題型16A: 特徵值與特徵向量的變化

１６A**01** 【 中正81資工[4] 】

(a) Prove that similar matrices have the same characteristic polynomial and the same eigenvalues.

(b) Suppose that B is similar to A with $B = P^{-1}AP$. Then x is an eigenvector of A associated with the eigenvalues λ if and only if $P^{-1}x$ is an eigenvector of B associated with the eigenvalue λ. Prove it.

【解】請參閱綜線CH16定理1a.

１６A**02** 【 交大85資工[4] 】

$$\text{Let}\quad A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 1 & 4 & -3 & 0 \\ -1 & -2 & 0 & -3 \end{bmatrix}$$

(a) Diagonalize the matrix A if possible, that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. (8%)

$$\text{(b) Let}\quad B = \begin{bmatrix} 5 & 2 & 4 & 3 \\ 1 & 7 & 2 & 2 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

and let $C = AB$. Find the determinant of matrix C. (3%)

(c) Let $Q=\begin{bmatrix}3&0&0&0\\1&2&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}$

and let $R=QAQ^{-1}$. Find the eigenvalues and eigenvectors of matrix R. (3%)

【解】(a) (細節略, 請參閱題型12C)

$\det(A-xI)=(x-5)^2(x+3)^2$

令 $P=\begin{bmatrix}-8&-16&0&0\\4&4&0&0\\1&0&1&0\\0&1&0&1\end{bmatrix}$, $D=\begin{bmatrix}5&0&0&0\\0&5&0&0\\0&0&-3&0\\0&0&0&-3\end{bmatrix}$.

則 $A=PDP^{-1}$. #

(b) (請參閱題型04B)

∵ B的第三,四列成比例,

∴ $\det B=0$ (綜線CH4定理7)

∴ $\det C=\det(AB)=\det A\det B$ (綜線CH4定理6)

$=(\det A)\cdot 0 = 0$.

(c) $R=QAQ^{-1}=QPDP^{-1}Q^{-1}=(QP)D(QP)^{-1}$

$$QP=\begin{bmatrix}3&0&0&0\\1&2&0&0\\0&0&1&0\\0&0&0&-1\end{bmatrix}\begin{bmatrix}-8&-16&0&0\\4&4&0&0\\1&0&1&0\\0&1&0&1\end{bmatrix}=\begin{bmatrix}-24&-48&0&0\\0&-8&0&0\\1&0&1&0\\0&-1&0&-1\end{bmatrix}$$

∴ R的eigenvalue 爲 5, –3 .

$$5的eigenvector為\ s\begin{bmatrix}-24\\0\\1\\0\end{bmatrix}+t\begin{bmatrix}-48\\-8\\0\\-1\end{bmatrix},\quad s,t不全為零.$$

$$-3的eigenvector為\ s\begin{bmatrix}0\\0\\1\\0\end{bmatrix}+t\begin{bmatrix}0\\0\\0\\-1\end{bmatrix},\quad s,t不全為零.$$

１６A**03** 【 中正83資工[4] 】

(a) If A is diagonalizable , show that A^n is diagonalizable, where n is any positive integer. (5%)

(b) Let $A=\begin{bmatrix}2&0&0\\1&0&0\\3&2&1\end{bmatrix}$. Find the eigenvalues of the matrices A and A^n , where n is any positive integer. (5%)

【解】(a) 若A可對角化, 則∃可逆矩陣P, 使得$P^{-1}AP=D$, D為對角矩陣.

$\therefore\ P^{-1}A^nP=(P^{-1}AP)^n=D^n$

D^n仍為對角矩陣, ∴ A^n可對角化.

(b) $\det(A-xI)=(2-x)(-x)(1-x)$

∴ A的特徵值為2, 0, 1

∴ A^n的特徵值為2^n, 0, 1. (綜線CH16定理1a)

１６A**04** 【 清大81工工[7.4] 】

Which statement is true?

(a) Similar matrices have the same eigenvalues and eigenvectors.

(b) If A and C are $n\times n$ matrices and C is invertible and v is an eigenvector of A, then $C^{-1}v$ is an eigenvector of $C^{-1}AC$.

(c) Any two $n\times n$ diagonal matrices are similar.

(d) Two similar $n\times n$ matrices represent the same linear transformation of $\mathbb{R}^n$ into itself relative to two suitably chosen bases for $\mathbb{R}^n$.

(e) none of the above.

【解】選(b)(d).

分別解說如下:

(a) 相似矩陣的eigenvalue相等，但eigenvector通常不等。
eigenvector 之間的關係如(b)所示，相差一個可逆矩陣的因子。

(b) 證明請參閱綜線CH16定理1a.

(c) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ 與 $\begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$ 的行列式不等，因而不相似. (綜線CH4定理6要訣4)

(d) 此為相似矩陣的基本性質。 (綜線CH7定理21a)

１６A**05** 【 大同82資工[8] 】

Identify which of the following statements is true:

(a) similar matrices always have the same eigenvalues;

(b) similar matrices always have the same eigenvectors;

(c) linear operators on infinite-diemnsional vector space has n distinct eigenvalues.

【解】 選 a

【說明】(a) 此為定理. (綜線CH16定理1a)

(b) 特徵向量差一個矩陣因子. (綜線CH6定理1a)

(c) 舉反例如下:

考慮實係數多項式空間$\mathbb{R}[x]$，及線性映射

$T: \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$, $T(p(x)) = x \cdot p(x)$

對任意$p(x)$, $T(p(x))$與 $p(x)$ 的degree不同，不可能成倍數 。

$\therefore$ T無eigenvector，無eigenvalue.

１６A**06** 【 交大82工工[6](d) 】

Label the following statements as being true or false.

(d) Similar matrices always have the same eigenvectors. (T, F).

