定理20)

$\therefore$ $x_1\in E_{\lambda_1}, x_2\in E_{\lambda_2}, x_1+x_2=o \Longrightarrow x_1=x_2=o$. (綜線CH11定理3)

(3) True. 此為定理. (綜線CH12定理29)

(4) False.

例如 $T:\mathbb{R}^3\longrightarrow\mathbb{R}^3$,

$$T\left(\begin{bmatrix} x\\ y\\ z\end{bmatrix}\right)=\begin{bmatrix} 2&0&0\\ 0&2&0\\ 0&0&3\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix}$$

令 $g(t)=(t-2)(t-3)$, 則 $g(T)=O$.

１２B**１５** 【 元智84工工Y[2] 】

Let $A=\{a_{ij}\}$ be a $n\times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

(a) what is det$A$? the determinant of A.

(b) trace(A) = $\sum_{i=1}^{n} a_{ii} = f(\lambda_1, \lambda_2, \ldots, \lambda_n)$, what is f ?

(c) Assume that $A + \alpha I$ is invertible, what are the eigenvalues of $(A + \alpha I)^{-1}$?

(d) If $B = M^{-1}AM$, show that A and B have the same eigenvalues, what are the relationships between the eigenvectors of A and the eigenvectors of B ?

(e) Suppose A has $n-p+1$ distinct eigenvalues with eigenvalue λ_1 repeated p times. How do you check if A is diagonalizable ?

【解】(a) $\det A = \lambda_1\lambda_2 \ldots \lambda_n$ (綜線CH12定理13要訣)

(b) $f(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$ (綜線CH12定理13要訣)

(c) (請參閱題型16A)

$(\lambda_1 + \alpha)^{-1}, (\lambda_2 + \alpha)^{-1}, \ldots, (\lambda_n + \alpha)^{-1}$. (綜線CH16定理1a)

(d) (請參閱題型16A)

設v爲A對於λ的特徵向量： $Av = \lambda v$

$\therefore\ MBM^{-1}v = v$, $\therefore\ BM^{-1}v = M^{-1}v$

$\therefore\ M^{-1}v$爲B對於λ的特徵向量.

結論: A的特徵向量左乘M^{-1}就是B的特徵向量. (綜線CH16定理1a)

(e) 特徵多項式爲n次多項式, 已有一個根是p重根,

$\therefore$另外$n-p$個特徵值 $\lambda_2, \ldots, \lambda_{n-p+1}$ 都是特徵多項式的一重根.

$\therefore\ \dim(\mathrm{Ker}(A-\lambda_i I)) = 1$ (綜線CH12定理19)

A可對角化 $\iff \dim(\ker(A-\lambda_1 I)) = p$ (綜線CH12定理21)

$\iff \mathrm{rank}(A-\lambda_1 I) = n-p$ (綜線CH8定理8)

１２B１６ 【 元智83工工[14] 】

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of matrix A. Suppose $PA = LDU$ and $d_1, d_2, \ldots, d_n$ are the pivots. Answer the following questions.

(1) How is the determinant of A related to A's eigenvalues ? A's pivots ?

(2) If A has inverse, what are the eigenvalues of A^{-1} ?

(3) Suppose A has $n-p+1$ distinct eigenvalues with eigenvalues λ_1 repeated p times. How do you check if A is diagonalizable ?

【分析】$PA=LDU$分解可參閱綜線CH3定理27，定理29．本題是假設A為$n\times n$矩陣．L為單位上三角矩陣，U為單位下三角矩陣，D為對角矩陣．

【解】(1) (本題(1)屬於題型03E)

矩陣A必可三角化為$S\Gamma S^{-1}$ ，

$\therefore\ \det A=\det S\det\Gamma\det S^{-1}=\det\Gamma=\lambda_1\lambda_2\ldots\lambda_n$

另一方面，由$PA=LDU$，兩邊取行列式得:

$\det P\det A=\det L\det D\det U=1\cdot\det D\cdot\det U$

而$\det P=\pm1$， $\therefore\ \det A=\pm\det D=\pm d_1d_2\ldots d_n$．

(2) (本題(2)屬於題型16A)

$\because A$可逆，

$\therefore\ \det A\neq0$． (綜線CH4定理17)

設λ為特徵值，則$\lambda\neq0$． (綜線CH14定理17)

$\because\ \det(A-\lambda I)=0$

$\therefore\ \det(\lambda(\lambda^{-1}I-A^{-1})A)=0$

$\therefore\ \lambda^n\det(\lambda^{-1}I-A^{-1})\det A=0$

$\therefore\ \det(\lambda^{-1}I-A^{-1})=0$

$\therefore\ \lambda^{-1}$ 為A^{-1} 的特徵值．

$\therefore\ A^{-1}$ 的特徵值為$\lambda_1^{-1},\lambda_2^{-1},\ldots,\lambda_n^{-1}$ ．

(3) A的特徵值共有n個(可重複)值．扣除λ_1的p個值，剩下$n-p$個．而題目說共有$n-p+1$個相異特徵值．所以其它的特徵值都是一重值．所以:

A可對角化 $\Longleftrightarrow$ $\dim\ker(A-\lambda_1I)=p$． (綜線CH12定理19,定理21)

12B17 【中正84資工[4](a)】

(a) Let A be an $n\times n$ matrix with real entries such that $A^{1995}-I=O$． Show that A is diagonalizable in $\mathbb{C}$．

【解】(a) 令$f(x)=x^{1995}-1$ ， 並將A的minimal polynomial記為$m(x)$．

$f(x)$的根為 $\exp(2k\pi i/1995)$， $k=0,1,\ldots,1994$． (高中數學)

$\because\ f(A)=O$， $\therefore\ m(x)$整除$f(x)$． (綜線CH16定理22)

$\therefore\ m(x)$無重根．

$\therefore\ A$在$\mathbb{C}$可對角化． (綜線CH6定理26)

１２B**１８**【 交大85資料[6] 】

[複選題]

In the following statements, which ones are correct ?

(a) If 0 is an eigenvalue of matrix A, then A is singular.

(b) The matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ can be decomposed as $A = P\Lambda P^{-1}$ with diagonal Λ.

(c) $A = \begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$ have the same eigenvalues.

(d) A and B are similar if $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

【解】選(a)(b)(c).

【說明】本題(a)(c)屬於題型12A. 本題(b)屬於題型12B. 本題(d)屬於題型07D.

(a) 此爲定理. (綜線CH16定理12)

(b) $\det(A - xI) = \begin{vmatrix} 1-x & 1 \\ 1 & 2-x \end{vmatrix} = (1-x)(2-x)$

A有完全相異的eigenvalue 1 , 2

$\therefore$ A 可對角化. (綜線CH12定理23)

(c) $\det(A - xI) = \begin{vmatrix} 2-x & -3 \\ 1 & -1-x \end{vmatrix} = x^2 - x + 1$

$\det(B - xI) = \begin{vmatrix} -x & -1 \\ 1 & 1-x \end{vmatrix} = x^2 - x + 1$

$\therefore$ A, B的eigenvalue相同.

