題型13A: 正交(單式)矩陣

１３A01 【 淡江84資工[2] 】

Show the following conditions are equivalent for an $n\times n$ matrix P.

(a) P is invertible and $P^{-1}=P^{T}$, where P^{T} is the transpose of P.

(b) The rows of P are orthonormal (with respect to the dot-product).

(c) The columns of P are orthonormal (with respect to the dot-product).

【分析】此題需在實數系討論，否則將不成立.

【解】此題考證明，請參閱綜線CH13定理3，此處不再重複.

１３A02 【 中央86資工[1](j) 】

[是非論證題]

(j) If matrix A has orthonormal columns, then $AA^{T}=I$.

【解】False.

例如 $A=\begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix}$ 具有orthonormal的columns，但 $AA^{T}\neq I$.

【討論】對實數矩陣A,

A has orthonormal columns $\Longleftrightarrow$ $A^{T}A=I$.

A has orthonormal rows $\Longleftrightarrow$ $AA^{T}=I$. (綜線CH13定理3)

１３A03 【 元智85電資[5] 】

True or false ? Explain your answers.

(1) $(A+B)^{T}=A^{T}+B^{T}$.

(2) $(AB)^{T}=A^{T}B^{T}$.

(3) $\det(A^{T})=\det(A)$.

(4) $A^{T}=A^{-1}$ if and only if A is orthogonally diagonalizable .

(5) $A^{\mathrm{T}}=A^{-1}$ if A is an orthogonal matrix .

【解】(1) True.

$\forall i, j$,

$(A+B)^{\mathrm{T}}$的(i, j)位置$=(A+B)$的(j, i)位置$=(A$的(j,i)位置$)+(B$的(j,i)位置$)$

$=(A^{\mathrm{T}}$的(i,j)位置$)+(B^{\mathrm{T}}$的(i,j)位置$)=(A^{\mathrm{T}}+B^{\mathrm{T}})$的$(i,j)$位置.

(2) False.

應是 $(AB)^{\mathrm{T}}=B^{\mathrm{T}}A^{\mathrm{T}}$ (綜線CH2定理23)

反例只須取不可交換的A,B即可. 例如 $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

(3) True. 此爲定理. (綜線CH4定理5)

(4) False.

例如$A=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $A^{-1}=A^{\mathrm{T}}$, 但A(在實數系)還不能對角化.

在複數系時(可單式對角化, 但)也不能正交對角化. (綜線CH13定理15)

另外, 像$A=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$可正交對角化, 但$A^{-1}\neq A^{\mathrm{T}}$.

(5) True.

$A^{\mathrm{T}}=A^{-1} \Longleftrightarrow AA^{\mathrm{T}}=I=A^{\mathrm{T}}A$ (綜線CH2定義10)

$\Longleftrightarrow$ A爲orthogonal (綜線CH13定義1)

１３A**04** 【 中正82資工[3] 】

If P (permutation matrix) is a $p\times p$ matrix with each row of P contains exactly one nonzero entry, namely 1; and each column of P contains exactly one nonzero entry, namely 1, then

(a) P is nonsingular,

(b) $P^{\mathrm{T}}P=PP^{\mathrm{T}}=I$. where P^{T} is the transpose of P and I is the identity matrix. Prove it! (10%)

【分析】 此題屬於題型13A. 請參閱綜線CH3定義14要訣2, 及CH2定義25要訣1.

【解】 ((a)小題可由(b)小題得出, 所以先做(b))

(b) 令P的行向量依序爲$c_1, c_2, \ldots, c_p$

對$i \neq j$, c_i 與c_j 的零位於不同的列, 所以 $c_i^T c_j = 0$

對 $i = j$, $c_i^T c_j = 1$

而 $P^T P$的第i,j位置爲 $c_i^T c_j$,

$\therefore\ P^T P = I$

令P的列向量依序爲$r_1, r_2, \ldots, r_p$

對$i \neq j$, r_i 與r_j 的零位於不同的行, 所以 $r_i r_j^T = 0$

對 $i = j$, $r_i r_j^T = 1$

而 PP^T 的第i,j位置爲 $r_i r_j^T$,

$\therefore\ PP^T = I$

(a) 由以上即知P爲non-singular. (且$P^{-1} = P^T$) (綜線CH2定義10)

１３A**05** 【 元智83工工[2] 】

[是非題]

If both Q_1 and Q_2 are orthogonal, then $Q_1^T Q_2^T$ may not be orthogonal.

【分析】本題雖沒說Q_1, Q_2 是什麼, 但顯然是矩陣.

【解】×, 解說如下:

Q_1, Q_2 都是orthogonal matrix時,

$(Q_1^T Q_2^T)(Q_1^T Q_2^T) = Q_2 Q_1 Q_1^T Q_2^T = Q_2 Q_2^T = I$

$\therefore\ Q_1^T Q_2^T$ 必是orthogonal matrix.

１３A**06** 【 大同84資工[5] 】

Verify that $P = \begin{bmatrix} 2/3 & -2/3 & 1/3 \\ 2/3 & 1/3 & -2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$ is an orthogonal matrix.

【解】

$$P^TP=\begin{bmatrix}2/3 & 2/3 & 1/3\\ -2/3 & 1/3 & 2/3\\ 1/3 & -2/3 & 2/3\end{bmatrix}\begin{bmatrix}2/3 & -2/3 & 1/3\\ 2/3 & 1/3 & -2/3\\ 1/3 & 2/3 & 2/3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

同樣可算得 $PP^T=I$

∴ P 爲 orthogonal matrix.

１３A**０７** 【 清大82工工[2] 】

Identify Hermitian, Normal, Orthogonal, Singular, Unitary, similarity matrices from the following matrices.

$$A=\begin{bmatrix}1+i & -1+i\\ 1+i & 1-i\end{bmatrix},\qquad B=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$C=(1/3)\begin{bmatrix}1 & 2 & -2\\ 2 & -2 & -1\\ 2 & 2 & 2\end{bmatrix},\quad D=\begin{bmatrix}4 & 5 & 6\\ 1 & 2 & 3\\ 7 & 8 & 9\end{bmatrix},\quad E=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}.$$

【解】1°

$$A^H=\begin{bmatrix}1+i & -1+i\\ 1+i & 1-i\end{bmatrix}^H=\begin{bmatrix}1-i & 1-i\\ -1-i & 1+i\end{bmatrix}\neq A$$

∴ A不是Hermitain matrix

$$A^HA=\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix},\quad AA^H=\begin{bmatrix}4 & 0\\ 0 & 4\end{bmatrix}$$

∴ A是Normal matrix($A^HA=AA^H$), A不是Unitary matrix($A^HA\neq I$)

$$A^TA=\begin{bmatrix}4i & 0\\ 0 & -4i\end{bmatrix}$$

∴ A不是Orthogonal matrix　($A^TA \neq I$)

$\det A = (1+i)(1-i) - (1+i)(-1+i) = 4 \neq 0$

∴ A不是Singular matrix

2°

$$B^H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = B$$

∴ B是Hermitain matrix

$B^HB = B^2 = BB^H$　,　∴ B是Normal matrix

$$B^HB = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \neq I$$

∴ B不是Unitary matrix

$B^TB = B^2 \neq I$,　∴ B不是Orthogonal matrix

$\det B = (-1) - 1 = -2 \neq 0$　,　∴ B不是Singular matrix

3° $C^H \neq C$　,　∴ C不是Hermitain matrix

$$C^HC = (1/9)\begin{bmatrix} 9 & 2 & 0 \\ 2 & 12 & 2 \\ 0 & 2 & 9 \end{bmatrix},\quad CC^H = (1/9)\begin{bmatrix} 9 & 0 & * \\ * & * & * \\ * & * & * \end{bmatrix}.$$

∴ C不是Normal matrix，不是Unitary matrix.