【解】(d) False. (綜線CH16定理1a)

１６A**07** 【 成大81資工丙[2] 】

$$A=\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$$

(a) Please find the eigenvalues of A. (5%)

(b) Please find the eigenvectors of A. (5%)

(c) What are the eigenvalues of A^2. (5%)

【解】(a) $\det(A-xI)=\ldots=-(x-5)(x+3)^2$

$\therefore$ eigenvalues 爲 5, –3, –3

(b) 解 $(A-5I)v=o$, 可得5的eigenvector $k[-1, -2, 1]^T$, $k\neq0$,

解 $(A+3I)v=o$, 可得–3的eigenvector $k[-2, 1, 0]^T+h[3, 0, 1]^T$, h, k不全爲零.

(c) eigenvalues 爲 $5^2, (-3)^2, (-3)^2$,

即 25, 9, 9 (綜線CH16定理1a)

１６A**08** 【 師大83資教[2] 】

Let $A=\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$. If we shift the matrix by subtracting $5I$, that is $B=A-5I=\begin{bmatrix} -4 & -1 \\ 2 & -1 \end{bmatrix}$,

what are the eigenvalues and eigenvectors of B, and how are they related to those of A ? (14%)

【解】1°求B的特徵值,特徵向量:

$\det(B-xI)=(x+2)(x+3)$

$\therefore$ B的特徵值爲–2, –3.

$$B+2I=\begin{bmatrix}-2 & -1\\ 2 & 1\end{bmatrix}\sim\begin{bmatrix}2 & 1\\ 0 & 0\end{bmatrix}$$

$\therefore$ B相對於–2的特徵向量為 $t\begin{bmatrix}1\\ -2\end{bmatrix}, t\neq 0$.

$$B+3I=\begin{bmatrix}-1 & -1\\ 2 & 2\end{bmatrix}\sim\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$$

$\therefore$ B相對於–3的特徵向量為 $t\begin{bmatrix}1\\ -1\end{bmatrix}, t\neq 0$.

2° $A=B+5I$,

A的特徵值為3, 2. (CH16定理1a,CH16定理3).

A相對於3的特徵向量為$t\begin{bmatrix}1\\ -2\end{bmatrix}, t\neq 0$.

A相對於2的特徵向量為$t\begin{bmatrix}1\\ -1\end{bmatrix}, t\neq 0$.

１６A**09** 【 交大80工工[12] 】

$A=\begin{bmatrix}1 & 2 & -1\\ 1 & 0 & 1\\ 4 & -4 & 5\end{bmatrix}$, $P(A)=3A^5+2A^3+I$, 求$P(A)$的eigenvalues.

【要訣】設λ為A的eigenvalue, 則$p(\lambda)$為$p(A)$的eigenvalue (綜線CH16定理3)

【解】 det($A-\lambda I$)

$$=\begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix} \overset{(1)}{=} \begin{vmatrix} 2-\lambda & 2-\lambda & 0 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix} = \ldots = (2-\lambda)(\lambda-3)(\lambda-1)$$

∴ A的eigenvalues爲1, 2, 3

$P(1)=3\cdot 1^5+2\cdot 1^3+1=6$, $P(2)=113$, $P(3)=784$

∴ $P(A)$的eigenvalues爲6, 113, 784 (綜線CH16定理1a)

１６A**１０** 【 中正80資工[5] 】

Let $p(\lambda)=a_0+a_1\lambda+a_2\lambda^2+\ldots+a_n\lambda^2$. Define the matrix $p(A)$ by $p(A) = a_0I+a_1A+a_2A^2+ \ldots +a_nA^n$, where I is the $n\times n$ identity matrix. Show that if $\lambda_1, \lambda_2, \lambda_3,\ldots,\lambda_n$ are eigenvalues of A, then $p(\lambda_1), p(\lambda_2), p(\lambda_3), \ldots , p(\lambda_n)$ are eigenvalues of $p(A)$.

【分析】此題多項式p的次數記爲n, 而A列出n個特徵值, 表示A是n階方陣. 這兩個n通常應該要不一樣, 這是本題命題上的暇疵.

【解】∵ λ_i爲A的eigenvalue , ∴ $\exists v_i\neq 0$ 使 $Av_i=\lambda_i v_i$

$\forall\ k=2, 3, \ldots,\quad A^k v_i=A^{k-1}Av_i=A^{k-1}\lambda_i v_i=\lambda_i A^{k-1}v_i=\ldots=\lambda_i\lambda_i^{k-1}\ v_i=\lambda_i^k v_i$

∴ $p(A)v_i=(a_0I+a_1A+\ldots+a_nA^n)v_i=a_0v_i+a_1Av_i+\ldots+a_nA^nv_i$

$=a_0v_i+a_1\lambda_iv_i+\ldots+a_n\lambda_i^nv_i=(a_0+a_1\lambda_1+\ldots+a_n\lambda_i^n)v_i=p(\lambda_i)v_i$

∴ $p(\lambda_i)$爲A的eigenvalue

１６A**１１** 【 淡江82資工[1] 】

Show that $A=\begin{bmatrix} a_0 & a_1 & a_2 & a_3 \\ a_3 & a_0 & a_1 & a_2 \\ a_2 & a_3 & a_0 & a_1 \\ a_1 & a_2 & a_3 & a_0 \end{bmatrix}$ can be written $a_0I+a_1C+a_2C^2+a_3C^3$, where C is a constant matrix. Also find the eigenvalues and eigenvectors of A.

【解】1°

$$令\quad C=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\1&0&0&0\end{bmatrix},\ 則A=a_0I+a_1C+a_2C^2+a_3C^3\,.$$

2°先對C做處理:

$\det(C-xI)=\ldots=x^4-1=(x-1)(x+1)(x-i)(x+i)$

∴ C的特徵值爲$1, i, -1, -i$.