(d) $\text{tr}A = 5$, $\text{tr}B = 4$

$\because$ $\text{tr}A \neq \text{tr}B$ $\therefore$ A, B不相似. (綜線CH7定理22)

１２B**１９** 【 清大81工工[7.5] 】

An $n\times n$ matrix is diagonalizable if and only if it is similar to a diagonal matrix.

Which of the following matrices are diagonalizable ?

(a) $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 9 & -12 \\ 7 & -11 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(e) none of the above.

【解】選(a)(b)(c)(d). 解說如下:

分別計算特徵多項式:

$$\begin{vmatrix} -x & 0 \\ 1 & -1-x \end{vmatrix} = x^2+x = x(x+1)$$

$$\begin{vmatrix} 3-x & -2 \\ 1 & -x \end{vmatrix} = x^2-3x+2 = (x-1)(x-2)$$

$$\begin{vmatrix} 9-x & -12 \\ 7 & -11-x \end{vmatrix} = x^2+2x-15 = (x+5)(x-3)$$

前三個矩陣各別都具有完全相異的eigenvalues, 所以都可以對角化。

第四個矩陣本來就已是對角矩陣. (綜線CH12定理23,定理13)

１２B**２０** 【 大同80資工[3](k) 】

True or False:

(k) If A and B are diagonalizable, so is AB.

【解】(k) False, 反例如下:

令 $A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

A的特徵多項式爲 $x^2-1=(x-1)(x+1)$

∵ 特徵值完全相異 ∴ A可對角化. (綜線CH12定理23)

B本身已是對角矩陣, ∴ B可對角化

$$令C = AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

∵ C已是Jordan form. ∴ C不可對角化.

題型12C: 對角化計算題

１２C**01** 【 雲技84工工[10] 】

計算下列正方矩陣(square matrix)A的eigenvalues.

$$A=\begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$$

【解】

$$\det(A-xI)=\begin{vmatrix} -2-x & 0 & 1 \\ -6 & -2-x & 0 \\ 19 & 5 & -4-x \end{vmatrix}$$

$=-x^3-8x^2-x-8$ (綜線CH12定理13)

$=(x+8)(-x^2-1)=-(x+8)(x-i)(x+i)$

$\therefore$ eigenvalue爲 $-8, i, -i$.

１２C**02** 【 中原86工工[1] 】

Find:

(a) the eigenvalues;

(b) a basis for eigenspaces of the matrix

$$\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

【解】(a) $\det(A-tI)=t^2-4t-5=(t-5)(t+1)$,

$\therefore$ eigenvalue爲 $5, -1$.

(b)

$$A-5I=\begin{bmatrix}-4 & 4\\ 2 & -2\end{bmatrix}\sim\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

可取 $\begin{bmatrix}1\\ 1\end{bmatrix}$ 爲5的eigenspace的基底.

$$A+I=\begin{bmatrix}2 & 4\\ 2 & 4\end{bmatrix}\sim\begin{bmatrix}1 & 2\\ 0 & 0\end{bmatrix}$$

可取 $\begin{bmatrix}-2\\ 1\end{bmatrix}$ 爲–1的eigenspace的基底.

１２C**03** 【 交大85工工[3] 】

Find the eigenvalues and the corresponding eigenspaces for the following matrix A.

$$A=\begin{bmatrix}6 & -4\\ 3 & -1\end{bmatrix}$$

【解】

$$\det(A-xI)=\begin{vmatrix}6-x & -4\\ 3 & -1-x\end{vmatrix}=x^2-5x+6=(x-2)(x-3)$$

∴ eigenspace 爲 2, 3 .

$$A-2I=\begin{bmatrix}4 & -4\\ 3 & -3\end{bmatrix}\sim\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

∴ 2 對應的eigenspace爲 $\left\{ t\begin{bmatrix}1\\ 1\end{bmatrix} \;\middle|\; t\in\mathbb{R} \right\}$.

$$A-3I=\begin{bmatrix}3 & -4\\3 & -4\end{bmatrix}\sim\begin{bmatrix}3 & -4\\0 & 0\end{bmatrix}$$

$\therefore$ 3 對應的eigenspace為 $\left\{ t\begin{bmatrix}4\\3\end{bmatrix} \,\middle|\, t\in\mathbb{R} \right\}$.

12C04 【朝陽85工工[8]】

Let $A=\begin{bmatrix}1 & 2 & -1\\1 & 0 & 1\\4 & 4 & 5\end{bmatrix}$. Find an eigenvalue and its associated eigenvector of A.

【解】

$$\det(A-xI)=\begin{vmatrix}1-x & 2 & -1\\1 & -x & 1\\4 & 4 & 5-x\end{vmatrix}=\begin{vmatrix}2-x & 2-x & 0\\1 & -x & 1\\4 & 4 & 5-x\end{vmatrix}$$

$$=(2-x)\begin{vmatrix}1 & 1 & 0\\1 & -x & 1\\4 & 4 & 5-x\end{vmatrix}=(2-x)\begin{vmatrix}1 & 0 & 0\\1 & -1-x & 1\\4 & 0 & 5-x\end{vmatrix}=(2-x)(-1-x)(5-x)$$

A的特徵值為2, –1, 5 .

$$A-2I\sim\ \ldots\ \sim\begin{bmatrix}1 & 0 & 5/6\\0 & 1 & -1/12\\0 & 0 & 0\end{bmatrix}$$，可取得2的特徵向量 $\begin{bmatrix}-10\\1\\12\end{bmatrix}$

$$A+I\sim\ \ldots\ \sim\begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$，可取得–1的特徵向量 $\begin{bmatrix}-1\\1\\0\end{bmatrix}$

$$A-5I \sim \ \dots \ \sim \begin{bmatrix} 1 & 0 & 1/6 \\ 0 & 1 & 1/6 \\ 0 & 0 & 0 \end{bmatrix}$$，可取得5的特徵向量 $\begin{bmatrix} -1 \\ -1 \\ 6 \end{bmatrix}$

１２Ｃ**05** 【 雲技84電資Ｘ[10] 】

Find the characteristic polynomial, eigenvalues, and eigenvectors of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

【解】設此矩陣為A.

