題型15A: Jordan form 的理論

15A01 【精編加強題】

對矩陣 $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$，若$a,b,c,d\in\{0,1\}$，問所組成的16種可能情形中,那些是可對角化矩陣?

【解說】本題對舉反例很有幫助.

【解】

1° $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ 已對角化

2° $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ 可對角化爲 $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

3° $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ 可對角化爲 $\begin{bmatrix} (1+\sqrt{5})/2 & 0 \\ 0 & (1-\sqrt{5})/2 \end{bmatrix}$

4° $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 可對角化爲 $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

5° $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ 可對角化爲 $\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$

6° $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

不可對角化(已是Jordan form)

15A02 【 精編加強題 】

Prove or give a counter example:

If A is similar to A' and B to B' (and all $p\times p$), then $A+B$ is similar to $A'+B'$.

【解】 反例可取爲

$$A=\begin{bmatrix}1&0\\1&0\end{bmatrix}, A'=\begin{bmatrix}0&0\\0&1\end{bmatrix}, B=\begin{bmatrix}-1&0\\0&0\end{bmatrix}, B'=\begin{bmatrix}-1&0\\0&0\end{bmatrix}$$

$$A+B=\begin{bmatrix}0&0\\1&0\end{bmatrix}, A'+B'=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$$

15A03 【 清大78工工[5](ⅱ) 】

If there exists a nonsingular matrix P such that $P^{-1}AP=B$, then B is said to be similar to A.

Then,

(ⅱ) Either prove or give a counter example:

If A is similar to A' and B to B' (and all are $p\times p$), then AB is similar to $A'B'$.

【分析】 (1) 若$A=P^{-1}A'P$, $B=P^{-1}B'P$

則$AB=P^{-1}A'B'P$，於是AB與$A'B'$ similar.

在$A=P^{-1}A'P$, $B=Q^{-1}B'Q$的情形下, AB未必與$A'B'$相似.

(2) 本題的反例較不容易找. 思考的線索是找可對角化矩陣A, B使AB不可對角化.

【解】 令$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$，則$A$的特徵多項式$\det(A-\lambda I)=\lambda^2-1$

$\therefore$ A相似於$A' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (綜線CH12定理23)

再令$B = B' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, 則$B$相似於$B'$

$$AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

但$A'B' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\therefore$ AB與$A'B'$不相似.

１５A**04** 【 台大81資工[4] 】

Let T be a linear operator on a finite-dimensional vector space V. If $\text{rank}(T-\lambda I)^m = \text{rank}(T-\lambda I)^{m+1}$ for some positive integer m , then null space of $(T-\lambda I)^m$ denoted $N(T-\lambda I)^m$ is the generalized eigenspace of T associated with λ.

【參考章節】綜線CH15定義3, CH14定理4, CH14定理5.

【說明】 λ的廣義特徵子空間爲 $K_\lambda = \bigcup_{j=1}^{\infty} N((T-\lambda I)^j)$. (綜線CH15定義3)

【解】 $\because \begin{cases} \dim V = \dim N(T-\lambda I)^m + \text{rank}(T-\lambda I)^m \\ \dim V = \dim N(T-\lambda I)^{m+1} + \text{rank}(T-\lambda I)^{m+1} \end{cases}$ (綜線CH8定理8)

而已知 $\text{rank}(T-\lambda I)^m = \text{rank}(T-\lambda I)^{m+1}$,

$\therefore$ $\dim N(T-\lambda I)^m = \dim N(T-\lambda I)^{m+1}$

又因$N(T-\lambda I)^m$爲$N(T-\lambda I)^{m+1}$之子空間, (綜線CH14定理4)

$\therefore$ $N(T-\lambda I)^m = N(T-\lambda I)^{m+1}$ (綜線CH6定理22a)