順便得知C可對角化. (綜線CH12定理23)

3°令$p(x)=a_0+a_1x+a_2x^2+a_3x^3$,

∵ $A=p(C)$， 而 C 的特徵值爲$1, i, -1, -i$.

∴ A的特徵值爲 $p(1), p(i), p(-1), p(-i)$, (綜線CH16定理3)

即: $a_0+a_1+a_2+a_3,\ a_0+ia_1-a_2-ia_3,\ a_0-a_1+a_2-a_3,\ a_0-ia_1-a_2+ia_3$.

(若要將本題的A對角化, 也是先對C做.)

【加強演練】

(1) 在複數系將下列的A對角化：

$$A=\begin{bmatrix}a&b&c\\c&a&b\\b&c&a\end{bmatrix}$$

(2) 將 $a^3+b^3+c^3-3abc$ 分解爲一次式的乘積

[解] (1)

$$令C=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix},\quad p(x)=a+bx+cx^2,\quad 則A=p(C)\,.$$

C可對角化: $P^{-1}CP=\text{diag}(1, \omega, \omega^2)$

其中 $P=\begin{bmatrix}1&1&1\\1&\omega&\omega^2\\1&\omega^2&\omega\end{bmatrix}$, $\omega=\dfrac{-1+i\sqrt{3}}{2}$.

$\therefore\ P^{-1}AP=\ \mathrm{diag}(a+b+c, a+\omega b+\omega^2c, a+\omega^2b+\omega c)$

(2) 由(1)所得結果兩邊取行列式得

$a^3+b^3+c^3-3abc=(a+b+c)(a+\omega b+\omega^2c)(a+\omega^2b+\omega c)$

１６A**12** 【 交大81資工[7](a) 】

(a) (4%) Let A be a nonsingular matrix and let λ be an eigenvalue of A. Show that λ^{-1} is an eigenvalue of A^{-1} .

【分析】本題請參看綜線CH16定理12. 注意本題並未假設A是可對角化.

【解】(a) 先證明$\lambda\neq 0$:

$\because$ A nonsingular, $\therefore$ det$A\neq 0$ (綜線CH4定理17)

$\therefore$ $\det(A-0I)\neq 0$

而 $\det(A-\lambda I)=0$, $\therefore$ $\lambda\neq 0$

再證明λ^{-1} 爲A^{-1} 的eigenvalue:

設v爲λ的eigenvector, 則

$v=Iv=A^{-1}Av=A^{-1}\lambda v=\lambda A^{-1}v$

$\therefore$ $\lambda^{-1}v=A^{-1}v$

$\therefore$ λ^{-1}爲A^{-1}的eigenvalue. (綜線CH12定義1)

[另解] 也可經由特徵多項式, 證明$\det(A^{-1}-\lambda^{-1}I)=0$. 讀者可自行嘗試.

１６A**13** 【 朝陽85工工[9] 】

Suppose λ is an eigenvalue of the nonsingular matrix A with associated eigenvector X.
Find an eigenvalue and its associated eigenvector of the matrix A^{-1} .

【解】 A有一個特徵值爲$1/\lambda$, 配特徵向量X . (綜線CH16定理1a)

１６Ａ**１４**【 清大84資料[2] 】

(a) Find the eigenvalues and a set of eigenvectors of the following matrix, which happens to be degenerate:

$$M=\begin{bmatrix} 3/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

(b) Find the spectrum decomposition of matrix M .

(c) Find the eigenvalues and corresponding eigenvectors of M^{-1} .

【解】(a) (細節略)

$\det(M-xI)=-(x-2)(x-1)^2$ $\quad\therefore$ M的特徵值為 2,2,1 .

解 $(M-2I)v=o$ 得特徵值2的一組特徵向量 $[1,-1,0]^T$, $[0,0,1]^T$.

解 $(M-I)v=o$ 得特徵值1的一組特徵向量 $[1,1,0]^T$.

(b) 接(a):

令 $S=\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, 則

$$M = S\,\mathrm{diag}(2,2,1)\,S^{-1}$$

$$= S\left(2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}+1\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)S^{-1}$$

$$= 2S\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}S^{-1}+S\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}S^{-1}= \;....$$

$$= 2\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 1\begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ #

(c) M^{-1}的特徵值爲M的特徵值的倒數，即 1/2, 1/2, 1 .

特徵向量與M相同. (綜線CH16定理1a)

１６Ａ**１５** 【 清大81資料[18] 】

Let $A\in\mathbb{R}^{3\times3}$ have eigenvalues {1, 2, 5}. What are the eigenvalues of matrix $I-A^{-1}$?

(a) {1, 0.2, 0.5} (b) {–1, –2, –5} (c) {0, 0.5, 0.8}

(d) {0, 1, 4} (e) {1, 2, 5}

【解】選(c). 解說如下:

∵ A的eigenvalues爲{ 1, 2, 5 } , ∴ A可對角化.

∴ A可逆且A^{-1}的eigenvalues爲

{ 1/1, 1/2, 1/5 } ＝ { 1, 0.5, 0.2 } (綜線CH16定理12)

令 $f(x)=1-x$

則 $I-A^{-1}=f(A^{-1})$ (綜線CH2定義15)

∴ $I-A^{-1}$ 的eigenvalues 爲

$\{ f(1), f(0.5), f(0.2) \} = \{ 0, 0.5, 0.8 \}$ (綜線CH16定理3)

１６Ａ**１６** 【 中央84資工[5] 】

Find $A^{33}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, where $A=\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$.