特徵多項式 $\det(A-xI)=(x-1)(x+1)(x-3)(x-2)$

特徵值為 $1, -1, 3, 2$.

$$A-I \sim \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$，1的特徵向量為$t\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, t \neq 0$

$$A+I \sim \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$，-1的特徵向量為$t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, t \neq 0$

$$A-2I\sim\begin{bmatrix}1&0&0&29/3\\0&1&0&7/3\\0&0&1&3\\0&0&0&0\end{bmatrix},\quad 2\text{的特徵向量爲}t\begin{bmatrix}-29\\-7\\-9\\3\end{bmatrix},\ t\neq 0$$

$$A-3I\sim\begin{bmatrix}1&0&-9/4&0\\0&1&-3/4&0\\0&0&0&1\\0&0&0&0\end{bmatrix},\quad 3\text{的特徵向量爲}t\begin{bmatrix}9\\3\\4\\0\end{bmatrix},\ t\neq 0$$

12C06【交大83資科[2]】

Let $A=\begin{bmatrix}3&0&0&0&0\\0&2&1&0&0\\0&0&2&0&0\\0&0&0&3&0\\0&0&0&0&2\end{bmatrix}$. Answer the following questions.

(a) Find all eigenvalues of A and their algebraic multiplicities. (3%)

(b) For each eigenvalue of A, find its associated linearly independent eigenvectors. (3%)

【解】(a)

$$\det(A-xI)=\begin{vmatrix}3-x&0&0&0&0\\0&2-x&1&0&0\\0&0&2-x&0&0\\0&0&0&3-x&0\\0&0&0&0&2-x\end{vmatrix}=(3-x)^2(2-x)^3$$

∴ eigenvalue 爲 3，具algebraic multiplicity 2，

及 2，具algebraic multiplicity 3.

(b) 對eigenvalue 3 ,

$$A-3I=\begin{bmatrix}0&0&0&0&0\\0&-1&1&0&0\\0&0&-1&0&0\\0&0&0&0&0\\0&0&0&0&-1\end{bmatrix}\sim\ldots\sim\begin{bmatrix}0&1&0&0&0\\0&0&1&0&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

$\therefore$ 可取得線性獨立的eigenvectors: $\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\0\\1\\0\end{bmatrix}$

對eigenvalue 2 ,

$$A-3I=\begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{bmatrix}\sim\ldots\sim\begin{bmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

$\therefore$ 可取得線性獨立的eigenvectors: $\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$

１２C**07**【 成大85資工[3] 】

Let A be the real matrix $A=\begin{bmatrix} 6 & -2 & -3 \\ -2 & 3 & -6 \\ -3 & -6 & -2 \end{bmatrix}$

Find $P\in \mathrm{Mat}_3(\mathbb{R})$ such that $P^{-1}AP$ is a diagonal matrix.

【解】$\det(A-xI)=\ldots=-(x-7)^2(x+7)$

$$A-7I=\begin{bmatrix} -1 & -2 & -3 \\ -2 & -4 & -6 \\ -3 & -6 & -9 \end{bmatrix}\sim\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$,解得特徵向量 $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$

$$A+7I=\begin{bmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{bmatrix}\sim\begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{bmatrix}$$, 解得特徵向量 $\begin{bmatrix} 1/3 \\ 2/3 \\ 1 \end{bmatrix}$.

令 $P=\begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$, 則 $P^{-1}AP=\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -7 \end{bmatrix}$

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１２C**08**【 中央85資工[6] 】

Diagonalize the matrix A to PDP^{-1} and find P and D, where

$$A=\begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}.$$

【解】

$$\det(A-xI)=\begin{vmatrix}-1-x & 4 & -2\\ -3 & 4-x & 0\\ -3 & 1 & 3-x\end{vmatrix}(-1)=\begin{vmatrix}-1-x & 4 & -2\\ -3 & 4-x & 0\\ 0 & -3+x & 3-x\end{vmatrix}$$

$$=(3-x)\begin{vmatrix}-1-x & 4 & -2\\ -3 & 4-x & 0\\ 0 & -1 & 1\end{vmatrix}=(3-x)\begin{vmatrix}-1-x & 2 & -2\\ -3 & 4-x & 0\\ 0 & 0 & 1\end{vmatrix}=(3-x)\begin{vmatrix}-1-x & 2\\ -3 & 4-x\end{vmatrix}$$

$$=-(x-3)(x-2)(x-1)$$

$$A-I=\begin{bmatrix}-2 & 4 & -2\\ -3 & 3 & 0\\ -3 & 1 & 2\end{bmatrix}\sim\begin{bmatrix}-1 & 0 & 1\\ -1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$，解得1的 eigenvector $\begin{bmatrix}t\\ t\\ t\end{bmatrix}$

$$A-2I=\begin{bmatrix}-3 & 4 & -2\\ -3 & 2 & 0\\ -3 & 1 & 1\end{bmatrix}\sim\begin{bmatrix}-3/2 & 0 & 1\\ -3/2 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$，解得2的 eigenvector $\begin{bmatrix}t\\ 3t/2\\ 3t/2\end{bmatrix}$

$$A-3I=\begin{bmatrix}-4 & 4 & -2\\ -3 & 1 & 0\\ -3 & 1 & 0\end{bmatrix}\sim\begin{bmatrix}-4 & 0 & 1\\ -3 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$，解得3的 eigenvector $\begin{bmatrix}t\\ 3t\\ 4t\end{bmatrix}$

令 $P=\begin{bmatrix}1 & 2 & 1\\ 1 & 3 & 3\\ 1 & 3 & 4\end{bmatrix}$，$D=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$， 則 $A=PDP^{-1}$

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１２C09 【 元智85電資[1] 】

Consider the symmetric 4×4 matrix

$$A=\begin{bmatrix}4&0&0&0\\0&3&0&1\\0&0&4&0\\0&1&0&3\end{bmatrix}$$

(1) find its characteristic polynomial

(2) find its eigenvalues

(3) find its associated eigenvectors

(4) find P such that $P^{-1}AP=D$, where D is its diagonal matrices.

【解】(1) $\det(A-xI)$

$$=\begin{vmatrix}4-x&0&0&0\\0&3-x&0&1\\0&0&4-x&0\\0&1&0&3-x\end{vmatrix}=(4-x)\begin{vmatrix}3-x&0&1\\0&4-x&0\\1&0&3-x\end{vmatrix}$$

$$=(4-x)^2\begin{vmatrix}3-x&1\\1&3-x\end{vmatrix}=\ldots=(x-4)\,(x-2)^3$$

(2) eigenvalue爲4,4,4,2

(3)

$$A-4I=\begin{bmatrix}0&0&0&0\\0&-1&0&1\\0&0&0&0\\0&1&0&-1\end{bmatrix}\sim\begin{bmatrix}0&1&0&-1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

4的eigenvector爲 $\begin{bmatrix} r \\ t \\ s \\ t \end{bmatrix}$, r, s, t是不全爲零的純量.

$$A-2I=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2的eigenvector爲 $\begin{bmatrix} 0 \\ -t \\ 0 \\ t \end{bmatrix}$, t是不爲零的純量.