$C^TC = C^HC \neq I$　∴ C不是Orthogonal matrix

$\det C \neq 0$　∴ C不是Singular matrix

4° $D^H \neq D$　,　∴ D不是Hermitain matrix

$$D^HD = \begin{bmatrix} 66 & * & * \\ * & * & * \\ * & * & * \end{bmatrix},\quad DD^H = \begin{bmatrix} 77 & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

∴ D不是Normal matrix($D^HD \neq DD^H$), 不是Unitary matrix($D^HD \neq I$)

$D^T D = D^H D \neq I$ ∴ D不是Orthogonal matrix

$\det D = 0$ ∴ D是Singular matrix

5° $E^H = E$, ∴ E是Hermitain matrix

$E^H E = E^2 = E E^H$, ∴ E是Normal matrix

$$E^H E = \begin{bmatrix} 5 & * \\ * & * \end{bmatrix} \neq I$$ ∴ E不是Unitary matrix

$E^T E = E^2 \neq I$, ∴ E不是Orthogonal matrix

$\det E \neq 0$, ∴ E不是Singular matrix

6° 不同size的矩陣不可能similar.

$\text{tr}A = 2$, $\text{tr}B = 0$, $\text{tr}E = 4$. $\text{tr}C = 1$, $\text{tr}D = 15$.

∴ A, B, C, D, E彼此都不similar.

１３A**08** 【 師大83資教[1] 】

If A is an n by n orthogonal matrix, prove $\det(A) = \pm 1$. (8%)

【解】∵ A爲orthogonal matrix , ∴ $A^T A = I$ (CH2定義25)

∴ $\det(A^T A) = \det I$

∴ $\det(A^T)\det A = 1$ (CH4定理6)

∴ $\det(A)\det A = 1$ (CH4定理5)

∴ $\det(A) = \pm 1$

１３A**09** 【 師大84資教[13] 】

Let Q be an orthogonal matrix. What is $\det(Q)$?

【解】1或–1. 證明同上題.

１３A**10** 【 交大80資工[1](f) 】

True (T) or False (F): (1 for each)

(f) If A is an orthogonal matrix then $\det(A) = \pm 1$.

【解】(f) True. 證明同上題.

１３A 11 【 中正81資工[1] 】

Prove that

(a) the magnitude of determinant of a unitary matrix is unity, and

(b) all the eigenvalues of a unitary matrix have unity magnitude.

【解】(a) 設A為unitary matrix, 則 $A^H A = I$. (綜線CH13定義1)

$\det(A^H A) = (\det A^H)(\det A) = \overline{(\det A)}(\det A)$, (綜線CH4定理5,6)

$\therefore\ \overline{(\det A)}(\det A) = 1$, 即 $|\det A| = 1$

(b) 設A為unitary matrix, 則 $A^H A = I$. (綜線CH13定義1)

若$v \neq o$使$Av = \lambda v$. (綜線CH12定義8)

$v^H v = v^H I v = v^H (A^H A) v = (Av)^H (Av) = (\lambda v)^H (\lambda v) = \bar{\lambda}\lambda v^H$ (綜線CH2定理23)

而 $v^H v \neq 0$, ($\because\ v \neq o$)

$\therefore\ 1 = \bar{\lambda}\lambda$, 即 $|\lambda| = 1$

１３A 12 【 交大80工工[13] 】

說明Hermitian matrix, skew-Hermitian matrix, 及unitary matrix的eigenvalues的特性.

【分析】這三種矩陣都是可單式對角化的矩陣. 請參閱綜線CH13定理17c.

【解】(a) Hermitian matrix的eigenvalue必為實數. 證明請參閱綜線CH13定理14.

(b) Skew-Hermitian matrix 的eigenvalue必為純虛數. 證明如下：

設$A^H = -A$, $Ax = \lambda x$, $x \neq 0$

則 $x^H A x = x^H \lambda x = \lambda x^H x$

$$x^H A x = -x^H A^H x = -(Ax)^H x = -(\lambda x)^H x = -\bar{\lambda}\, x^H x$$

$\therefore\ \lambda x^H x = -\bar{\lambda} x^H x$

$\therefore\ \lambda = \bar{\lambda}$

(c) unitary matrix的eigenvalue的絕對值必為1. 證明見上題(中正81資工[1]).

１３A**13** 【 元智81工工[2] 】

若A爲一 real orthogonal matrix 且λ爲A之一real eigenvalue, 則λ有無可能爲2?

【解】不可能. 理由見上題.

【加強演練】

True or False?

"orthogonal matrix 的eigenvalue可能爲2"

[解] True.

設 $A=\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$, 其中 $a^2+b^2=1$

$\det(A-\lambda I)=\lambda^2-2a\lambda+1$

將$\lambda=2$ 代入上式得 $4-4a+1=0$, 解出$a=5/4$

再由 $(5/4)^2+b^2=1$ 可解出 $b=\pm(3/4)i$

令 $A=\dfrac{1}{4}\begin{bmatrix} 5 & 3i \\ -3i & 5 \end{bmatrix}$

則 $A^TA=I$, 且A的eigenvalue爲2, 1/2.

１３A**14** 【 清大81資科[20] 】

Let $Q\in\mathbb{R}^{n\times n}$ be an orthogonal matrix and let $a=[1, 1, \ldots ,1]^t\in\mathbb{R}^n$

Define $\|x\|=(\sum_{i=1}^{n} x_i^2)^{1/2}$ for any $x=[x_1, x_2, \ldots, x_n]^t\in\mathbb{R}^n$. What is $\|Qa\|$?

(a) 1 (b) n (c) $\sqrt{n}$ (d) n^2 (e) $2n$.

【參考章節】 綜線CH13定理4

【解】選(c). 解說如下:

$\|Qa\|=\|a\|=\sqrt{n}$. (綜線CH13定理4)

１３A**１５**【 交大82資工[3](d) 】

[是非倒扣題]

(d) Let Q be an $m\times n$ matrix with orthonormal columns, A an $n\times n$ matrix, and $B=QA$. Then the column vectors of A are linearly dependent if and only if the column vectors of B are linearly dependent.