【解】

$$A\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}3&0&-1\\2&3&1\\-3&4&5\end{bmatrix}\begin{bmatrix}1\\1\\-1\end{bmatrix}=4\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

$$A^{33}\begin{bmatrix}1\\1\\-1\end{bmatrix}=4^{33}\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}4^{33}\\4^{33}\\-4^{33}\end{bmatrix}$$

【討論】本題型通常是先求出A^{33},再做相乘. 但計算到中途發現特徵多項式爲

$$-x^3+11x^2-32x+16 =(x-4)(x^2-7x+4)=(x-4)(x-(7+\sqrt{45}))(x-(7-\sqrt{45}))$$

因特徵値爲根數，用對角化法或用Cayley-Hamilton法都無法化簡.

幸好所給的向量剛好是特徵向量，否則就會亂成一團.

題型16B: 計算 f(A)

１６B**01** 【 交大79工工[12] 】

Let $A=\begin{bmatrix}3 & -5\\1 & -3\end{bmatrix}$. Compute A^9.

(Hint: Find a matrix P such that $P^{-1}AP$ is a diagonal matrix D and show that $A^9=PD^9P^{-1}$.)

【解】 1°先做對角化(細節略, 解法參閱題型12C):

$$令\ P=\begin{bmatrix}5 & 1\\1 & 1\end{bmatrix},\ 則\ P^{-1}AP=D=\begin{bmatrix}2 & 0\\0 & -2\end{bmatrix}.$$

2° $A^9=(PDP^{-1})^9=PD^9P^{-1}$

$$=\begin{bmatrix}5 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}512 & 0\\0 & -512\end{bmatrix}\left((1/4)\begin{bmatrix}1 & -1\\-1 & 5\end{bmatrix}\right)=\begin{bmatrix}768 & -1280\\256 & -768\end{bmatrix}$$ #

１６B**02** 【 雲技84電資Ｘ[6] 】

Let $A=\begin{bmatrix}3 & -5\\1 & -3\end{bmatrix}$. Compute A^9.

(Hint: Find a matrix P such that $P^{-1}AP$is a diagonal matrix.

【解】同上題.

１６B**03** 【 交大81資工[6] 】

Let

$$A=\begin{bmatrix}5 & 6\\-2 & -2\end{bmatrix}$$

(a) Find the eigenvalues and the corresponding eigenvectors for A.

(b) Factor the matrix A into a product SDS^{-1} , where D is diagonal.

(c) Use (b) to compute A^{10} .

【解】(a) (細節略，解法參閱題型12C)

$\det(A-xI)=(t-2)(t-1)$

∴ 解$(A-2I)v=o$，得2的eigenvectors $t[-2, 1]^T$， $t\neq 0$

∴ 解$(A-I)v=o$，得1的eigenvectors $t[-3, 2]^T$， $t\neq 0$

(b)

令 $S=\begin{bmatrix}-2 & -3\\ 1 & 2\end{bmatrix}$ ，則 $S^{-1}AS=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}=D$

∴ $A=SDS^{-1}$

(c) $A^{10}=(SDS^{-1})^{10}=SD^{10}S^{-1}$ (綜線CH16定理3)

$$=\begin{bmatrix}-2 & -3\\ 1 & 2\end{bmatrix}\begin{bmatrix}1024 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}-2 & -3\\ 1 & 2\end{bmatrix}=\begin{bmatrix}4093 & 6138\\ -2046 & -3068\end{bmatrix}$$

16B**04** 【 中央83資工[2] 】

Let matrix $A=\begin{bmatrix}2 & 0 & 1\\ -1 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}$

(a) Factor A into $S\Lambda S^{-1}$ to find S and Λ, where S is the eigenvector matrix and Λ is the eigenvalue matrix of A. (10%)

(b) Find A^{69} . (4%)

【解】 (a) (細節略)

$\det(A-xI)$

$$=\begin{vmatrix} 2-x & 0 & 1 \\ -1 & 2-x & 0 \\ 1 & 0 & 2-x \end{vmatrix} = \begin{vmatrix} 3-x & 0 & 3-x \\ -1 & 2-x & 0 \\ 1 & 0 & 2-x \end{vmatrix} = \ldots = (3-x)(2-x)(1-x)$$

∴ A的特徵值爲3, 2, 1.

解$(A-3I)v=o$, 得3的eigenvectors $t[1, -1, 1]^T$, $t\neq 0$

解$(A-2I)v=o$, 得2的eigenvectors $t[0, 1, 0]^T$, $t\neq 0$

解$(A-1I)v=o$, 得1的eigenvectors $t[1, 1, -1]^T$, $t\neq 0$

令 $S=\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$, $\Lambda=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, 則 $A=S\Lambda S^{-1}$.

(b) $A^{69}=(S\Lambda S^{-1})^{69}=S\Lambda^{69}S^{-1}$

$$=\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3^{69} & 0 & 0 \\ 0 & 2^{69} & 0 \\ 0 & 0 & 1^{69} \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}^{-1}$$

$$=\begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 3^{69} & 0 & 0 \\ 0 & 2^{69} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 1 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$

$$=(1/2)\begin{bmatrix} 3^{69}+1 & 0 & 3^{69}-1 \\ -3^{69}+1 & 2^{70} & -3^{69}+2^{70}-1 \\ 3^{69}-1 & 0 & 3^{69}+1 \end{bmatrix}$$

１６Ｂ**05** 【 大同80資工[6] 】

Find A^n if $A=\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}$.