以下只須再證 $\forall i = 1,2,\ldots$, $N(T-\lambda I)^m = N(T-\lambda I)^{m+i}$

即可得知: $\bigcup_{j=1}^{\infty} N(T-\lambda I)^j = \bigcup_{j=1}^{m} N(T-\lambda I)^j = N(T-\lambda I)^m$

對i進行數學歸納法:

$i=1$時已得證，現假設$i=k$時成立，

對$i=k+1$：

對$v \in N(T-\lambda I)^{m+i}$，

$o=(T-\lambda I)^{m+i}(v)=(T-\lambda I)^{m+k+1}(v)=(T-\lambda I)^{m+k}(T-\lambda I)(v)$

$\therefore (T-\lambda I)(v) \in N(T-\lambda I)^{m+k}=N(T-\lambda I)^m$ (由歸納假設)

$\therefore (T-\lambda I)^m(T-\lambda I)(v)=o$ $\therefore (T-\lambda I)^{m+1}(v)=o$

$\therefore v \in N(T-\lambda I)^{m+1}=N(T-\lambda I)^m$

此即 $N(T-\lambda I)^{m+i} \subseteq N(T-\lambda I)^m$

又顯然有 $N(T-\lambda I)^m \subseteq N(T-\lambda I)^{m+i}$

$\therefore N(T-\lambda I)^{m+i}=N(T-\lambda I)^m$

１５A05 【 台大80資工[11] 】

[True or False Problem]

Let T be a linear operator on a finite dimensional vector space V, and λ be an eigenvalue of T. If rank$\{(T-\lambda I)^m\}$=rank$\{(T-\lambda I)^{m+1}\}$ for some integer m, then the generalized eigenspace $K_\lambda=N\{(T-\lambda I)^{m+1}\}$.

【參考章節】CH15定義3. CH14定理4,5.

【解】 True, 證明如上題, 注意: $N\{(T-\lambda I)^{m+1}\}=N\{(T-\lambda I)^m\}$

１５A06 【 台大85資工[10] 】

[複選題]

Which of the following are true.

(1) Eigenvectors of a linear operator T are also generalized eigenvector of T.

(2) Let T be a linear operator on an n-dimensional vector space whose characteristic polynomials can be factored. Then, for any eigenvalue λ of T, K_λ=nullspace($(T-\lambda I)^n$), generalized eigenspace of T corresponding to λ.

(3) The Jordan canonical form of a diagonal matrix is the matrix itself.

(4) If an operator has a Jordan canonical form, then there is a unique Jordan basis for the operator.

【解】選(1)(2)

【討論】(1) True.

此爲基本概念. (綜線CH15定義3)

(2) True. 證明如下:

$K_\lambda = \bigcup_j \mathrm{Ker}(T-\lambda I)^j = \mathrm{Ker}(T-\lambda I)^d$,

$T-\lambda I$爲K_λ上的nilpotent, 且以d爲index. (綜線CH15定義3)

$d \le \dim K_\lambda \le \dim V = n$ (綜線CH14定理11a)

$\therefore\ K_\lambda = \mathrm{Ker}(T-\lambda I)^n$ (綜線CH14定理4)

(3) False. 反例如下:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{的Jordan form 應是} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(Jordan form的eigenvalue一定是排在一起.) (綜線CH15定理8)

(4) False.

$$\text{例如} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{可取} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{爲Jordan basis}$$

$$\text{但} \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \right\} \text{也是Jordan basis.}$$

１５A**07** 【 台大83資工[10] 】

[複選題]

Which of the following statements is (or are) true ?
(1) The Jordan canonical form of a diagonal matrix is the matrix itself.
(2) Matrices having the same Jordan canonical form are similar.
(3) Every matrix is similar to its Jordan canonical form.
(4) If an operator has a Jordan canonical form, then there is a unique Jordan canonical basis for that operator.

【解】 選(2).

【討論】(1) Jordan form 的 eigenvalue 一定是排在一起. (綜線CH15定理8)
例如diag(2,3,2) 的Jordan form 應是 diag(2,2,3)或diag(3,2,2).