【解】(d) True. 證明如下:

令線性映射$T:\mathbb{C}^{n\times 1}\longrightarrow\mathbb{C}^{m\times 1}$，定義爲 $Tx=Qx$

$\because$ Q的行orthonormal,

$\therefore$ $Q^HQ=I$. (綜線CH13定理3)

$\therefore$ $\|Tx\|^2=(Qx)^H(Qx)=x^HQ^HQx=x^Hx=\|x\|^2$

$\therefore$ $Tx=o \Longrightarrow \|x\|=\|Tx\|=o \Longrightarrow Tx=o$

$\therefore$ $\ker T=\{o\}$

而B的第j行$=Q\cdot(A$的第j行$)=T(A$的第j行$)$ (綜線CH2定理6)

$\therefore$ A的行線性線性相關 $\Longleftrightarrow$ B的行線性線性相關 . (綜線CH8定理11b)

題型13B: 正交(單式)對角化的理論

１３B01【 清大79工工[6] 】

(a) Show that a $m\times m$ matrix A is normal iff A has a linear independent set of m eigenvectors that may be chosen so as to form an orthonormal set. (15%)

(b) Verify statement (a) by $A=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$ (10%)

【解】(a) 1° A為normal matrix

$\Longleftrightarrow$ ∃矩陣U, 使得 $\begin{cases} U^{-1}AU\text{為對角線矩陣} \\ U^HU=I=UU^H. \end{cases}$ (綜線CH13定理15)

2° ∃矩陣U, 使得$U^{-1}AU$為對角線矩陣

$\Longleftrightarrow$ U的columns都是A的eigenvector, 且形成independent set (綜線CH12定理16)

3° $U^HU=I=UU^H$

$\Longleftrightarrow$ U的columns形成orthonormal set (綜線CH13定理3)

4° 由以上1°2°3°即得證.

(b) 1°

$$A^HA=\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

$$AA^H=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$$

∴ A為normal matrix.

2° $\det(A-xI)=x^2-2x+2=(x-\lambda_1)(x-\lambda_2)$,

其中$\lambda_1=1+i$, $\lambda_2=1-i$.

$$A-\lambda_1 I \sim\ldots\sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$，可取得特徵向量 $v_1=\frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$

$$A-\lambda_2 I \sim\ldots\sim \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$，可取得特徵向量 $v_2=\frac{1}{\sqrt{2}}\begin{bmatrix} -i \\ 1 \end{bmatrix}$

檢驗得知 $v_1^H v_2=0$, $v_1^H v_1=1$, $v_2^H v_2=1$.

∴ $\{v_1 , v_2\}$為orthonormal set.

∴ $\{v_1 , v_2\}$也是independent set. (綜線CH9定理15)

１３B02 【 清大80工工[1] 】

(a) A square matrix A is normal iff A has full set of eigenvectors that may be chosen to form an orthonormal set. (15%)

(b) Given an example to specify that nonreal symmetric matrices need not have a full set of eigenvectors. (10%)

【參考章節】CH13定理15, CH13定理15a, CH13習題15.2.

【解】 (a) 同上題.

(b) 設 $A=\begin{bmatrix} 2i & 1 \\ 1 & 0 \end{bmatrix}$，則 A 為symmetric matrix

$$\det(A-\lambda I)=\lambda^2-2i\lambda-1=(\lambda-i)^2$$

$$A-iI\sim \begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$$

[i的geometric multiplicity] $=2-\text{rank}(A-iI)=2-1=1$

< [i的algebraic multiplicity]

∴ A不可對角化 (綜線CH12定理21)

∴ A無full set of eigenvectors. (綜線CH12定理16)

１３B**03** 【 中正79資工[6] 】

Let A be a unitarily diagonable matrix, i.e., $A = UDU^*$, where D is diagonal, U is unitary, and U^* is the conjugate transpose of U. Show that $A^*A = AA^*$.

【解】請參閱綜線CH13定理

１３B**04** 【 交大86資工[6](f) 】

[是非倒扣題]

Every real symmetric matrix is diagonalizable.

【解】True. (綜線CH13定理15)

１３B**05** 【 台大86資工[1](d) 】

[是非題]

(d) $A = \begin{pmatrix} 6 & -3 & 1 \\ -3 & 1 & 4 \\ 1 & 4 & 2 \end{pmatrix}$, A可以對角化.

【解】True.

【說明】實數對稱矩陣可正交對角化. (綜線CH13定理15)

１３B**06** 【 清大86資科[8] 】

Let $\{u_1, u_2, \ldots, u_n\}$ be an orthonormal basis for $\mathbb{R}^n$. Define

$$A = \lambda_1 u_1 u_1^t + \lambda_2 u_2 u_2^t + \ldots. + \lambda_n u_n u_n^t .$$

Show that A is a symmetric matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and that u_i is an eigenvector corresponding to λ_i for each i.

【分析】本題的$\mathbb{R}^n$ 必須是$\mathbb{R}^{n\times 1}$. (否則A會變成是一個數)

【解】1° $\forall i$, $(u_i u_i^t)^t = u_i^{tt} u_i^t = u_i u_i^t$ (綜線CH2定理23)

$$\therefore A^t = (\lambda_1 u_1 u_1^t + \lambda_2 u_2 u_2^t + \ldots + \lambda_n u_n u_n^t)^t = \lambda_1 (u_1 u_1^t)^t + \lambda_2 (u_2 u_2^t)^t + \ldots + \lambda_n (u_n u_n^t)^t$$

$$= \lambda_1 (u_1 u_1^t) + \lambda_2 (u_2 u_2^t) + \ldots. + \lambda_n (u_n u_n^t) = A .$$

$\therefore$ A為對稱矩陣.

2° $\forall i,\quad (u_i u_i^t)u_i = u_i(u_i^t u_i) = u_i[1] = u_i$

$\forall i \neq j,\quad (u_j u_j^t)u_i = u_j(u_j^t u_i) = u_j[0] = o$

$\therefore\ \forall i,\quad Au_i = (\lambda_1 u_1 u_1^t + \lambda_2 u_2 u_2^t + \ldots + \lambda_n u_n u_n^t)u_i$

$= \lambda_1 u_1 u_1^t u_i + \lambda_2 u_2 u_2^t u_i + \ldots + \lambda_n u_n u_n^t u_i = \lambda_i u_i$

１３B**07** 【 元智85電資[2] 】

True or False ? Explain your answers.

(1) The characteristic equation of a symmetric matrix has only non-negative solutions.

(2) A square matrix is orthogonally diagonalizable if it is symmetric.

(3) A square matrix is symmetric if it is orthogonally diagonalizable.

(4) If A^T is orthogonal, A is also orthogonal.

(5) If a square matrix has n distinct eigenvalues, then it is diagonalizable.

【分析】第(2)小題若在複數系則爲False,

例如 $\begin{bmatrix} 2i & 1 \\ 1 & 0 \end{bmatrix}$ 爲對稱矩陣，但卻不可對角化. (綜線CH12定理21)

依整份試題觀察，命題者是指實數矩陣.

第(5)小題未指明n是什麼，但依一般習慣，應指A爲$n\times n$矩陣.

【解】(1) False.

例如對稱矩陣 $\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ 的特徵方程式的解爲–1, –2.

(2) True. (3) True

對實數矩陣，可正交對角化 $\Longleftrightarrow$ 對稱 (綜線CH13定理15)

(4) True.

A^T 爲orthogonal $\Longleftrightarrow (A^T)^T(A^T) = I = (A^T)(A^T)^T$ (綜線CH13定義1)

$\Longleftrightarrow AA^T = I = A^TA \Longleftrightarrow A$ 爲orthogonal

(5) True.