【解】 $\det(A-xI)=\begin{vmatrix}1-x & 1\\ 1 & -x\end{vmatrix}=x^2-x-1=(x-\lambda_1)(x-\lambda_2)$,

其中$\lambda_1=\dfrac{1+\sqrt{5}}{2}$, $\lambda_2=\dfrac{1-\sqrt{5}}{2}$

令 $x^n=q(x)(x^2-x-1)+ax+b$

以 λ_1, λ_2 分別代入上式得 $\lambda_1^n=a\lambda_1+b$, $\lambda_2^n=a\lambda_2+b$

解得:

$$a=\frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}=\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$b=\frac{\lambda_1\lambda_2^n-\lambda_2\lambda_1^n}{\lambda_1-\lambda_2}=\frac{-1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right)$$

由Cayley-Hamilton定理可知 $f(A)=O$

$$\therefore\ A^n=aA+bI=a\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}+b\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}a+b & a\\ a & b\end{bmatrix}$$

１６Ｂ**06** 【 清大76資科[6] 】

Let $A=\begin{bmatrix}2 & 1\\ -1 & 4\end{bmatrix}$

(1) Find an invertible matrix P such that $P^{-1}AP$ is a triangular matrix. (10%)

(2) Compute A^n for some positive integer n. (10%)

【解】 (1) $\det(A-xI)=x^2-6x+9=(x-3)^2$

解$(A-3I)v=o$, 得3的eigenvectors $t[1, 1]^T$, $t\neq 0$

$$A\begin{bmatrix}1\\1\end{bmatrix}=3\begin{bmatrix}1\\1\end{bmatrix}$$

$$A\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}2&1\\-1&4\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\4\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}+3\begin{bmatrix}0\\1\end{bmatrix}$$

let $P=\begin{bmatrix}1&0\\1&1\end{bmatrix}$, then $P^{-1}AP=\begin{bmatrix}3&1\\0&3\end{bmatrix}$. (綜線CH7定理19②)

(2) $$\begin{bmatrix}3&1\\0&3\end{bmatrix}^n=\left(3\begin{bmatrix}1&0\\0&1\end{bmatrix}+\begin{bmatrix}0&1\\0&0\end{bmatrix}\right)^n$$

$$=3^n\begin{bmatrix}1&0\\0&1\end{bmatrix}^n+n\cdot 3^{n-1}\begin{bmatrix}1&0\\0&1\end{bmatrix}^{n-1}\begin{bmatrix}0&1\\0&0\end{bmatrix}$$ (二項式定理)

$$=\begin{bmatrix}3^n&n3^{n-1}\\0&3^n\end{bmatrix}$$

$$\therefore A^n=\left(P\begin{bmatrix}3&1\\0&3\end{bmatrix}P^{-1}\right)^n=P\begin{bmatrix}3&1\\0&3\end{bmatrix}^nP^{-1}$$

$$=\begin{bmatrix}1&0\\1&1\end{bmatrix}\begin{bmatrix}3^n&n3^{n-1}\\0&3^n\end{bmatrix}\begin{bmatrix}1&0\\-1&1\end{bmatrix}=\begin{bmatrix}3^n-n3^{n-1}&n3^{n-1}\\-n3^{n-1}&3^n+n3^{n-1}\end{bmatrix}$$

【加強演練】

設$A=\begin{bmatrix}2&1&0\\0&2&-1\\0&0&2\end{bmatrix}$, 求$A^n$.　　　Ans: $\begin{bmatrix}2^n&n2^{n-1}&-n(n-1)2^{n-3}\\0&2^n&-n2^{n-1}\\0&0&2^n\end{bmatrix}$

１６B**07** 【 清大85資料[2] 】

Let A be a matrix defined below.

$$A=\begin{bmatrix}-3&1&0\\1&-3&0\\0&0&3\end{bmatrix}$$

(a) Find the eigenvalues and corresponding eigenvectors of matrix A^5 .

(b) Find the eigenvalues and corresponding eigenvectors of matrix A^{-1} .

(c) Evaluate e^A , where e^A is defined as $\sum_{k=0}^{\infty}\frac{A^k}{k!}$.

【註】本題(a)(b)屬於題型16A.

【解】先求A的eigenvalues及eigenvectors:

$\det(A-xI)=(3-x)(x+2)(x+4)$　　　∴ A的eigenvalues爲 3, –2, –4 .

解$(A-3I)v=o$, 得3的eigenvectors $t[0, 0, 1]^T$, $t\neq 0$

解$(A+2I)v=o$, 得–2的eigenvectors $t[1, 1, 0]^T$, $t\neq 0$

解$(A+4I)v=o$, 得–4的eigenvectors $t[-1, 1, 0]^T$, $t\neq 0$

(a) A^5 的eigenvalues爲 3^5 , $(-2)^5$, $(-4)^5$.

3^5的eigenvectors $t[0, 0, 1]^T$, $t\neq 0$

$(-2)^5$的eigenvectors $t[1, 1, 0]^T$, $t\neq 0$

$(-4)^5$的eigenvectors $t[-1, 1, 0]^T$, $t\neq 0$

(b) A^{-1} 的eigenvalues爲 1/3 , –1/2 , –1/4.

1/3的eigenvectors $t[0, 0, 1]^T$, $t\neq 0$

–1/2的eigenvectors $t[1, 1, 0]^T$, $t\neq 0$

–1/4的eigenvectors $t[-1, 1, 0]^T$, $t\neq 0$

(c)

$$A=\begin{bmatrix}0&1&-1\\0&1&1\\1&0&0\end{bmatrix}\begin{bmatrix}3&0&0\\0&-2&0\\0&0&-4\end{bmatrix}\begin{bmatrix}0&1&-1\\0&1&1\\1&0&0\end{bmatrix}^{-1}$$

$$e^{A}=\begin{bmatrix}0&1&-1\\0&1&1\\1&0&0\end{bmatrix}\begin{bmatrix}e^3&0&0\\0&e^{-2}&0\\0&0&e^{-4}\end{bmatrix}\begin{bmatrix}0&1&-1\\0&1&1\\1&0&0\end{bmatrix}^{-1}$$

$$=(1/2)\begin{bmatrix}e^{-2}+e^{-4}&e^{-2}-e^{-4}&0\\e^{-2}-e^{-4}&e^{-2}+e^{-4}&0\\0&0&2e^3\end{bmatrix}$$

16B08 【 交大83工工[9] 】

Let $A=\begin{bmatrix}-2&-6\\1&3\end{bmatrix}$

(a) Compute e^A . (5%)

(b) Find the eigenvalues and the corresponding eigenvectors of e^A . (8%)

(c) Compute A^{100} . (4%)

【解】先對A作對角化(細節略, 解法參閱題型12C):

$\det(A-I)=x(x-1)$.