(2) 此為定理. (綜線CH15定理14)

(3) 矩陣必須在它的特徵多項式在所指定的數系可完全分解才會有Jordan canonical form. (綜線CH15定理8)
例如在實數系討論時，矩陣 $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ 根本就沒有Jordan canonical form. 但這題語意不明，也許命題者心目中是討論有Jordan form的矩陣.

(4) Jordan basis 並不唯一. (綜線CH15習題11.2)

題型15B Jordan form 的計算

15B01 【 元智82工工[8] 】

Find all possible Jordan canonical matrices B with characteristic polynomial $(x-1)^3x^4$ and minimal polynomial $(x-1)^2x^2$. (For this problem only, simply write down your answer without any explanation.)

【解】

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 \end{array}\right]$$

，其中 d 為 0 或 1，共兩種可能.

【說明】由特徵多項式$(x-1)^3x^4$即知其Jordan matrix 如下形

$$\left[\begin{array}{ccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ ? & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & ? & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & ? & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & ? & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & ? & 0 \end{array}\right]$$

再由最小多項式$(x-1)^2x^2$即知左上角的3×3矩陣再細分，其中最大的一塊應是2×2，所以可確定左上角分為2×2及1×1的兩個Jordan basic matrix,

即 $\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right]$

另外，右下角的4×4矩陣再細分，其中最大的一塊應是2×2，所以可有2×2, 2×2 以及 2×2, 1×1, 1×1的兩種分法，即

$$\left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right], \text{或} \left[\begin{array}{cc|c|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right],$$

15B02 【交大80資料[4]】

$A = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 2 & 0 \\ 1 & -1 & 3 \end{bmatrix}$ Find the Jordan form J of A.

【解】 $\det(A-\lambda I) = ... = -(\lambda^3 - 5\lambda^2 + 7\lambda - 1)$

令 $f(\lambda) = \lambda^3 - 5\lambda^2 + 7\lambda - 1$

由 $f(1) \neq 0, f(-1) \neq 0$ 可知 $f(x)$無有理根. (綜線CH12定理9b)

$f'(x) = 3\lambda^2 - 10\lambda + 7$

由輾轉相除法(Euclid's algorithm)可測知$f(\lambda)$與$f'(\lambda)$無公因式

∴ $f(\lambda)$無重根 (綜線CH12定理9a)

令 $f(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda-\lambda_3)$, 其中$\lambda_1, \lambda_2, \lambda_3$ 相異

∴ A可對角化 (綜線CH12定理23)

$$\therefore\ A\text{之Jordan form爲}\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix}$$

其中 $\lambda_1, \lambda_2, \lambda_3$ 爲A的三個(相異)特徵值. (綜線CH12定理9b)

１５B**03** 【 清大77資科[7] 】

Let $A = \begin{bmatrix}3 & 1 & -3\\ -7 & -2 & 9\\ -2 & -1 & 4\end{bmatrix}$

(1). Find eigenvalues of A.

(2). Find the Jordan canonical form of A.

【解】(1) $\begin{vmatrix}3-x & 1 & -3\\ -7 & -2-x & 9\\ -2 & -1 & 4-x\end{vmatrix} = \begin{vmatrix}1-x & 0 & 1-x\\ -7 & -2-x & 9\\ -2 & -1 & 4-x\end{vmatrix} = \ldots = -(x-1)(x-2)^2$

$\therefore$ eigenvalues of A are 1 , 2 , 2.

