題型14A: 冪零的性質

１４A**01** 【 台大79資工[4](iii) 】

(Yes or No question and explain the reason:)

(ⅲ) Let V be the n-dimensional vector space, $T:V\longrightarrow V$ a linear transformation such that $T^2=O$, then $T=O$.

【解】NO

考慮線性映射$T:\mathbb{R}^2\longrightarrow\mathbb{R}^2$， $T(x, y)=(0, x)$

則 $T(T(x, y))=T(0, x)=(0, 0)$

$\therefore$ $T^2=O$, $(\forall v, T^2v=o)$

但$T\neq O$. (∃v使得$Tv\neq o$)

１４A**02** 【 元智82電資[15] 】

By the order of a nilpotent matrix N is meant the least integer k such that $N^k=O$. Observe that if $k=1$, then $N=O$. Verify that :

(a). If M is similar to a nilpotent matrix N, then M is nilpotent and has the same order as N.

(b). A nilpotent matrix is singular.

(c). No nonsingular matrix A is similar to a nilpotent matrix.

【分析】這order通常稱爲index. (綜線CH14定義1)

【解】(a). M is similar to N 表示存在可逆方陣P, 使得 $N=P^{-1}MP$.

而 $N^i=(P^{-1}MP)^i=P^{-1}M^iP$

$\therefore$ $N^i=O \Longleftrightarrow M^i=O$

$\therefore$ N爲nilpotent $\Longleftrightarrow$ M爲nilpotent, 且其order必相同.

(b). 考慮order爲k的nilpotent N,

$(\det N)^k=\det(N^k)=\det O=0$

$\therefore$ $\det N=0$

$\therefore$ N爲singular matrix (綜線CH4定理17)

(c). 假設有nonsingular matrix A 相似於nilpotent matrix,

由(a)部份得知A也必定是nilpotent matrix,

再由(b)部份得知A必須是singular matrix, 這就造成矛盾。

所以不可能有這種A.

１４A03 【 交大78資工[4](a) 】

(a) Can a nilpotent matrix be nonsingular ? Explain.

【解】(a) 不可能.

[證法１] 如上題.

[證法２] 若A為non-singular(可逆),

則$A^2 = A \cdot A$為non-singular (綜線CH2定理12①)

$\therefore$ $A^3 = A^2 \cdot A$為non-singular

....

$\therefore$ 對任意正整數k, A^k 都是non-singular

$\therefore$ $A^k = O$永不成立. ($\because$ O為singular)

１４A04 【 清大86工工[6] 】

Recall that an $n \times n$ matrix N is nilpotent if $N^m = O$ for some integer m.

(a) Show that the eigenvalues of a nilpotent matrix N are all zero.

(b) Show conversely that if the eigenvalues of a matrix N are all zero, then N is nilpotent.

【解】本題考定理證明. 請參閱綜線CH14定理17, 此處不再重複.

題型14B: 循環子空間

１４B**01** 【 台大80資工[6] 】

[True or False Problem]

Let T be a linear operator on a finite-dimensional vector space V, and let $x, y \in V$. If W is the T-cyclic space generated by x and W' is the T-cyclic space generated by y, and $W = W'$, then $x = y$.

【分析】循環分解定理中所選取的向量並非唯一. (綜線CH14定理12)

將A化為Jordan form的P也非唯一.

【解】 False, 反例如下:

$$V = \mathbb{R}^{2\times 1},\quad T: V \longrightarrow V \text{ 定義為 } T\left(\begin{bmatrix} r \\ s \end{bmatrix}\right) = \begin{bmatrix} 0 \\ r \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

則 $$Tx = T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\quad T^2x = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$Ty = T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, T^2y = T\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

此例 $W = W' = \mathbb{R}^{2\times 1}$, 但$x \neq y$.

１４B**02** 【 台大83資工[8] 】

[複選題]

Let V be a finite-dimension vector space. Which of the following statements are true ?

(1) There exists a linear operator T with no T-invariant subspace.