此爲定理. (綜線CH12定理23)

１３Ｂ08 【 大同80資工[2] 】

If A has eigenvalues 0 and 1, corresponding to the eigenvectors

$$\begin{bmatrix}1\\2\end{bmatrix} \text{ and } \begin{bmatrix}2\\-1\end{bmatrix}.$$

How can you tell in advance that A is symmetric ? What are its trace and determinant ? What is A ?

【解】1° eigenvector爲2×1向量, ∴ A爲2×2矩陣

∵ 2×2矩陣有2個獨立之eigenvector

∴ A可對角化 (綜線CH6定理22,CH12定理16)

又此二eigenvector正交 ∴ A可正交對角化 (綜線CH13定義19)

∴ A爲對稱矩陣 (綜線CH13定理15①)

2° tr$A$爲eigenvalue之和, ∴ tr$A$＝0＋1＝1 (綜線CH13定理8)

det$A$爲eigenvalue之積 ∴ det$A$＝0・1＝0 (綜線CH13定理8)

3°

$$A\begin{bmatrix}1\\2\end{bmatrix}=0\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1&2\\2&-1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix},$$

$$A\begin{bmatrix}2\\-1\end{bmatrix}=1\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}1&2\\2&-1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}$$

(綜線CH12定義1)

$$\therefore A\begin{bmatrix}1&2\\2&-1\end{bmatrix}=\begin{bmatrix}1&2\\2&-1\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

(綜線CH2定理6)

$$\therefore A=\begin{bmatrix}1&2\\2&-1\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}\begin{bmatrix}1&2\\2&-1\end{bmatrix}^{-1}=\cdots\cdots=\begin{bmatrix}4/5&-2/5\\-2/5&1/5\end{bmatrix}$$

#

１３B**09** 【 清大80資料[6] 】

Let $A\in\mathbb{R}^{n\times n}$ be a symmetric matrix with n distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and corresponding unit eigenvectors $u_1, u_2, ..., u_n$. Denote $P_i = u_i u_i^t$ for $1\le i\le n$. Show that

(a) u_i is orthogonal to u_j for $i\neq j$.

(b) $\sum_{i=1}^{n} P_i = I$

(c) $A = \sum_{i=1}^{n} \lambda_i P_i$

【解】(a) 請參閱題型10B, 或由綜線CH13定理14改寫.

(b) 由已知條件及(a)可知$\{u_1, u_2, ..., u_n\}$爲正交單位集.

將這n個行矩陣排成$n\times n$矩陣Q, 即$Q=\begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix}$

則Q爲orthogonal matrix, $\therefore\ QQ^T = I$, (綜線CH13定理3)

$\therefore\ \sum_{i=1}^{n} u_i u_i^T = I$, 故得證. (綜線CH2定理8)

(c) 令$D=\text{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$, 由(a)(b)得知

$A = QDQ^{-1} = QDQ^T$

$$=\begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\begin{bmatrix} u_1^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$=\begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix}\begin{bmatrix} \lambda_1 u_1^T \\ \vdots \\ \lambda_n u_n^T \end{bmatrix} = \sum_{i=1}^{n} u_i(\lambda_i u_i)^T = \sum_{i=1}^{n} \lambda_i P_i$$

(綜線CH2定理8)

１３B**１０**【 台大83資工[9] 】

[複選題]

Which of the following statements is (or are) true ?

(1) Every self-adjoint operator is normal.

(2) The eigenvalues of a self-adjoint operator must all be real.

(3) Every normal operator is diagonalizable.

(4) Every self-adjoint operator is diagonalizable.

【分析】將各個operator想成matrix即可作答. 請參閱綜線CH10定義31,CH13定理12a.

【解】 選(1)(2)(3)(4)

【討論】(1) $T^*=T \implies T^*T=TT^*$.

(2) 此爲定理. (綜線CH13定理14)

(3) 此爲定理. (綜線CH13定理15)

(4) normal家族都可單式對角化. (綜線CH13定理12a, CH13定理15)

１３B**１１**【 交大83資料[3] 】

Show that if matrix A is real and normal and has real eigenvalues, then A is symmetric. (6%)

【解】∵ A爲normal matrix

∴ 存在unitary matrix$U$使得$A=UDU^H$, $D=\mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

由題意得知$\lambda_1, \lambda_2, \dots, \lambda_n$ 都是實數. ∴ $D^H=D$.

∴ $A^H=(UDU^H)^H=U^{HH}D^HU^H=UDU^H=UDU^H=A$

而已知A爲實數矩陣, 所以 $A^T=A^H=A$.

題型13C: 正交(單式)對角化的計算

１３C**０１**【 成大81資工甲乙[2] 】

For the following symmetric matrix A,

$$A=\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

(a) Find a matrix P such that $P^{-1}AP$ is diagonal.

(b) Determine an orthogonal matrix P, such that $P^{-1}AP$ is diagonal.

【解】(a) $\begin{vmatrix} 3-\lambda & -1 & -1 \\ -1 & 3-\lambda & -1 \\ -1 & -1 & 3-\lambda \end{vmatrix} \begin{matrix} \\ (1) \\ (1) \end{matrix} = \begin{vmatrix} 1-\lambda & 1-\lambda & 1-\lambda \\ -1 & 3-\lambda & -1 \\ -1 & -1 & 3-\lambda \end{vmatrix}$

$=(1-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ -1 & 3-\lambda & -1 \\ -1 & -1 & 3-\lambda \end{vmatrix} \begin{matrix} (1)\ (1) \\ \\ \\ \end{matrix} =(1-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 4-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix}$

$=(1-\lambda)(4-\lambda)^2$

$$A-4I=\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

解得特徵向量$v_1=\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $v_2=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

$$A-I=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}\sim\begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\end{bmatrix}$$，解得特徵向量$v_3=\begin{bmatrix}1\\1\\1\end{bmatrix}$.

令 $P=\begin{bmatrix}-1&-1&1\\1&0&1\\0&1&1\end{bmatrix}$，則 $P^{-1}AP=\begin{bmatrix}4&0&0\\0&4&0\\0&0&1\end{bmatrix}$

(b) 對 v_1, v_2 執行Gram-Schmidt process:

$v_1'=v_1$，$\langle v_1', v_1'\rangle=2$，$\langle v_2, v_1'\rangle=1$

$$v_2'=v_2-(1/2)\,v_1'=\begin{bmatrix}-1/2\\-1/2\\1\end{bmatrix}$$

再對v_1', v_2', v_3 做單位化, 得正交單位集v_1'', v_2'', v_3'' .

$$令\ P=\begin{bmatrix}-1/\sqrt{2}&-1/\sqrt{6}&1/\sqrt{3}\\1/\sqrt{2}&-1/\sqrt{6}&1/\sqrt{3}\\0&2/\sqrt{6}&1/\sqrt{3}\end{bmatrix}$$

則 P 為 orthogonal matrix, 且 $P^{-1}AP=\begin{bmatrix}4&0&0\\0&4&0\\0&0&1\end{bmatrix}$

１３C**02** 【 交大80資工[2] 】

For $A=\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, find (if possible)

(a) (2%) The characteristic polynomial of A.

(b) (2%) The eigenvalues of A.

(c) (2%) The eigenvectors of A for eigenvalues of A.

(d) (2%) A matrix P, such that $P^{-1}AP$ is a diagnoal matrix.