(4)

令$P=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$, 則 $P^{-1}AP=\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

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１２Ｃ**１０** 【 雲技85工工[7] 】

試求一非奇異(non–singular)矩陣P和一對角線(diagonal)矩陣D, 使A與D相似(similar),

其中 $A=\begin{bmatrix} -1 & -4 & -8 \\ -4 & -7 & 4 \\ -8 & 4 & -1 \end{bmatrix}$

【解】

$$\det(A-xI)=\begin{vmatrix} -1-x & -4 & -8 \\ -4 & -7-x & 4 \\ -8 & 4 & -1-x \end{vmatrix}=\begin{vmatrix} -9-x & 0 & -9-x \\ -4 & -7-x & 4 \\ -8 & 4 & -1-x \end{vmatrix}$$

$$=(-9-x)\begin{vmatrix} 1 & 0 & 1 \\ -4 & -7-x & 4 \\ -8 & 4 & -1-x \end{vmatrix}=(-9-x)\begin{vmatrix} 1 & 0 & 0 \\ -4 & -7-x & 8 \\ -8 & 4 & 7-x \end{vmatrix}$$

$$=(-9-x)\begin{vmatrix} -7-x & 8 \\ 4 & 7-x \end{vmatrix}=\ldots=-(x-9)(x+9)^2$$

$A-9I\sim\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}$，解得特徵向量 $\begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix}$

$A+9I\sim\begin{bmatrix} 1 & -1/2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$，解得特徵向量 $\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$，$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

令 $P=\begin{bmatrix} -1 & 1/2 & 1 \\ 1/2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, $D=\begin{bmatrix} 9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$，則 $P^{-1}AP=D$.

１２Ｃ**１１**【 中央82資電[7] 】

Factor the matrix

$$A=\begin{bmatrix}1.5 & -1.0 & -2.0\\ -0.4 & 1.2 & 0.4\\ -0.3 & -0.6 & -0.2\end{bmatrix}$$

into $S\Lambda S^{-1}$ to find S and Λ, where S is the eigenvector matrix of A and Λ is the eigenvalue matrix of A.

【分析】一大堆小數點算起來不方便又容易弄錯，所以先調成整數矩陣再算.

【解】

令 $B=\begin{bmatrix}15 & -10 & -20\\ -4 & 12 & 4\\ -3 & -6 & -2\end{bmatrix}$，則 $A=\dfrac{1}{10}B$.　先對B作對角化:

$\det(B-\lambda I)$

$$=\begin{vmatrix}15-\lambda & -10 & -20\\ -4 & 12-\lambda & 4\\ -3 & -6 & -2-\lambda\end{vmatrix}=\begin{vmatrix}15-\lambda & -10 & -5-\lambda\\ -4 & 12-\lambda & 0\\ -3 & -6 & -5-\lambda\end{vmatrix}$$

$$=(-5-\lambda)\begin{vmatrix}15-\lambda & -10 & 1\\ -4 & 12-\lambda & 0\\ -3 & -6 & 1\end{vmatrix}=(-5-\lambda)\begin{vmatrix}18-\lambda & -4 & 0\\ -4 & 12-\lambda & 0\\ -3 & -6 & 1\end{vmatrix}$$

$$=\ \dots\ =-(x-10)(x-20)(x+5)$$

$$B-10I\sim\ \dots\ \sim\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}$$，解得特徵向量 $\begin{bmatrix}0\\ -2\\ 1\end{bmatrix}$

$$B-20I\sim\ldots\sim\begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}$$，解得特徵向量 $\begin{bmatrix}-2\\1\\0\end{bmatrix}$

$$B+5I\sim\ldots\sim\begin{bmatrix}1&0&-1\\0&1&0\\0&0&0\end{bmatrix}$$，解得特徵向量 $\begin{bmatrix}1\\0\\1\end{bmatrix}$

令 $S=\begin{bmatrix}0&-2&1\\-2&1&0\\1&0&1\end{bmatrix}$， $\Lambda=\begin{bmatrix}1&0&0\\0&2&0\\0&0&-0.5\end{bmatrix}$

則 $S^{-1}BS=10\Lambda$ ， 移項得 $B=10S\Lambda S^{-1}$

∴ $A=S\Lambda S^{-1}$

１２C**12** 【 元智85工工甲[3] 】

Let $A=\begin{bmatrix}0.8&0.2&0.1\\0.1&0.7&0.3\\0.1&0.1&0.6\end{bmatrix}$. Suppose that $\lambda_1=0.5$ and $\lambda_2=0.6$ are two eigenvalues of A.

Show that A can be diagonalized. That is, find M and D so that $A=MDM^{-1}$, where D is a diagonal matrix.

【分析】本題已給了兩個特徵值，另一個可從trace推得，所以不必求算特徵多項式.

【解】設第三個特徵值爲λ_3，

$$\lambda_1+\lambda_2+\lambda_3=0.5+0.6+\lambda_3=\mathrm{tr}A=0.8+0.7+0.6 .$$

∴ $\lambda_3=1.0$.

$$A-0.5I=\begin{bmatrix}0.3&0.2&0.1\\0.1&0.2&0.3\\0.1&0.1&0.1\end{bmatrix}\sim\ldots\sim\begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}$$，取特徵向量 $\begin{bmatrix}1\\-2\\1\end{bmatrix}$

$$A-0.6I=\begin{bmatrix}0.2&0.2&0.1\\0.1&0.1&0.3\\0.1&0.1&0.0\end{bmatrix}\sim\ldots\sim\begin{bmatrix}1&1&0\\0&0&1\\0&0&0\end{bmatrix}$$，取特徵向量 $\begin{bmatrix}-1\\1\\0\end{bmatrix}$

$$A-1.0I=\begin{bmatrix}-0.2&0.2&0.1\\0.1&-0.3&0.3\\0.1&0.1&-0.4\end{bmatrix}\sim\ldots\sim\begin{bmatrix}1&0&-9/4\\0&1&-7/4\\0&0&0\end{bmatrix}$$，取特徵向量 $\begin{bmatrix}9/4\\7/4\\1\end{bmatrix}$

令 $M=\begin{bmatrix}1&-1&9\\-2&1&7\\1&0&4\end{bmatrix}$，$D=\begin{bmatrix}0.5&0&0\\0&0.6&0\\0&0&1.0\end{bmatrix}$，則 $A=MDM^{-1}$.

１２C13 【清大84工工[6]】

$$A=\begin{bmatrix}.5&.2&.3\\.3&.8&.3\\.2&.0&.4\end{bmatrix},\ u_1=\begin{bmatrix}.3\\.6\\.1\end{bmatrix},\ u_2=\begin{bmatrix}1\\-3\\2\end{bmatrix},\ v_0=\begin{bmatrix}1/3\\1/3\\1/3\end{bmatrix}.$$

u_1 and u_2 are eigenvectors of A.