(2) Let T be a linear operator on a finite dimension vector space V, and let x and y be elements of V. If W is T-cyclic subspace generated by x, W' is the T-cyclic subspace generated by y, and $W = W'$, then $x = y$.

(3) Let T be a linear operator on a finite dimension vector space, then there exists a polynomial $g(t)$ of degree n such that $g(T) = O$ (zero transformation).

(4) If T is a linear operator on V, then for any $x \in V$, then T-cyclic subspace generated by x is the same as the T-cyclic subspace generated by $T(x)$.

【勘誤】第(3)小題漏列$n = \dim V$的條件.

【分析】此題較難，須對cyclic subspace有較強的幾何直覺.

【解】 選(3)

【討論】(1) 任何線性算子T都有不變子空間. 至少V自己就是一個. (綜線CH11定理30)

(2) 例如 $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, $T\left(\begin{bmatrix} p \\ q \\ r \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$.

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ T(x) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\quad T^2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

$$y = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ T(y) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\quad T^2(y) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

x和y的T-cyclic subspace都是xy平面. 但$x \neq y$.

(3) 由Cayley-Hamilton定理即得. (綜線CH16定理18)

(4) 如第(2)小題的例子. $T(x)$的T-cyclic subspace是y軸，與x的T–cyclic subspace並不相同.

１４B**03** 【 元智82電資[13] 】

Let A be an $n\times n$ matrix and let x_1 be a nonzero vector in $\mathbb{R}^n$.

(a). Show that there is a first integer k such taht the vectors

$$x_1 , Ax_1 , ..., A^{k-1}x_1 , A^k x_1$$

are linearly dependent. Why is $k\le n$?

(b). Conclude that there is a polynomial $m_1(\lambda)$ of the form

$$m_1(\lambda)=c_0+c_1\lambda+ \; .... \; + \; c_{k-1}\lambda^{k-1}+\lambda^k$$

such that $m_1(A)x_1=o$. Why is $m_1(\lambda)$ a polynomial of lowest degree such that $m_1(A)x_1=o$?

【解】(a).假設 $x_1 , Ax_1 , ..., A^{n-1}x_1 , A^n x_1$ 線性獨立, 則在n維空間$\mathbb{R}^n$ 中

將有內含$n+1$個向量的獨立集, 此為矛盾. (綜線CH6習題19.2)

$\therefore$ $x_1, Ax_1, ... , A^{n-1}x_1, A^n x_1$ 線性相關

$\therefore$ $\exists\, i$ 使得 $x_1, Ax_1, ... , A^{i-1}x_1, A^i x_1$ 線性相關

依題意, $k=\min\{i \mid x_1, Ax_1, ... , A^{i-1}x_1, A^i x_1$線性相關 $\}$,

$\therefore$ $k\le n$

(b).由前述k的定義, 可知存在不全為零的 $a_0, a_1, ... , a_k$ 使得

$$a_0x_1+a_1Ax_1+ \; ... \; +a_{k-1}A^{k-1}x_1+a_kA^kx_1=o .$$

顯然$a_k\neq 0$. (否則將違背k的最小性)

令 $c_i=a_i/a_k$, $i=0,1,...,k-1$.

則 $c_0x_1+c_1Ax_1+....+c_{k-1}A^{k-1}x_1+A^kx_1=o$,

此即 $(c_0+c_1A+....+c_{k-1}A^{k-1}+A^k)x_1=o$,

$\therefore$ $m_1(A)x_1=o$

假設有多項式$p(x)$滿足$p(A)x_1=o$, 且 $\deg p(x)<\deg m_1(x)$,

令 $p(x)=b_0+b_1x+....+b_hx^h$, $b_h\neq 0$

則有 $h<k$, 且 $(b_0+b_1A+....+b_{h-1}A^{h-1}+b_hA^h)x_1=o$,

$\therefore$ $b_0x_1+b_1Ax_1+....+b_{h-1}A^{h-1}x_1+b_hA^hx_1=o$,

$\therefore$ $x_1 , Ax_1 , ..., A^{h-1}x_1 , A^h x_1$ 線性相關

此與k的最小性矛盾.

$\therefore$ 不可能有這種 $p(x)$.