(e) (2%) An orthogonal matrix Q, such that $Q^{T}AQ$ is a diagonal matrix

【解】 (a) $\det(A-\lambda I)=\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix}=\lambda^2-4\lambda+3$

(b) $\lambda^2-4\lambda+3=(\lambda-3)(\lambda-1)$, ∴ A之eigenvalues 爲3, 1

(c) 對$\lambda=3$, 解 $(A-3I)x=o$ 得特徵向量 $x=[-t, t]^T, t\neq 0$ (細節略)

對$\lambda=1$, 解 $(A-I)x=o$ 得特徵向量 $x=[t, t]^T, t\neq 0$ (細節略)

(d) 取 $P=\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, 則 $P^{-1}AP=\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$.

(e) 取 $Q=\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, 則$Q^{-1}AQ=Q^{T}AQ=\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$,

且 $Q^{T}Q=I=QQ^{T}$

１３C**03** 【 清大75資科[4] 】

Let $A=\begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

(1) Find eigenvalues of A.

(2) Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal.

【解】 (1)

$$\det(A-\lambda I)=\begin{vmatrix} 5-\lambda & -1 & 0 \\ -1 & 5-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix}=-(\lambda-4)^2(\lambda-6)$$

the eigenvalues of A are 6,4,4 (綜線CH12定理7(b))

(2) (細節略)

對$\lambda=6$, 解 $(A-6I)x=o$ 得特徵向量 $[\,1,-1,0\,]^T$,

對$\lambda=4$, 解 $(A-4I)x=o$ 得特徵向量 $[\,1,1,0\,]^T$, 及$[\,0,0,1\,]^T$

(此二向量恰好已正交)

$$\text{Let } P=\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then, P is orthogonal matrix, and $P^{-1}AP=\begin{bmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

１３C**04** 【 交大83資工[7](b) 】

Let L be the linear operator mapping $\mathbb{R}^3$ into $\mathbb{R}^3$ defined by $L(x)=Ax$,

where $A=\begin{bmatrix} 0 & 2 & -1 \\ 2 & 3 & -2 \\ -1 & -2 & 0 \end{bmatrix}$. Thus the matrix A represents L with respect to the standard basis $[e_1, e_2, e_3]$.

(b) Find an orthonormal basis such that the matrix representing L with respect to this basis is a diagonal matrix. (6%)

【解】(b) (細節略)

$\det(A-xI)=-(x-5)(x+1)^2$

解得–1的特徵向量 $v_1=[\,1, 0, 1\,]^T$, $v_2=[-2, 1, 0\,]^T$

解得5的特徵向量 $v_3=[\,1, 2, -1\,]^T$.

對v_1 , v_2 執行Gram-Schmidt正交化:

$$u_1=v_1=\begin{bmatrix}1\\0\\1\end{bmatrix},\quad u_2=v_2-\frac{v_2\cdot u_1}{u_1\cdot u_1}\,u_1=\begin{bmatrix}-1\\1\\1\end{bmatrix}.$$

則u_1 , u_2 , u_3 爲正交的特徵向量. 再個別做單位化, 將所求基底取爲

$$\left\{\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix},\begin{bmatrix}-1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix},\begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\-1/\sqrt{6}\end{bmatrix}\right\}$$

L對此基底的矩陣表示爲 $\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&5\end{bmatrix}$.

１３C**05** 【 清大86工工[2] 】

Orthogonally diagonalize the following two matrices.

(a) $A=\begin{bmatrix}2&0&1\\0&2&0\\1&0&2\end{bmatrix}$

(b) $B=\begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}$ Hint: Its characteristic equation is $-(\lambda-7)^2(\lambda+2)=0$

【解】(細節略)

(a) $\det(A-\lambda I)=\ \ldots\ =-(\lambda-1)(\lambda-2)(\lambda-3)$

分別解得λ=1,2,3的特徵向量$[-1, 0, 1]^T$, $[0, 1, 0]^T$, $[1, 0, 1]^T$.

$$令U=\begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix},$$

則 $U^{-1}=U^T$, 且 $U^{-1}AU=\text{diag}(1,2,3)$. #

(b) $$\det(B-\lambda I)=\begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix}=\begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 7-\lambda & 0 & 7-\lambda \end{vmatrix}$$

$$=(7-\lambda)\begin{vmatrix} 3-\lambda & -2 & 4 \\ -2 & 6-\lambda & 2 \\ 1 & 0 & 1 \end{vmatrix}=\ \ldots\ =-(\lambda-7)^2(\lambda+2)$$

解 $(B-7I)v=o$ 得λ=7的特徵向量 $v_1=[1, -2, 0]^T$, $v_2=[0, 2, 1]^T$.

進行Gram-Schmidt process:

$$v_1'=v_1,\quad v_2'=v_2-\frac{\langle v_2, v_1'\rangle}{\langle v_1', v_1'\rangle}v_1'=(1/5)[4, 2, 5]^T$$

$$\therefore\ 在7的eigenspace取得orthonormal basis\ \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \\ 0/\sqrt{5} \end{bmatrix},\ \begin{bmatrix} 4/\sqrt{45} \\ 2/\sqrt{45} \\ 5/\sqrt{45} \end{bmatrix}$$

解 $(B+2I)v=o$ 得$\lambda=-2$的特徵向量 $v_3=[2, 1, -2]^T$

∴ 在–2的eigenspace取得orthonormal basis $\begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

令$U=\begin{bmatrix} 1/\sqrt{5} & 4/\sqrt{45} & 2/3 \\ -2/\sqrt{5} & 2/\sqrt{45} & 1/3 \\ 0 & 5/\sqrt{45} & -2/3 \end{bmatrix}$,

則 $U^{-1}=U^T$, 且 $U^{-1}AU=\text{diag}(7,7,-2)$. #

１３C**06** 【 元智85工工乙[2] 】

Let $A=\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}$. Find an unitary matrix U and a triangular matrix W so that $U^{-1}AU=W$.

【分析】請參閱綜線CH13範例11.

【解】$\det(A-xI)=(x-2)(x-1)$

$$A-2I=\begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$, 解得特徵向量 $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 5 & -3 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}=\begin{bmatrix} -8 \\ -6 \end{bmatrix}=-7\begin{bmatrix} 1 \\ 1 \end{bmatrix}+\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\therefore \begin{cases} A\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}=2\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix} \\ A\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}=-7\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}+\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\end{bmatrix}\end{cases}$$

令 $U=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$，則$U$爲unitary matrix，且 $U^{-1}AU=\begin{bmatrix}2 & -7\\0 & 1\end{bmatrix}=W$

13C07【師大84資教[15]】

Given $A=\begin{bmatrix}2 & -3\\2 & -5\end{bmatrix}$,

find an orthogonal matrix U formed from the eigenvectors of A which diagonalizes A.

【解】因A非對稱矩陣，所以不能正交對角化，本題無解. (綜線CH13定理15)

【說明】 $\det(A-xI)=x^2+3x-4=(x+4)(x-1)$

$$A-I=\begin{bmatrix}1 & -3\\2 & -6\end{bmatrix}\sim\begin{bmatrix}1 & -3\\0 & 0\end{bmatrix},\quad \text{解得eignevector } t\begin{bmatrix}3\\1\end{bmatrix}.$$

$$A+4I=\begin{bmatrix}6 & -3\\2 & -1\end{bmatrix}\sim\begin{bmatrix}2 & -1\\0 & 0\end{bmatrix},\quad \text{解得eignevector } t\begin{bmatrix}1\\2\end{bmatrix}.$$

A的eigenspace之間並未正交.