解$(A-I)v=o$, 得1的eigenvectors $t[-2, 1]^T$, $t\neq 0$

解$(A-0I)v=o$, 得0的eigenvectors $t[-3, 1]^T$, $t\neq 0$

令$P=\begin{bmatrix} -2 & -3 \\ 1 & 1 \end{bmatrix}$, 則 $A=P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1}$.

(a)

$$e^A=P\begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix}P^{-1}=\ldots=\begin{bmatrix} -2e+3 & -6e+6 \\ e-1 & 3e-2 \end{bmatrix}$$

(b) e^A 的 eigenvalues爲 $e, 1$.

得e的eigenvectors $t[-2, 1]^T$, $t\neq 0$

得1的eigenvectors $t[-3, 1]^T$, $t\neq 0$

(c)

$$A^{100}=P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}^{100}P^{-1}=P\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}P^{-1}=A.$$

16B09【交大85工工[4]】

Let $A=\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$, compute e^A .

【解】

$$A^2=\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}=\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$e^A=\sum_{k=0}^{\infty}\frac{1}{k!}A^k=I+A+O+O+\ldots.$$

$$=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}+\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

１６Ｂ**１０**【 中正79資工[5] 】

Let $A=\begin{bmatrix}3 & 2\\ 2 & 0\end{bmatrix}$. Compute e^A .

【解】1°先做對角化: (過程略)

$$令P=\begin{bmatrix}2 & 1\\ 1 & -2\end{bmatrix},\ 則P^{-1}AP=D=\begin{bmatrix}4 & 0\\ 0 & -1\end{bmatrix}$$

2° $\exp(A)=\exp(PDP^{-1})=P\exp(D)P^{-1}$

$$=\begin{bmatrix}2 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}e^4 & 0\\ 0 & e^{-1}\end{bmatrix}\begin{bmatrix}2 & 1\\ 1 & -2\end{bmatrix}^{-1}=\ldots=(1/5)\begin{bmatrix}4e^4+e^{-1} & 2e^4-2e^{-1}\\ 2e^4-2e^{-1} & e^4+4e^{-1}\end{bmatrix}$$

１６Ｂ**１１**【 元智81工工[5] 】

$$令\quad A=\begin{pmatrix}3 & -3 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}$$

(1) 試求一 4×4 matrix P, 使 $P^{-1}AP=J$, a Jordan matrix.

(2) 並求e^A , 用矩陣表之.

【參考章節】綜線CH16範例9.

【解】(1) $\det(A-xI)=x^2(3-x)^2$

解 $(A-0I)v=o$ 得 $\mathrm{Ker}A=\{(0,0,0,t)^T \mid t\in\mathbb{R}\}$

解 $(A-0I)^2v=o$ 得 $\mathrm{Ker}(A^2)=\{(t,t,0,u)^T \mid t,u\in\mathbb{R}\}$

解 $(A-3I)v=o$ 得 $\mathrm{Ker}(A-3I)=\{(t,0,0,0)^T \mid t\in\mathbb{R}\}$

解 $(A-3I)^2v=o$ 得 $\mathrm{Ker}((A-3I)^2)=\{(t,0,u,0)^T \mid t,u\in\mathbb{R}\}$

$v_1=(1,1,0,0)^T\in \mathrm{Ker}(A^2)\backslash\mathrm{Ker}(A)$

$v_2 = Av_1 = [0, 0, 0, 1]^T$

$v_3 = (0, 0, 1, 0)^T \in \text{Ker}((A-3I)^2) \backslash \text{Ker}(A-I)$

$v_4 = (A-3I)v_3 = [1, 0, 0, 0]^T$.

令 $P = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$，則 $P^{-1}AP = J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{pmatrix}$

(2)

$$e^A v_1 = \sum_{n=0}^{\infty} \frac{1}{n!} A^n v_1 = v_1 + Av_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$e^A v_2 = \sum_{n=0}^{\infty} \frac{1}{n!} A^n v_2 = v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$e^A v_3 = e^3 e^{A-3I} v_3 = e^3 \sum_{n=0}^{\infty} \frac{1}{n!} (A-3I)^n v_3 = e^3 (v_3 + (A-3I)v_3)$$

$$= e^3 \left(\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right) = e^3 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} e^3 \\ 0 \\ e^3 \\ 0 \end{pmatrix}$$

$$e^A v_4 = e^3 e^{A-3I} v_4 = e^3 \sum_{n=0}^{\infty} \frac{1}{n!} (A-3I)^n v_4 = e^3 v_4 = e^3 \begin{pmatrix} e^3 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore\ e^A P=\begin{pmatrix}1&0&e^3&e^3\\1&0&0&0\\0&0&e^3&0\\1&1&0&0\end{pmatrix}$$ (綜線CH2定理6)

$$\therefore\ e^A=\begin{pmatrix}1&0&e^3&e^3\\1&0&0&0\\0&0&e^3&0\\1&1&0&0\end{pmatrix}P^{-1}=\ \dots\ =\begin{pmatrix}e^3&1-e^3&e^3&0\\0&1&0&0\\0&0&e^3&0\\0&1&0&1\end{pmatrix}$$

１６B**１２**【 交大82工工[8] 】

Compute e^A using the following matrices.