(2)

$$A-2I=\begin{bmatrix}1 & 1 & -3\\ -7 & -4 & 9\\ -2 & -1 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 1 & -3\\ 0 & 1 & -4\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore$ rank$(A-2I)=2$

$\therefore$ dim Ker$(A-2I)=3-2=1$ (綜線CH8定理8)

又eigenvalue 2 的 algebraic multiplicity 爲 2 , (綜線CH12定義18)

$$\therefore\ A\text{的Jordan form爲}\begin{bmatrix}1&0&0\\0&2&0\\0&1&2\end{bmatrix}.\qquad\text{(綜線CH15定理8③)}$$

15B04【交大82資料[5]】

Find the Jordan form of the following matrix and find P such that $P^{-1}AP$ is in Jordan form

$$A=\begin{bmatrix}2&2&-1\\-1&-1&1\\-1&-2&2\end{bmatrix}$$

【解】 $\begin{vmatrix}2-x&2&-1\\-1&-1-x&1\\-1&-2&2-x\end{vmatrix}\ (-1)\ =\begin{vmatrix}2-x&2&-1\\-1&-1-x&1\\0&-1+x&1-x\end{vmatrix}\ \ldots\ =(1-x)^3$

解 $(A-I)v=o$ 可得 $\ker(A-I)=\{\,t[-2,1,0]^T+u[1,0,1]^T\mid t,u\in\mathbb{R}\,\}$

$\therefore\ A-I$的循環基底關係圖如下:

$v_1\longrightarrow v_2\longrightarrow o$

$v_3\longrightarrow o$

$$\therefore\ A\text{的Jordan form爲}\ J=\begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix}$$

取 $v_1=\begin{bmatrix}1\\0\\0\end{bmatrix}\notin\ker(A-I)$， 則 $v_2=(A-I)v_1=\begin{bmatrix}1\\-1\\-1\end{bmatrix}$

再取 $v_3=\begin{bmatrix}1\\0\\1\end{bmatrix}\in\ \ker(A-I)\setminus Z(v_1, A-I)$

令 $P=\begin{bmatrix}1&1&1\\0&-1&0\\0&-1&1\end{bmatrix}$，則$P$可逆，且 $P^{-1}AP=J$.

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１５B０５【 交大79資料[9] 】

Find the Jordan form for the matrix

$$A=\begin{bmatrix}2&-1&0&0\\1&2&0&0\\0&0&2&1\\0&0&-1&2\end{bmatrix}$$

【解說】本題雖聲稱要找Jordan form，但其實只是對角化的問題.

【解】

$$\det(A-\lambda I)=\begin{vmatrix}2-\lambda&-1&0&0\\1&2-\lambda&0&0\\0&0&2-\lambda&1\\0&0&-1&2-\lambda\end{vmatrix}=\begin{vmatrix}2-\lambda&-1\\1&2-\lambda\end{vmatrix}\begin{vmatrix}2-\lambda&1\\-1&2-\lambda\end{vmatrix}$$

(綜線CH4定理20)

$=(\lambda^2-4\lambda+5)^2=(\lambda-\lambda_1)^2(\lambda-\lambda_2)^2$ ，其中$\lambda_1=2+i$, $\lambda_2=2-i$

$A-(2\pm i)I=$

$$\begin{bmatrix} \mp i & -1 & 0 & 0 \\ 1 & \mp i & 0 & 0 \\ 0 & 0 & \mp i & 1 \\ 0 & 0 & -1 & \mp I \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & \mp i & 0 & 0 \\ 0 & 0 & 1 & \pm i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ $\text{rank}(A-(2\pm i)I)=2$ (綜線CH6定理23)

∴ $\dim \text{Ker}(A-(2\pm i)I)=4-2=2$ (綜線CH8定理8)

∵ λ_1的 algebraic multiplicity 與 geometric multiplicity 相等,

λ_2的 algebraic multiplicity 與 geometric multiplicity 相等

∴ A可對角化 (綜線CH12定理21)

∴ A的Jordan form爲 $\begin{bmatrix} 2+i & 0 & 0 & 0 \\ 0 & 2+i & 0 & 0 \\ 0 & 0 & 2-i & 0 \\ 0 & 0 & 0 & 2-i \end{bmatrix}$