題型13D: 正交(單式)對角化的應用

13D01 【 中正83資工[2] 】

Let A be a matrix that commutes with its transpose (i.e., $A^TA = AA^T$). Such a matrix is said to be normal.

(a) Show that $A^k = O$ if and only if $A = O$, where k is any positive integer. (5%)

(b) Show that $AX = O$ if and only if $A^T = O$. (5%)

【分析】通常normal matrix是指滿足$A^HA = AA^H$ 的複數矩陣，本題原意應是只討論實數矩陣.

【解】(a) 若$A = O$，顯然$A^k = O$.

若$A^k = O$，則A的特徵值全爲0 . (綜線CH14定理17)

$\because A$ normal，$\therefore \exists$unitary matrix P使得$P^{-1}AP = D$, (綜線CH13定理15)

D的對角線元素爲A的特徵值, (綜線CH12定理16)

$\therefore\ D = O$, $\therefore\ A = PDP^{-1} = POP^{-1} = O$

(b) $A^T = O \implies A = O \implies AX = O$

另一方向題意不明，應有筆誤.

若"$\forall X, AX = O$"，則$A = O$，當然$A^T = O$

若"$\exists X$使得$AX = O$"，則未必$A = O$，也就未必 $A^T = O$. (反例自舉)

13D02 【 清大79資科[1](3,4) 】

For each of the following statements,explain whether it is true or false

(3) There is a nonzero real symmetric matrix N such that $N^3 = O$.

(4) If A is a real matrix such that $A = -A^t$, where A^t is the transpose of A, then $I-A$ is invertible.

【解】(3) False. 證明如下:

設N爲非零實數對稱矩陣，且$N^3 = O$.

存在矩陣U，使得 $N = UDU^{-1}$，其中U爲對角線矩陣. (綜線CH13定理15)

$\therefore\ (UDU^{-1})^3 = O$, $\therefore\ UD^3U^{-1} = O$, (綜線CH16定理2)

$\therefore\ D^3 = O$ $\therefore\ D = O$ (註1)

∴ $N=UDU^{-1}=O$ ， 此爲矛盾.

(4) True. 證明如下:

由已知條件可得 $A^H=(-A)$

∴ 存在矩陣U使得$A=UDU^{-1}$, 其中$D=\text{diag}(\lambda_1, \ldots, \lambda_n)$,

且$\lambda_1, \ldots, \lambda_n$ 都是純虛數. (綜線CH13定理15,定理17c)

∴ $I-A=I-UDU^{-1}=U(I-D)U^{-1}$,

∴ $\det(I-A)=\det(U(I-D)U^{-1})=\det(I-D)$ (綜線CH4定理6)

$=(1-\lambda_1)(1-\lambda_2)....(1-\lambda_n)\neq 0$

∴ $I-A$可逆. (綜線CH4定理17)

(4) [另證] 使用矛盾證法加以證明:

若$I-A$不可逆, 則 $\ker(I-A)\neq\{o\}$,

∴ 存在$x\neq o$使得$(I-A)x=o$.

∴ $x^Tx\neq 0$, 且$x=Ax$

∴ $x^Tx=x^TAx=(x^TAx)^T$ (∵1×1矩陣)

$=x^TA^Tx=-x^TAx=-x^Tx$,

∴ 卻又導致$x^Tx=0$ ∴ 得出矛盾.

[註1] 設 $D=\text{diag}(d_1, \ldots, d_n)$, 則 $D^3=\text{diag}(d_1^3, \ldots, d_n^3)$

由 $D^3=O$ 即得 $d_i^3=0, i=1, 2, \ldots, n$

∴ $d_i=0, i=1, 2, \ldots, n$

∴ $D=O$

１３D**03** 【 清大75資料[6] 】

If A is a real symmetric matrix satifying $A^k=I$ for some $k\geq 1$, prove that $A^2=I$.

(Hint: Using diagonalization)

【解】 ∵ A 爲 real symmetric matrix,

∴ $\exists P$, 使得$P^{-1}=P^T$, $A=P\,\text{diag}(\lambda_1, \ldots, \lambda_n)P^{-1}$, $\lambda_i\in\mathbb{R}$. (綜線CH13定理15)

$I=A^k=(P\,\text{diag}(\lambda_1, \ldots, \lambda_n)P^{-1})^k$

$=P\,(\text{diag}(\lambda_1, \ldots, \lambda_n))^kP^{-1}$ (綜線CH16定理2)

$=P\,\text{diag}(\lambda_1^k, \ldots, \lambda_n^k)P^{-1}$ (綜線CH16定理2)

∴ $\text{diag}(\lambda_1^k, \ldots, \lambda_n^k)=P^{-1}IP=I$

$\therefore\ \lambda_1^{\,k}=\lambda_2^{\,k}=\ldots\ldots=\lambda_n^{\,k}=1$

$\therefore\ \lambda_i=\pm 1 \qquad (\because\ \lambda_i\in\mathbb{R})$

$\therefore\ A^2=(P\,\mathrm{diag}(\pm 1,\ldots,\ \pm 1)P^{-1})^2=P\,(\mathrm{diag}(\pm 1,\ldots,\ \pm 1))^2\,P^{-1}=PIP^{-1}=I$

【加強演練】

(a) 若本題的2改爲3，是否仍成立 ？

(b) 若本題的real改爲complex，是否仍成立 ？

[解] (a) 不成立，反例如下:

$$A=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},\quad A^2=I,\ 但A^3\neq I.$$

(b) 不成立，反例如下:

$$A=\begin{bmatrix}(-1+i\sqrt{3})/2 & 0\\ 0 & (-1-i\sqrt{3})/2\end{bmatrix},\quad A^3=I,\ 但A^2\neq I.$$

13D04 【 台大75資工[10] 】

Let $A=\begin{bmatrix}4 & 0 & 0\\ 0 & 1 & i\\ 0 & -i & 1\end{bmatrix}$, Find a matrix B such that $B^{H}B=A$, where $i=\sqrt{-1}$,

B^{H} the conjugate transpose of B.

【解】請參閱綜線CH13範例19，此處不再重複.

１３D**05** 【 交大86資料[3] 】

Given $A=\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & 1 \end{bmatrix}$. Find a matrix B such that $B^H B=A$.

【解】本題與上題(台大75資工所[10])完全相同.

１３D**06** 【 台大82資工[5] 】

Find new coordinates x', y' so that the following forms can be written as $\lambda_1(x')^2+\lambda_2(y')^2$.

$x^2-12xy-4y^2$.