(a) Find another eigenvector of A: u_3, so that u_1, u_2 and u_3 form a basis for $\mathbb{R}^3$. (5%)

(b) Write v_0 as a linear combination of u_1, u_2 and u_3. (5%)

(c) For $k=1, 2, 3,\ldots$, define a dynamic system: $v_k=A^k v_0$. Find the (vector) value that v_k will converge to when k increase. (5%)

【分析】本題已給出兩個特徵向量，這使我們不必由求特徵多項式，也不必做分解因式.

【解】(a)

$$Au_1=\begin{bmatrix}.5 & .2 & .3\\ .3 & .8 & .3\\ .2 & .0 & .4\end{bmatrix}\begin{bmatrix}.3\\ .6\\ .1\end{bmatrix}=\begin{bmatrix}.3\\ .6\\ .1\end{bmatrix},$$ u_1的特徵值爲1.

$$Au_2=\begin{bmatrix}.5 & .2 & .3\\ .3 & .8 & .3\\ .2 & .0 & .4\end{bmatrix}\begin{bmatrix}1\\ -3\\ 2\end{bmatrix}=\begin{bmatrix}.5\\ -1.5\\ 1\end{bmatrix}=(.5)\begin{bmatrix}1\\ -3\\ 2\end{bmatrix},$$ u_2的特徵值爲0.5

$\because\ 1+0.5+\lambda_3=\mathrm{tr}A=0.5+0.8+0.4$ (綜線CH12定理13要訣1)

$\therefore\ \lambda_3=0.2$

$$A-0.2I=\begin{bmatrix}.3 & .2 & .3\\ .3 & .6 & .3\\ .2 & .0 & .2\end{bmatrix}\sim\ .....\ \sim\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

取得0.2的特徵向量 $u_3=\begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$

相異特徵值對應的特徵向量u_1 ,u_2 ,u_3 必定線性獨立. (綜線CH12定理20)

$\therefore$ 可成爲 $\mathbb{R}^3$ 的基底. (綜線CH6定理22)

(b) (請參閱題型03A)

令 $x(10u_1)+yu_2+zu_3=3v_0$, (調成整數較好算)

$$\text{即}\begin{bmatrix}3 & 1 & 1\\ 6 & -3 & 0\\ 1 & 2 & -1\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$$ (左直切:綜線CH2定理6)

$$\left[\begin{array}{ccc|c}3 & 1 & 1 & 1\\ 6 & -3 & 0 & 1\\ 1 & 2 & -1 & 1\end{array}\right]\sim\overset{\text{列運算}}{\cdots\cdots}\sim\left[\begin{array}{ccc|c}1 & 0 & 0 & 3/10\\ 0 & 1 & 0 & 4/15\\ 0 & 0 & 1 & -1/6\end{array}\right]$$

$\therefore$ $(3/10)(10u_1)+(4/15)u_2-(1/6)u_3=3v_0$,

即 $v_0=u_1+(4/45)u_2-(1/18)u_3$

(c) (請參閱題型16E)

接續(a)的結果,

$$\text{令 } P=\begin{bmatrix}3 & 1 & 1\\ 6 & -3 & 0\\ 1 & 2 & -1\end{bmatrix},\ D=\begin{bmatrix}1 & 0 & 0\\ 0 & 0.5 & 0\\ 0 & 0 & 0.2\end{bmatrix}\quad \text{則 } A=PDP^{-1}$$

#

$$\therefore\ A^k=(PDP^{-1})^k=PD^kP^{-1}=P\begin{bmatrix}1 & 0 & 0\\ 0 & 0.5^k & 0\\ 0 & 0 & 0.2^k\end{bmatrix}P^{-1}$$

$$\lim A^k=P\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}P^{-1}=\ ....=\begin{bmatrix}.3 & .3 & .3\\ .6 & .6 & .6\\ .1 & .1 & .1\end{bmatrix}$$

$$\lim v_k = \lim (A_k v_0) = \begin{bmatrix} .3 & .3 & .3 \\ .6 & .6 & .6 \\ .1 & .1 & .1 \end{bmatrix} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} .3 \\ .6 \\ .1 \end{bmatrix}$$

#

１２C**１４**【 中原86工工[4] 】

Suppose $B = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$. Find:

(a) the characteristic polynomial $\Delta(t)$ and eigenvalues of B;

(b) a maximal set S of linearly independent eigenvectors of B;

(c) Is B diagonalizable ? If yes, find P such that $P^{-1}BP$ is diagonal.

【解】(a)

$\Delta(t) = \det(B - tI) = \ldots = -t^3 + 12t + 16 = -(t+2)^2(t-4)$

∴ B的eigenvalues爲–2,–2,4.

(b)

$$B + 2I \sim \ldots \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ 解得特徵向量 } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$B - 4I \sim \ldots \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \text{ 解得特徵向量 } \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

∴ a maximal set S of linearly independent eigenvectors of B 爲

$$\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\1\end{bmatrix}\right\}$$

(c) B不可對角化，因eigenvector不足以形成basis.

１２C15【 台大84資工[2] 】

Let $P_2[x]=\{a_0+a_1x+a_2x^2 \mid a_i \text{ is real number}\}$, define a linear transformation $T: P_2[x] \longrightarrow P_2[x]$ as $T(a_0+a_1x+a_2x^2)=a_2+a_1x+a_0x^2$. Find a basis of $P_2[x]$, so that $[T]_\beta$ is a diagonal matrix. $[T]_\beta$ = the matrix of T using β as a basis to represent).

【分析】本題在幾何上是一個鏡射，幾何觀念清楚時可以不必求特徵多項式，直接得知所求的基底.

【解】先取標準基底$S=\{1, x, x^2\}$.

由T的定義得:

$$T(1)=0\cdot 1+0\cdot x+1\cdot x^2$$
$$T(x)=0\cdot 1+1\cdot x+0\cdot x^2,$$
$$T(x^2)=1\cdot 1+0\cdot x+0\cdot x^2.$$

$$\therefore \quad [T]_S=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$ (綜線CH7定義9)

對此矩陣求特徵多項式:

$$\begin{vmatrix}-x&0&1\\0&1-x&0\\1&0&-x\end{vmatrix}=-(x-1)^2(x+1)$$

$\therefore$ T的特徵值爲1, –1.

對特徵值1求特徵向量:

$$[T]_S - I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

得出齊次解 $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ 及 $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

∴ 得出兩個線性獨立特徵向量 $1+x^2, x$.

對特徵值–1求特徵向量:

$$[T]_S - (-1)I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

得出齊次解 $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

∴ 得出特徵向量 $1-x^2$.

取 $\beta = \{ 1+x^2 , x, 1-x^2 \}$, 則

$$[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ (綜線CH12定義15)

【另解】(適用於填充題, 簡答題或作答時間緊迫的場合.)