(a). $A=\begin{bmatrix}1&1\\-1&-1\end{bmatrix}$ (b). $A=\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}$ (c). $A=\begin{bmatrix}-2&-6\\1&3\end{bmatrix}$

【解】(a) $\det(A-xI)=x^2$

$$A^2=\begin{bmatrix}1&1\\-1&-1\end{bmatrix}\begin{bmatrix}1&1\\-1&-1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

$$e^A=\sum_{n=0}^{\infty}\frac{1}{n!}A^n=\sum_{n=0}^{1}\frac{1}{n!}A^n=I+A=\begin{bmatrix}2&1\\-1&0\end{bmatrix}$$

(b) $\det(A-xI)=(1-x)^3$

$$(A-I)=\begin{bmatrix}0&0&-1\\0&0&0\\0&0&0\end{bmatrix},\quad (A-I)^2=O$$

$$e^{A-I} = \sum_{n=0}^{\infty} \frac{1}{n!}(A-I)^n = \sum_{n=0}^{1} \frac{1}{n!}(A-I)^n = I+(A-I)=A$$

$$\therefore\ e^{A}=e^{1}\,e^{(A-I)}=eA=\begin{bmatrix} e & 0 & -e \\ 0 & e & 0 \\ 0 & 0 & e \end{bmatrix}$$

(c) $\det(A-xI)=x^2-x=x(x-1)$

解 $(A-0I)v=o$ 得 0的eigenvector $[-3, 1]^T$

解 $(A-I)v=o$ 得 1的eigenvector $[-2, 1]^T$

令 $P=\begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix}$, 則 $PAP^{-1}=D=\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\therefore\ e^{A}=e^{PDP^{-1}}=P\,e^{D}\,P^{-1}$$

$$=\begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}\begin{bmatrix} -3 & -2 \\ 1 & 1 \end{bmatrix}^{-1}=\begin{bmatrix} 3-2e & 6-6e \\ -1+e & -2+3e \end{bmatrix}$$

１６B**13** 【 大同83資工[7] 】

Let $A\in M_{3\times3}(\mathbb{R})$ be defined by $A=\begin{bmatrix} 3 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 0 \end{bmatrix}$. Find $A^{1/3}$.

【分析】$A^{1/3}$ 是A的立方根, 任何X滿足$X^3=A$都合乎要求.

【解】$\det(A-xI)=-(x-4)(x-2)(x+1)$

解 $(A-4I)v=o$ 得 4的eigenvector $[-2, 0, 1]^T$

解 $(A-2I)v=o$ 得 4的eigenvector $[0, 1, 0]^T$

解 $(A+I)v=o$ 得 -1的eigenvector $[1, 0, 2]^T$

令$P=\begin{bmatrix}-2&0&1\\0&1&0\\1&0&2\end{bmatrix}$, $D=\begin{bmatrix}4&0&0\\0&2&0\\0&0&-1\end{bmatrix}$. 則 $A=PDP^{-1}$.

$\therefore\ A^{1/3}=PD^{1/3}P^{-1}$

$$=\begin{bmatrix}-2&0&1\\0&1&0\\1&0&2\end{bmatrix}\begin{bmatrix}4^{1/3}&0&0\\0&2^{1/3}&0\\0&0&-1\end{bmatrix}\begin{bmatrix}-2&0&1\\0&1&0\\1&0&2\end{bmatrix}^{-1}$$

$=$..... (讀者自行化簡)

【討論】以上是限在實數系作答, 若要考慮複數, 則非常繁瑣:

令$\omega=\dfrac{-1+\sqrt{3}\ i}{2}$, 則$\omega^2=\dfrac{-1+\sqrt{3}\ i}{2}$.

4的立方根有$4^{1/3}$, $4^{1/3}\omega$, $4^{1/3}\omega^2$, 2的立方根有$2^{1/3}$, $2^{1/3}\omega$, $2^{1/3}\omega^2$,
-1的立方根有-1, $-\omega$, $-\omega^2$.

D的立方根爲 $\begin{bmatrix}4^{1/3}\omega^{i}&0&0\\0&2^{1/3}\omega^{j}&0\\0&0&-\omega^{k}\end{bmatrix}$, $i, j, k\in\{0,1,2\}$

共有27個解. 於是A的立方根$PD^{1/3}P^{-1}$ 也有27個解.

題型16C: Cayley-Hamilton定理

１６Ｃ**０１**【 台大86資工[2] 】

$$A=\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 3 \end{pmatrix}$$

(a) $f(x)=x^3-4x^2-x+4$, 求 $f(A)$.

(b) 求A^{10}.

【解】(a) $\det(A-xI)=-(x-1)(x-4)(x+1)=x^3-4x^2-x+4=f(x)$

$\therefore\ f(A)=O$ (Cayley-Hamilton定理)

(b) 令 $x^{10}=q(x)f(x)+ax^2+bx+c$

則 $1^{10}=a+b+c,\quad (-1)^{10}=a-b+c,\quad 4^{10}=16a+4b+c$

解得 $a=(4^{10}-1)/15,\quad b=0,\quad c=(16-4^{10})/15$

$\therefore\ A^{10}=aA^2+bA+cI=aA^2+cI$

$$=a\begin{bmatrix} 1 & 1 & 2 \\ 0 & 4 & 6 \\ 0 & 6 & 13 \end{bmatrix}+c\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$=\frac{1}{15}\begin{bmatrix} 15 & M-1 & 2M-2 \\ 0 & 3M+12 & 6M-6 \\ 0 & 6M-6 & 12M+3 \end{bmatrix},\quad \text{其中}M=4^{10}=1048576$$

$$=\begin{bmatrix} 1 & 69905 & 139810 \\ 0 & 209716 & 419430 \\ 0 & 419430 & 838861 \end{bmatrix}$$