【解】

$$x^2-12xy-4y^2=\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & -6 \\ -6 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

令 $A=\begin{bmatrix} 1 & -6 \\ -6 & -4 \end{bmatrix}$ ，並對A做正交對角化: (細節略, 詳情見上個題型)

令 $U=\begin{bmatrix} 3/\sqrt{13} & 2/\sqrt{13} \\ -2/\sqrt{13} & 3/\sqrt{13} \end{bmatrix}$, 則 $U^{-1}AU=D=\begin{bmatrix} 5 & 0 \\ 0 & -8 \end{bmatrix}$, 且 $U^{-1}=U^T$

令 $v=\begin{bmatrix} x \\ y \end{bmatrix}$, $u=U^{-1}v=\begin{bmatrix} x' \\ y' \end{bmatrix}$ ，則

原式$=v^TAv=v^TUDU^{-1}v=(U^Tv)^TD(U^{-1}v)=u^TDu$

$$=\begin{bmatrix} x' & y' \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & -8 \end{bmatrix}\begin{bmatrix} x' \\ y' \end{bmatrix}=5(x')^2+(-8)(y')^2$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = U^{-1}v = U^{T}v = \begin{bmatrix} 3/\sqrt{13} & -2/\sqrt{13} \\ 2/\sqrt{13} & 3/\sqrt{13} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

即 $x' = (1/\sqrt{13})(3x-2y)$, $y' = (1/\sqrt{13})(2x+3y)$

１３D０７ 【 雲技84電資Ｘ[8] 】

Consider the conic section whose equation is $Q(X) = 2x^2 + 2xy + 2y^2 = 9$.

Represent the conic section with a new coordinate system x', y' such that

$$Q'(Y) = ax'^2 + by'^2 = c.$$

That is, find the constants a,b,and c, and express x',y' in terms of x,y.

(Hint: Write $Q(X) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ and consider rotation of the conic section).

【解】

令 $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, 則 $Q(X) = X^{T}AX$.

利用旋轉(行列式爲1的正交矩陣)將A對角化: (細節略, 詳情見上個題型)

令$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

則$U^{-1} = U^{T}$, 且 $A = UDU^{T}$.

令 $\begin{bmatrix} x' \\ y' \end{bmatrix} = Y = U^{-1}X = U^{T}X$,

即 $\begin{cases} x' = (1/\sqrt{2})x - (1/\sqrt{2})y \\ y' = (1/\sqrt{2})x + (1/\sqrt{2})y \end{cases}$

$Q(X) = X^T AX = X^T UDU^T X = Y^T DY$

即$Q'(Y) = Y^T DY = \begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = x'^2 + 3y'^2$

∴ $a = 1, b = 3, c = 9$ 爲一組解答.

１３D**08**【 雲技85工工[9] 】

令$S = \{(x,y) \mid xy = 1\}$, 而線性映射$T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ 定義爲 $(x,y) = (x+y, 2x-y)$.

(a) 求 S 經過 T 的映像(direct image) $T(S)$ 的方程式.

(b) 若利用座標旋轉將(a)所得的方程式簡化爲$Ax^2 + By^2 = 1$. 試求A與B的值.

【解】(a) (請參閱綜線CH8範例2, 及題型08A)

令$(u,v) = T(x, y) = (x+y, 2x-y)$,

由$u=x+y$, $v = 2x-y$可解得 $x = (u+v)/3$, $y = (2u-v)/3$.

$(u,v) \in T[S] \iff (x, y) \in S \iff xy=1$

$\iff ((u+v)/3)((2u-v)/3) = 1 \iff 2u^2+uv-v^2 = 9$

(b) 令$Q = 2u^2 + uv - v^2 = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} 2 & 1/2 \\ 1/2 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

∵ 實數對稱矩陣可正交對角化, (綜線CH13定理15)

∴ 可取得正交矩陣U, 使 $\begin{bmatrix} 2 & 1/2 \\ 1/2 & -1 \end{bmatrix} = U\text{diag}(\lambda_1, \lambda_2)U^{-1}$

前述正交矩陣U爲可取成$\det U = 1$, 即表示一個旋轉矩陣.

令 $\begin{bmatrix} x \\ y \end{bmatrix} = U^{-1} \begin{bmatrix} u \\ v \end{bmatrix}$, 則$Q = \begin{bmatrix} x & y \end{bmatrix} \text{diag}(\lambda_1, \lambda_2) \begin{bmatrix} x \\ y \end{bmatrix}$

∴ 經此座標旋轉後方程式變成 $\lambda_1 x^2 + \lambda_2 y^2 = 9$

$$\det\left(\begin{bmatrix} 2-\lambda & 1/2 \\ 1/2 & -1-\lambda \end{bmatrix}\right)=\lambda^2-\lambda-9/4$$，解得特徵值 $\lambda=\dfrac{1\pm\sqrt{10}}{2}$

$$\therefore\quad A=\frac{1+\sqrt{10}}{18}\quad,\quad B=\frac{1-\sqrt{10}}{18}\quad.$$

１３D**09**【 元智82工工[10] 】

Evaluate $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+2y^2+xy)}\,dxdy.$

(Hint: use principal-axis theorem, change of variables and the result of problem above.)

【參考章節】綜線CH13範例22

【解】令 $Q=x^2+2y^2+xy=[x\quad y]\begin{bmatrix} 1 & 1/2 \\ 1/2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

再取 $A=\begin{bmatrix} 1 & 1/2 \\ 1/2 & 2 \end{bmatrix}$，$v=\begin{bmatrix} x \\ y \end{bmatrix}$，則 $Q=v^TA\,v$

由特徵多項式 $\det(A-xI)=x^2-3x+7/4=(x-\lambda_1)(x-\lambda_2)$

可解出eigenvalues $\lambda_1,\lambda_2=\dfrac{3\pm\sqrt{2}}{2}$，並得 $\lambda_1\lambda_2=7/4$.

$\because$ A爲實數對稱矩陣, $\therefore$ 可做正交對角化, (綜線CH13定理15)

即存在矩陣P使得 $P^{-1}AP=D=\text{diag}(\lambda_1,\lambda_2)$, $P^{-1}=P^T$

令 $P^{-1}v=u=\begin{bmatrix} s \\ t \end{bmatrix}$，則 $u^T=(P^Tv)^T=v^TP$，

$$Q=v^TAv=v^TPDP^{-1}v=u^TDu=\lambda_1s^2+\lambda_2t^2$$

原式$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-Q}\,dxdy=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\lambda_1 s^2+\lambda_2 t^2)}\left|\dfrac{\partial(x,y)}{\partial(s,t)}\right| dsdt$

$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\lambda_1 s^2+\lambda_2 t^2)}\,dsdt=\int_{-\infty}^{\infty} e^{-\lambda_1 s^2}\,ds\int_{-\infty}^{\infty} e^{-\lambda_2 t^2}\,dt$

$=(\pi/\sqrt{\lambda_1})(\pi/\sqrt{\lambda_2})=(\pi/\sqrt{\lambda_1\lambda_2})=(\pi/\sqrt{7/4})=(2\pi/\sqrt{7})$

題型13E: Rayleigh原理

１３E01 【 台大77資工[7] 】

Let A be an $n\times n$ positive definite matrix, define

$$R(V)=\frac{V^{T}A\,V}{V^{T}V},\quad V\neq o,\text{ a vector,}$$

Prove that min{$R(V)$}＝the smallest eigenvalue of A. $V\neq$ o

【分析】此題由上下文可判知必須在實數系討論. 請參閱綜線CH13定理26.