利用同構的觀念, 將$a+bx+cx^2 \in P_2[x]$ 視爲$(a,b,c) \in \mathbb{R}^3$. 本題相當於是將 (a,b,c) 映到 (c,b,a). 這是一個鏡射, 鏡射面是 $\{ (s, t, s) \mid s,t \in \mathbb{R} \}$, 它通過

y軸並平分xy面與yz面的夾角. 所以可取鏡射面上的(1,0,1), (0,1,0)及垂直於鏡射面的(1,0,–1)組成所需的基底. 前兩個向量經映射不變, 第三個向量經映射反轉. 所以矩陣表示爲 diag(1, 1, –1) .

１２C**１６** 【 元智84電資[2] 】

Let $P^2(\mathbb{R})=\{a+bx+cx^2 \mid a,b,c\in\mathbb{R}\}$ which is known to be an $\mathbb{R}$-space.
Let $T: P^2(\mathbb{R})\longrightarrow P^2(\mathbb{R})$ be the linear operator defined by :

$$T(a+bx+cx^2)=(a+b)+(a+b)x+2cx^2.$$

Find a basis B for $P^2(\mathbb{R})$ such that the matrix of T with respect to B, $[T]_B$, is a diagonal matrix.

【解】$T(1)=1+x$, $T(x)=1+x$, $T(x^2)=2x^2$

$$\therefore \text{ 對標準基底 } \{1, x, x^2\},\ T\text{的矩陣表示}A=\begin{bmatrix}1&1&0\\1&1&0\\0&0&2\end{bmatrix}\text{ .(綜線CH7定義9)}$$

$\det(A-xI)=-x(x-2)^2$.

$$A-2I=\begin{bmatrix}-1&1&0\\1&-1&0\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&-1&0\\0&0&0\\0&0&0\end{bmatrix}\text{, 解得2的特徵向量}\begin{bmatrix}1\\1\\0\end{bmatrix},\ \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

$$A-0I=\begin{bmatrix}1&1&0\\1&1&0\\0&0&2\end{bmatrix}\sim\begin{bmatrix}1&1&0\\0&0&1\\0&0&0\end{bmatrix}\text{, 解得0的特徵向量}\begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

$\therefore$ $T(1+x)=2(1+x)$, $T(x^2)=2x^2$, $T(1-x)=0$

令 $B=\{1+x, x^2, 1-x\}$, 則

T對B的矩陣表示為 $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

１２C**17** 【 交大86資工[7] 】

Consider the vector space P_3 of polynomials of degree less than 3, and the ordered basis $B=(x^2, x, 1)$ for P_3 . Let $T:P_3 \longrightarrow P_3$ be the linear transformation such that $T(ax^2+bx+c)=(a-b-c)x^2+(b-c-a)x+(c-a-b)$

(a) Find the matrix A representing T with respect to the ordered basis B. (2%)

(b) Find the eigenvalues and the eigenvectors for the matrix A. (7%)

(c) Find $T^4(2x)$. (4%)

【解】(a) (請參閱題型07B) 由定義,

$$\begin{cases} T(x^2)=x^2-x-1 \\ T(x)=-x^2+x-1 \\ T(1)=-x^2-x+1 \end{cases}$$

$$\therefore\ A=\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$ (綜線CH7定義9)

(b) $\det(A-xI)=-(x+1)(x-2)^2$

$\therefore$ A的特徵值為$-1,2$.

$$A+I=\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

-1的特徵向量爲 $\begin{bmatrix} t \\ t \\ t \end{bmatrix}$, t爲非零常數.

$$A-2I=\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2的特徵向量爲 $\begin{bmatrix} -s-t \\ s \\ t \end{bmatrix}$, s,t爲不全爲零的常數.

(c) (請參閱題型07B)

$$A^2=\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}=\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$A^4=\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}=\begin{bmatrix} 11 & -5 & -5 \\ -5 & 11 & -5 \\ -5 & -5 & 11 \end{bmatrix}$$

(求低次乘冪用對角化法或Cayley-Hamilton法未必較快)

$[\,T^4(2x)\,]=[\,T^4]\,[\,2x\,]$ (綜線CH7定理15)

$=[\,T]^4\,[\,2x\,]$

$$= \begin{bmatrix} 11 & -5 & -5 \\ -5 & 11 & -5 \\ -5 & -5 & 11 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -10 \\ 22 \\ -10 \end{bmatrix}$$

$\therefore\ T^4(2x) = -10x^2 + 22x - 10$ (綜線CH6定義28)

１２C**18** 【 元智82工工[9] 】

Let $\varphi : \mathcal{M}_2(\mathbb{R}) \longrightarrow \mathcal{M}_2(\mathbb{R})$ be defined by

$$\varphi \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ 2d & c \end{bmatrix} \quad \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_2(\mathbb{R})$$

Find all eigenvalues of φ.

【解】 考慮 $\mathcal{M}_2(\mathbb{R})$ 的標準基底 $S = \{ E_{11}, E_{12}, E_{21}, E_{22} \}$，

對S求φ的矩陣表示:

$$\varphi(E_{11}) = \varphi\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = E_{11}$$

$$\varphi(E_{12}) = \varphi\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = E_{11} + E_{12}$$

$$\varphi(E_{21}) = \varphi\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = E_{12} + E_{22}$$

$$\varphi(E_{22}) = \varphi\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = 2E_{21}$$

$$\therefore\ [\varphi]_S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ (綜線CH定義)

$$\det([\varphi]_S - xI) = \begin{vmatrix} 1-x & 1 & 0 & 0 \\ 0 & 1-x & 1 & 0 \\ 0 & 0 & -x & 2 \\ 0 & 0 & 1 & -x \end{vmatrix} = \dots.$$

$$= (1-x)^2(x^2-2) = (x-1)^2(x-\sqrt{2})(x+\sqrt{2})$$

$\therefore$ φ 的 eigenvalues 為 $1, 1, \sqrt{2}, -\sqrt{2}$

【加強演練】

接本題, 找一個基底B, 使 $[\varphi]_B$ 為Jordan form.

[解] 如前述, 已求得 eigenvalue 為 $1, 1, \sqrt{2}, -\sqrt{2}$

求解 $(M-1I)v=o$： 取得特解 $v_2=[1, 0, 0, 0]^T$

再求解 $(M-1I)v=v_2$： 取得特解 $v_1=[0, 1, 0, 0]^T$

對eigenvalue $\lambda=\pm\sqrt{2}$, 求解 $(M-\lambda I)v=o$ ：

$$M-\lambda I = \begin{bmatrix} 1\mp\sqrt{2} & 1 & 0 & 0 \\ 0 & 1\mp\sqrt{2} & 1 & 0 \\ 0 & 0 & \mp\sqrt{2} & 2 \\ 0 & 0 & 1 & \mp\sqrt{2} \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 & -4\mp\sqrt{2} \\ 0 & 1 & 0 & -2\mp\sqrt{2} \\ 0 & 0 & 1 & \mp\sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

可解得特徵向量 $v=\begin{bmatrix}(4\pm\sqrt{2})\\(2\pm\sqrt{2})\\\pm\sqrt{2}\\1\end{bmatrix}$，取正號記爲 v_3，取負號記爲 v_4.