15B**06** 【元智83電資[5]】

Find the Jordan form J_A for $A=\begin{bmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{bmatrix}$

【解】

$$\det(A-xI)=\begin{vmatrix} -x & -3 & 1 & 2 \\ -2 & 1-x & -1 & 2 \\ -2 & 1 & -1-x & 2 \\ -2 & -3 & 1 & 4-x \end{vmatrix} \overset{(-1)}{\longleftarrow} = \begin{vmatrix} -x & -3 & 1 & 2 \\ -2 & 1-x & -1 & 2 \\ 0 & x & -x & 0 \\ -2 & -3 & 1 & 4-x \end{vmatrix}$$

$$= x\begin{vmatrix} -x & -3 & 1 & 2 \\ -2 & 1-x & -1 & 2 \\ 0 & 1 & -1 & 0 \\ -2 & -3 & 1 & 4-x \end{vmatrix} = x\begin{vmatrix} -x & -2 & 1 & 2 \\ -2 & -x & -1 & 2 \\ 0 & 0 & -1 & 0 \\ -2 & -2 & 1 & 4-x \end{vmatrix}$$

$$= -x\begin{vmatrix} -x & -2 & 2 \\ -2 & -x & 2 \\ -2 & -2 & 4-x \end{vmatrix} = -x\begin{vmatrix} -x & -2 & 2 \\ -2 & -x & 2 \\ 0 & x-2 & 2-x \end{vmatrix}$$

$$= \ \ldots = x^2(x-2)^2$$

經列運算可求得rank(A–2I)＝2,

∴ dim ker(A–2I)＝4–2＝2 (綜線CH8定理8)

∴ 特徵值2的基底關係圖爲

$$v_1 \xrightarrow{A-2I} o$$

$$v_2 \xrightarrow{A-2I} o$$

(綜線CH14範例14a)

經列運算可求得 rank(A–0I)＝3,

∴ dim ker(A–2I)＝4–3＝1

∴ 特徵值2的基底關係圖爲

$$v_3 \xrightarrow{A-0I} v_4 \xrightarrow{A-0I} o$$

(綜線CH14範例14a)

∴ 所求爲 $J=\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. (綜線CH15範例8a)

【討論】本題只需求Jordan form, 若還想要將A化爲Jordan form, 則做下述計算:

對特徵值2, 解$(A-2I)v=o$, 可得特徵向量$v_1=[-1, 1, 1, 0]^T$, $v_2=[1, 0, 0, 1]^T$.

對特徵值0, 解$(A-0I)v=o$, 可解得特徵向量$v_4=[1, 1, 1, 1]^T$.

再解 $(A-0I)v=v_4$，可得$v_3=[0,-1,-2,0]^T$.

$$\text{令}\quad P=\begin{bmatrix}-1 & 1 & 0 & 1\\ 1 & 0 & -1 & 1\\ 1 & 0 & -2 & 1\\ 0 & 1 & 0 & 1\end{bmatrix}\text{，則 } P^{-1}AP=J$$

１５Ｂ**07**【 元智80工工[4] 】

Let $P[x,y]=\{\,ax^2+bxy+cy^2+dx+ey+f \mid a,b,c,d,e,f\in\mathbb{R}\,\}$ and let T be the linear operator on $P[x,y]$ defined by

$$T(f(x,y))=\frac{\partial f}{\partial x},\quad \forall f(x,y)\in P[x,y].$$

Find the Jordan form for T. (Note: $x^2, xy, y^2, x, y, 1$ constitute a basis for $P[x,y]$)

【參考章節】綜線CH14範例15.