【證】Let P is an orthogonal matrix such that

$P^{-1}AP=P^{T}AP=\text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, (綜線CH13定理15)

we assume that $\lambda_1\geq\lambda_2\geq\dots\geq\lambda_n$,

Let $V=PX$, where X is a vector, then

$$R(V)=\frac{V^{T}A\,V}{V^{T}V}=\frac{X^{T}P^{T}APX}{X^{T}P^{T}PX}=\frac{X^{T}\text{diag}(\lambda_1,\dots,\lambda_n)\,X}{X^{T}X}$$

Let $X=(x_1, x_2, \dots, x_n)^{T}$, then

$$R(V)=\frac{\lambda_1x_1^2+\lambda_2x_2^2+\ \dots\ +\lambda_nx_n^2}{x_1^2+x_2^2+\ \dots\ +x_n^2}$$

$$\geq\frac{\lambda_nx_1^2+\lambda_nx_2^2+\ \dots\ +\lambda_nx_n^2}{x_1^2+x_2^2+\ \dots\ +x_n^2}=\lambda_n$$

Let V_n is an eigenvector of λ_n , then

$$R(V_n)=\frac{V_n^{T}AV_n}{V_n^{T}V_n}=\frac{V_n^{T}\lambda_nV_n}{V_n^{T}V_n}=\lambda_n$$

$$\therefore\ \lambda_n=\min_{v\neq o}\{R(V)\}$$

１３E02 【 交大80工工[10] 】

$x^2+y^2+z^2=1$, $c=x^2+3y^2+3z^2+4xy+4xz$, 求c之極大, 極小值.

【解】

令 $A=\begin{bmatrix}1&2&2\\2&3&0\\2&0&3\end{bmatrix}$, $x=\begin{bmatrix}x\\y\\z\end{bmatrix}$

$c=x^2+3y^2+3z^2+4xy+4xz=x^TAx$

$\det(A-\lambda I)=....=(3-\lambda)(\lambda-5)(\lambda+1)$

∴ eigenvalues爲5, 3, –1

∴ c之極大值爲5, 極小值爲–1 (綜線CH13定理26)

１３E**03** 【 交大82資工[4] 】

Define the operator $L:\mathbb{R}^3\longrightarrow\mathbb{R}^3$ by $L((x_1, x_2, x_3)^T)=(2x_2-x_3, 2x_1+3x_2-2x_3, -x_1-2x_2)^T$ for each $x=(x_1, x_2, x_3)^T\in\mathbb{R}^3$.

(a) Find the matrix A such that $L(x)=Ax$. (3%)

(b) Find the eigenvalues and the corresponding eigenvectors for the matrix A. (4%)

(c) For a vector $x\in\mathbb{R}^3$ with length 3, i.e., the 2-norm of the vector is 3, find the maximum possible length of the transformed vector $L(x)$. (3%)

(d) The matrix A represents L with respect to the ordered basis $[e_1, e_2, e_3]$. Find the matrix representing L with respect to $[y_1, y_2, y_3]$, where y_1, y_2, y_3 are three independent eigenvectors of the matrix A. (3%)

(e) Find the orthogonal matrix U such that U^TAU is a diagonal matrix. (5%)

【解】(a)

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}2x_2-x_3\\2x_1+3x_2-2x_3\\-x_1-2x_2\end{bmatrix}=\begin{bmatrix}0&2&-1\\2&3&-2\\-1&-2&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$\therefore\quad A=\begin{bmatrix}0&2&-1\\2&3&-2\\-1&-2&0\end{bmatrix}$$ (綜線CH7定理15)

(b) (細節略, 解法見題型12C)

$\det(A-\lambda I)=-(\lambda+1)^2(\lambda-5)$, $\quad\therefore$ A的eigenvalue爲–1, –1, 5

$\lambda=-1$ 所對應的eigenvector爲 $\begin{bmatrix} -2t+s \\ t \\ s \end{bmatrix}$, s,t爲不全爲0.

$\lambda=5$所對應的eigenvector爲 $\begin{bmatrix} -t \\ -2t \\ t \end{bmatrix}$, t不爲0.

(c) 考慮一切長度爲3的向量x,

$$\max \| L(x) \| = \max \| Ax \| = 3 \max \frac{\| Ax \|}{\| x \|}$$

$$= 3 \left(\max \frac{\| Ax \|^2}{\| x \|^2} \right)^{1/2} = 3 \left(\max \frac{x^T A^T A x}{x\, x^T} \right)^{1/2}$$

$= 3\,(A^TA$的最大eigenvalue $)^{1/2}$ (CH13定理26)

$\because$ A爲對稱矩陣, $\quad\therefore$ $A^TA=A^2$

$\because$ A的eigenvalue爲 –1, –1, 5

$\therefore$ A^2的eigenvalue爲 1, 1, 25

$\therefore$ $\max \| Ax \| = 3\,(25)^{1/2} = 15$

(d)

取$y_1=\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $y_2=\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$, $y_3=\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

則對基底 $[\, y_1 , y_2 , y_3 \,]$ 之矩陣表示爲

$$D=\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 5\end{bmatrix}$$

(e) 對(d)小題的y_1 , y_2 , 執行Gram-Schmidt process: (綜線CH9定理16)

$y_1'=y_1=[1 \quad 0 \quad 1]^T$

$$y_2'=y_2-\frac{y_2\cdot y_1'}{y_1'\cdot y_1'}y_1'=[1 \quad -1 \quad -1]^T$$

對y_1', y_2', y_3 執行單位化, 再排成矩陣U:

$$U=\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\\ 0 & -1/\sqrt{3} & 2/\sqrt{6}\\ 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6}\end{bmatrix}$$

則 $U^{-1}=U^T$, (即U為orthogonal) (綜線CH13定義1要訣4)

且 $U^TAU=D$. (綜線CH12定理16)

題型13F: 奇異值分解

13F01 【 交大78資料[3] 】

(a) State and Prove "Theorem of Singular-Value Decomposition".(10%)

(b) Use (a) on $A=\begin{bmatrix}1 & -1 & 2\\ -1 & 1 & -2\end{bmatrix}$, i.e., find the Singular-Value Decomposition of A. (8%)

【解】請參閱綜合線性代數28範例29, 此處不再重複.

13F02 【 清大81資料[29](a) 】

Let

$$B=\begin{bmatrix}-3 & 1 & 0\\ 1 & -3 & 0\\ 0 & 0 & 5\end{bmatrix}$$

(a) Give the singular value decomposition of matrix B.

【解】(a) 請參閱綜合線性代數28範例30, 此處不再重複.

13F03 【 清大83資料[2] 】

Let $X=\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}$, What are the singular values of matrix X?

【解】

$$X^TX=\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ 0 & -2 & 1\\ 0 & 1 & -2\end{bmatrix}=\begin{bmatrix}4 & 0 & 0\\ 0 & 5 & -4\\ 0 & -4 & 5\end{bmatrix}$$

$$\det(X^{\mathrm{T}}X-xI)=\begin{vmatrix} 4-x & 0 & 0 \\ 0 & 5-x & -4 \\ 0 & -4 & 5-x \end{vmatrix}=-(x-4)(x-9)(x-1)$$

∴ $X^{\mathrm{T}}X$的eigenvalues爲9, 4, 1.

∴ X的singular values爲3, 2, 1. (綜線CH13定理28)