令 $[A_i]_S=v_i,\ i=1,2,3,4$

亦即取 $A_1=\begin{bmatrix}0&1\\0&0\end{bmatrix}$，$A_2=\begin{bmatrix}1&0\\0&0\end{bmatrix}$，

$$A_3=\begin{bmatrix}4+3\sqrt{2}&2+\sqrt{2}\\\sqrt{2}&1\end{bmatrix},\quad A_4=\begin{bmatrix}4-3\sqrt{2}&2-\sqrt{2}\\-\sqrt{2}&1\end{bmatrix},$$

$$\because\begin{cases}Mv_1=v_1+v_2\\Mv_2=v_2\\Mv_3=\sqrt{2}\ v_3\\Mv_4=-2\sqrt{v_4}\end{cases}\qquad\therefore\begin{cases}\varphi(A_1)=A_1+A_2\\\varphi(A_2)=A_2\\\varphi(A_3)=\sqrt{2}\ A_3\\\varphi(A_4)=-\sqrt{2}\ A_4\end{cases}$$

令 $B=\{A_1,A_2,A_3,A_4\}$，則

$$[\varphi]_B=\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&0&\sqrt{2}&0\\0&0&0&-\sqrt{2}\end{bmatrix}$$

１２Ｃ**１９**【 清大81資料[29](b) 】

Let

$$B=\begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

(b) Give the spectrum decomposition of matrix B.

【解】(b) 1° 先對B作對角化:

$\det(B-xI)=-(x-5)(x+4)(x+2)$

由 $(B-5I)v=o$ 解得特徵向量 $[\,0, 0, 1\,]^T$

由 $(B+4I)v=o$ 解得特徵向量 $[\,1, -1, 1\,]^T$

由 $(B+2I)v=o$ 解得特徵向量 $[\,1, 1, 0\,]^T$

$$令\ S=\begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix},\quad 則\ S^{-1}BS=\begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

2° 計算分解式:

$$B=S\begin{bmatrix} 5 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix}S^{-1}$$

$$=S\left(5\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}-4\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}-2\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}\right)S^{-1}$$

$$=5S\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}S^{-1}-4S\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}S^{-1}-2S\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}S^{-1}$$

$$=\ ....$$

$$=5\begin{bmatrix}0&0&0\\0&0&0\\0&0&1\end{bmatrix}-4\begin{bmatrix}1/2&-/2&0\\-1/2&1/2&0\\0&0&0\end{bmatrix}-2\begin{bmatrix}1/2&1/2&0\\1/2&1/2&0\\0&0&0\end{bmatrix}$$

題型12D: 特種矩陣的特徵值

１２D01 【 交大80資工[5](a) 】

(a) (4%) Show that if $A^2 = A$ and λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = 1$.

【證】(a) 設 $Av = \lambda v$, $v \neq 0$,

則 $A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v$

$\therefore\ \lambda v = \lambda^2 v$ $\qquad \therefore\ (\lambda - \lambda^2)v = 0$

$\therefore\ \lambda - \lambda^2 = 0$ $\qquad \therefore\ \lambda = 0$ or 1

１２D02 【 交大82資科[3] 】

Suppose that $A^2 = -A$. Show that if λ is an eigenvalue for A, then λ is either 0 or -1.

【解】設λ為A的eigenvalue, 則存在向量$v \neq o$, 使得 $Av = \lambda v$

$A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda(\lambda v) = \lambda^2 v$

而 $A^2 v = -Av = -\lambda v$

$\therefore\ \lambda^2 v = -\lambda v$ $\qquad \therefore\ (\lambda^2 + \lambda)\, v = o$

$\therefore\ \lambda^2 + \lambda = 0$ $\qquad \therefore\ \lambda = 0$ 或 -1

１２D03 【 清大75資科[5](2) 】

Prove or disprove the following statements.

(2) If A is a real $n \times n$ matrix with $A^2 = -I$, then A has no real eigenvalues.

【解】(2) (prove)

Let λ is a eigenvalue of A with eigenvector v, then

$A^2 v = A(Av) = A(\lambda v) = \lambda A v = \lambda^2 v$,

but $A^2 v = -Iv = -v$

$\therefore\ \lambda^2 = -1$

$\therefore\ \lambda \notin \mathbb{R}$

１２D04 【 精編加強題 】

設 $A \in K^{n\times n}$, λ為A的特徵值. $f(x) \in K[x]$.

(1) 若 $f(A)=O$, 則 $f(\lambda)=0$.
(2) 若$A^2=I$, 則 $\lambda\in\{1,-1\}$.
(3) 若$A^k=O$, 則 $\lambda=0$.

【解】(1) 設λ為A的eigenvalue, $m(x)$為A的minimal polynomial,

則 $m(x)\mid f(x)$, 且 $x-\lambda\mid m(x)$ (綜線CH16定理22)

$\therefore\ x-\lambda\mid f(x)$

$\therefore\ f(\lambda)=0$ (因式定理)

(2) 由(1), 取$f(x)=x^2-1$ 即得.

(3) 由(1), 取$f(x)=x^k$ 即得.

１２D**05** 【 清大77資料[6](cd) 】

Prove or disprove the following statements.

(c) Let A be an nxn matrix over $\mathbb{R}$, if $A^t=-A$, then A has at least one negative eigenvalue. (A^t is the transpose of A)

(d) Let A be an $n\times n$ matrix over $\mathbb{R}$. Then A is invertible if 0 is not an eigenvalue of A.

【解】(c) [disprove]

[反例１] 取$A=O$ ($n\times n$零矩陣), 則 $A^t=-A$

但A的所有eigenvalue都是0, 沒有 negative eigenvalue

[反例２] 取$A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, 則 $A^t=-A$,

但 $\det(A-xI)=x^2+1$

可知A沒有negative eigenvalue (綜線CH12定理7)

(d) [prove] A is invertible

$\Longleftrightarrow$ $\det A\neq 0$ (綜線CH4定理17②)

$\Longleftrightarrow$ 0 不是特徵多項式 $\det(A-xI)$ 的根

$\Longleftrightarrow$ 0 不是A的eigenvalue (綜線CH12定理7)

１２D**06** 【 大同82資工[15] 】

Let A represent an $n\times n$ matrix. One of the following statements is not equivalent to the others.

Please identify it:

(a) A is invertible;

(b) A has distinct eigenvalues;

(c) $|A| \neq 0$;

(d) the system of equations $Ax=y$ has a solution x for each $n\times 1$ matrix y.