【解】考慮基底 $B=\{x^2, xy, y^2, x, y, 1\}$

$$\because\ \begin{cases}T(x^2)=2x=0\cdot x^2+0\cdot xy+0\cdot y^2+2\cdot x+0\cdot y+0\cdot 1\\ T(xy)=y=0\cdot x^2+0\cdot xy+0\cdot y^2+0\cdot x+1\cdot y+0\cdot 1\\ T(y^2)=0=0\cdot x^2+0\cdot xy+0\cdot y^2+0\cdot x+0\cdot y+0\cdot 1\\ T(x)=1=0\cdot x^2+0\cdot xy+0\cdot y^2+0\cdot x+0\cdot y+1\cdot 1\\ T(y)=0=0\cdot x^2+0\cdot xy+0\cdot y^2+0\cdot x+0\cdot y+0\cdot 1\\ T(1)=0=0\cdot x^2+0\cdot xy+0\cdot y^2+0\cdot x+0\cdot y+0\cdot 1\end{cases}$$

$$[T]_B=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\2&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&1&0&0\end{bmatrix}\quad \text{(綜線CH7定義9)}$$

$\therefore$ 由主對角線可看出 eigenvalues 為 0,0,0,0,0,0 (綜線CH13定理8要訣5)

dim Ker$T=6-\text{rank } T$ (綜線CH8定理8)

$=6-\text{rank}\,[T]_B$ (綜線CH8定義12②)

$=6-3=3$ .... (甲)

$$([T]_B)^2=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\2&0&0&0&0&0\end{bmatrix}$$

$\dim \text{Ker}(T^2) = 6-\text{rank}\,(T^2)$ (綜線CH8定理8)

$= 6-\text{rank}\,[T^2]_B$ (綜線CH8定義12②)

$= 6-\text{rank}([T]_B{}^2)$ (綜線CH8定理23)

$= 6-1 = 5$ ...... (乙)

由(甲)(乙)可知Jordan basis 之基底關係如下: (綜線CH14範例14a)

$$\begin{cases} v_1 \longrightarrow v_2 \longrightarrow v_3 \longrightarrow o \\ \qquad\quad v_4 \longrightarrow v_5 \longrightarrow o \\ \qquad\qquad\quad v_6 \longrightarrow o \end{cases}$$

∴ T之Jordan form如下: (綜線CH14範例12a)

$$\begin{bmatrix}0&0&0&0&0&0\\1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&0&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{bmatrix}$$

１５B**08** 【 元智84電資[5] 】

Let $V=L(e^{-x}, xe^{-x}, x^2e^{-x})$ be the $\mathbb{R}$-space generated by the three functions e^{-x}, xe^{-x} and x^2e^{-x} .

Let $\varphi \in L(V)$ be the linear operator on V defined by $\varphi(f(x))=f'(x)$ $\forall f(x)\in V$.

Find the Jordan form of φ .

【解】
$$\begin{cases} L(e^{-x}) = -e^{-x} \\ L(xe^{-x}) = e^{-x}-xe^{-x} \\ L(x^2 e^{-x}) = 2xe^{-x}-x^2 e^{-x} \end{cases}$$

以所給三函數當基底, L的矩陣表示爲$A=\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$ (綜線CH7定義9)

$\det(A-xI)=-(x+1)^3$

解$(A+I)v=o$. 取得一個特徵向量 $[1, 0, 0]^T$.

∴ 循環分解的基底關係圖爲

$X \longrightarrow X \longrightarrow X \longrightarrow o$, 至此已可判知Jordan form.

因本題簡單, 不妨繼續找出Jordan basis.

解 $(A+I)v=[1, 0, 0]^T$, 取得一特解$[1, 1, 0]^T$.

解 $(A+I)v=[1, 1, 0]^T$, 取得一特解$[1, 1, 1/2]^T$.

$$[1, 1, 1/2]^T \xrightarrow{A+I} [1, 1, 0]^T \xrightarrow{A+I} [1, 0, 0]^T \xrightarrow{A+I} o$$

以 $\{ e^{-x}+xe^{-x}+(1/2)x^2 e^{-x}, e^{-x}+xe^{-x}, e^{-x} \}$ 爲新基底, 則

在此新基底下, φ的矩陣表示爲 $\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

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