【解】 選 (b)

【說明】 可逆 $\Longleftrightarrow$ 0不是eigenvalue (綜線CH16定理12)

可逆 $\Longleftrightarrow$ 行列式不爲零. (綜線CH4定理17)

可逆 $\Longleftrightarrow$ $\forall y$, $Ax=y$有解. (綜線CH8定理17)

敘述(b)與可逆性並無關聯，例如 $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ 可逆, $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ 不可逆.

１２D０７ 【 交大81資料[3] 】

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$$

(a) Find all eigenvalues of A.

(b) Find an invariant subspace of dimension 3 of A.

【解】(a)

$$\det(A-\lambda I)=\begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 2 & 2-\lambda & 2 & 2 \\ 3 & 3 & 3-\lambda & 3 \\ 4 & 4 & 4 & 4-\lambda \end{vmatrix}$$

$$=\begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 2 & 2-\lambda & 2 & 2 \\ 3 & 3 & 3-\lambda & 3 \\ 10-\lambda & 10-\lambda & 10-\lambda & 10-\lambda \end{vmatrix}$$

$$=(10-\lambda)\begin{vmatrix} 1-\lambda & 1 & 1 & 1 \\ 2 & 2-\lambda & 2 & 2 \\ 3 & 3 & 3-\lambda & 3 \\ 1 & 1 & 1 & 1 \end{vmatrix} \quad (-3)\ (-2)\ (-1)$$

$$=(10-\lambda)\begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}=(10-\lambda)(-\lambda)^3$$

∴ A的eigenvalues爲0, 0, 0, 10

(b)

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix} \ (-2)\ (-3)\ (-4) \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\therefore\ \ker A=\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle|\ x_1+x_2+x_3+x_4=0 \right\}$$

$$=\left\{ t_2\begin{bmatrix}-1\\1\\0\\0\end{bmatrix}+t_3\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}+t_4\begin{bmatrix}-1\\0\\0\\1\end{bmatrix} \middle| \; t_2, t_3, t_4 \in \mathbb{R} \right\}$$

dim ker$A=3$, 且

ker$A$爲invariant subspace. (綜線CH11定理30)

【另解】(a)

$$A=\begin{bmatrix}1&1&1&1\\2&2&2&2\\3&3&3&3\\4&4&4&4\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\begin{bmatrix}1&1&1&1\end{bmatrix}$$

∴ rank$A=1$ (綜線CH8定理16a)

∴ dim Ker$A=4-$rank$A=4-1=3$ (綜線CH8定理8)

即 0爲eigenvalue，且geometric multiplicity是3 (綜線CH12定義18)

∴ eigenvalue 0 的 algebraic multiplicity 至少是3 (綜線CH12定理19)

$$A\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\begin{bmatrix}1&1&1&1\end{bmatrix}\begin{bmatrix}1\\2\\3\\4\end{bmatrix}=\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\begin{bmatrix}10\end{bmatrix}=10\begin{bmatrix}1\\2\\3\\4\end{bmatrix}$$

∴ 10 也是A的eigenvalue

∵ A頂多有4個eigenvalue, ∴ A的eigenvalues爲 0,0,0,10

(b) 由(a)中已知ker$A$爲3維子空間, 而它又是不變子空間,

∴ 取 ker$A$ 即可.

１２D**08** 【 清大80資料[5] 】

Find all eigenvalues of the matrix

$$A=\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 & -1 & 1 \end{bmatrix}$$

【解】

$$\begin{vmatrix} 1-\lambda & -1 & 1 & -1 & 1 & -1 \\ -1 & 1-\lambda & -1 & 1 & -1 & 1 \\ 1 & -1 & 1-\lambda & -1 & 1 & -1 \\ -1 & 1 & -1 & 1-\lambda & -1 & 1 \\ 1 & -1 & 1 & -1 & 1-\lambda & -1 \\ -1 & 1 & -1 & 1 & -1 & 1-\lambda \end{vmatrix} \begin{matrix} \leftarrow \\ (1) \\ (1) \\ (1) \\ (1) \\ (1) \end{matrix}$$

$$=\begin{vmatrix} -\lambda & -\lambda & -\lambda & -\lambda & -\lambda & -\lambda \\ -1 & 1-\lambda & -1 & 1 & -1 & 1 \\ 1 & -1 & 1-\lambda & -1 & 1 & -1 \\ -1 & 1 & -1 & 1-\lambda & -1 & 1 \\ 1 & -1 & 1 & -1 & 1-\lambda & -1 \\ -1 & 1 & -1 & 1 & -1 & 1-\lambda \end{vmatrix}$$

$$=(-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1-\lambda & -1 & 1 & -1 & 1 \\ 1 & -1 & 1-\lambda & -1 & 1 & -1 \\ -1 & 1 & -1 & 1-\lambda & -1 & 1 \\ 1 & -1 & 1 & -1 & 1-\lambda & -1 \\ -1 & 1 & -1 & 1 & -1 & 1-\lambda \end{vmatrix} \begin{matrix} (1)\ (-1) \\ \leftarrow (1) \\ \leftarrow (-1) \\ \leftarrow (1) \\ \leftarrow (-1) \\ \leftarrow (1) \end{matrix}$$

$$=(-\lambda)\begin{vmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 2-\lambda & 0 & 2 & 0 & 2 \\ 0 & -2 & -\lambda & -2 & 0 & -2 \\ 0 & 2 & 0 & 2-\lambda & 0 & 2 \\ 0 & -2 & 0 & -2 & -\lambda & -2 \\ 0 & 2 & 0 & 2 & 0 & 2-\lambda \end{vmatrix}$$

$$=(-\lambda)^3\begin{vmatrix} 2-\lambda & 2 & 2 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}=(-\lambda)^3\begin{vmatrix} 6-\lambda & 6-\lambda & 6-\lambda \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}$$

$$=(-\lambda)^3(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 2-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix}=(-\lambda)^3(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix}$$

$$=(-\lambda)^5(6-\lambda)$$

$\therefore\ \lambda=6, 0, 0, 0, 0, 0$ (本題化簡法很多, 請自行嘗試.)

【另解】令 $x=[\ 1, -1, 1, -1, 1, -1\ ]^T$,

則 $A=xx^T$, 而 $x^Tx=6$

$\therefore$ A的eigenvalue爲 0, 0, 0, 0, 0, 6 (綜線CH12定理26)

１２D09 【 元智83工工[9] 】

[是非題]

If every row of a square matrix adds to 1, then $A-I$ is non-singular, where Matrix I is an identity matrix.

【解】×, $A-I$必是singular. 解說如下:

令$u=[1\quad 1\quad \dots\quad 1]^T$, 則 $Au=u$ (乘乘看就會明白)

$\therefore\ (A-I)u=o$ $\therefore\ \ker(A-I)\neq\{o\}$.

$\therefore$ $A-I$爲singular. (綜線CH8定理17)