#

１６Ｃ**02** 【 清大86資料[10](b) 】

(b) Find $A^4-5A^3-A^2$, where $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$. (5%)

【解】(b) $\det(A-xI)=x^2-5x-2$

$\therefore\ A^2-5A-2I=O$ (Cayley-Hamilton定理)

$\therefore\ A^4-5A^3-A^2=A^2(A^2-5A-2I)+A^2=A^2$

$$=\ \dots\ =\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}\ .$$

１６Ｃ**03** 【 中央86資工[2] 】

Calculate A^{21} , where $A=\begin{bmatrix}7 & 2\\ -4 & 1\end{bmatrix}$

【解】 $ch(x)=\det(A-xI)=x^2-8x+15=(x-3)(x-5)$

令 $x^{21}=q(x)ch(x)+ax+b$,

以3, 5分別代入得 $3^{21}=3a+b,\quad 5^{21}=5a+b$

解得 $a=(5^{21}-3^{21})/2$, $b=(5\cdot 3^{21}-3\cdot 5^{21})/2$

$\therefore\ A^{21}=q(A)\cdot O+aA+bI$ (Cayley-Hamilton定理)

$$=\frac{5^{21}-3^{21}}{2}\begin{bmatrix}7 & 2\\ -4 & 1\end{bmatrix}+\frac{5\cdot 3^{21}-3\cdot 5^{21}}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}4\cdot 5^{21}-2\cdot 3^{21} & 2\cdot 5^{21}-2\cdot 3^{21}\\ -4\cdot 5^{21}+4\cdot 3^{21} & -2\cdot 5^{21}+4\cdot 3^{21}\end{bmatrix}$$

１６Ｃ**04** 【 中原86工工[3] 】

Let $A=\begin{bmatrix}1 & -2\\ 4 & 5\end{bmatrix}$, Find $f(A)$ where:

(a) $f(t)=t^2-3t+7$. Is A a root of $f(t)$?

(b) $f(t)=t^2-6t+13$. Is A a root of $f(t)$?

【解】(a) 特徵多項式 $\det(A-tI)=t^2-6t+13$

$\therefore\ A^2-A+13I=O$ (Cayley-Hamilton定理)

$f(A)=A^2-3A+7I=(A^2-6A+13I)+3A-6I$

$=3A-6I=\begin{bmatrix}-3 & -6\\ 12 & 9\end{bmatrix}$

A不是 $f(t)$的root.

(b) $f(A)=A^2-6A+13I=O$

A是$f(t)$的root.

１６Ｃ**05** 【 元智85電資[3]& 】

Find A^5 if $A=\begin{bmatrix}4 & -2 & 1\\ 2 & 0 & 1\\ 2 & -2 & 3\end{bmatrix}$

【解】 $\text{ch}(x)=\det(A-xI)$

$$=\begin{vmatrix}4-x & -2 & 1\\ 2 & -x & 1\\ 2 & -2 & 3-x\end{vmatrix}=\begin{vmatrix}3-x & -2 & 1\\ 3-x & -x & 1\\ 3-x & -2 & 3-x\end{vmatrix}=(3-x)\begin{vmatrix}1 & -2 & 1\\ 1 & -x & 1\\ 1 & -2 & 3-x\end{vmatrix}$$

$=\ \ldots=(3-x)(2-x)^2$

令 $x^5=q(x)(3-x)(2-x)^2+ax^2+bx+c$

以$x=3,2$分別代入得　$243=9a+3b+c$，　$32=4a+2b+c$

前式微分得 $5x^4=q_1(x)(2-x)+2ax+b$

以$x=2$代入得　$80=4a+b$

由前面三個a,b,c的方程式解得$a=131$, $b=-444$, $c=396$

$\therefore\ A^5=q(A)\cdot O+131A^2-444A+396I$

$$=131\begin{bmatrix}14 & -10 & 5\\ 10 & -6 & 5\\ 10 & -10 & 9\end{bmatrix}-444\begin{bmatrix}4 & -2 & 1\\ 2 & 0 & 1\\ 2 & -2 & 3\end{bmatrix}+396\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}454 & -422 & 211\\ 422 & -390 & 211\\ 422 & -422 & 243\end{bmatrix}$$

【另解】本題可對角化又有重根，利用最小多項式代替特徵多項式可簡化計算.
請參閱題型16D自解.

【另解】也可先對A做對角化再求算，請參閱題型16B

１６C**06** 【 雲技85工工[8] 】

若三階(3×3)方陣的特徵多項式(characteristic polynomial)為

$P(\lambda)=-\lambda^3+\lambda^2+14\lambda-24$,

(a) 試問A是否可對角化(diagonalizable)？理由為何 ？

(b) 試求 $(A^3-A^2-14A)^{-1}$ 的值.

【解】(a) 分解因式: $P(\lambda)=-(\lambda-2)(\lambda-3)(\lambda+4)$

$\because$ 特徵多項式分解為相異一次因式的乘積

$\therefore$ 可對角化.　　(綜線CH12定理23)

(b) $O=P(A)=-A^3+A^2+14A-24I$　　(綜線CH16定理18)

$\therefore\ (A^3-A^2-14A)=-24I$

$\therefore\ (A^3-A^2-14A)^{-1}=(-24I)^{-1}=(-1/24)I$.

１６Ｃ07 【 元智83工工[15] 】

Let $A=\begin{bmatrix}2 & 0 & -1\\ 1 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}$. Compute A^{100}.

【解】$\mathrm{ch}(x)=\det(A-xI)=(1-x)^2(2-x)$

令 $x^{100}=q(x)(1-x)^2(2-x)+ax^2+bx+c$ ....(A)

分別以$x=1, 2$ 代入，得

$1=a+b+c$, ...(B)

$2^{100}=4a+2b+c$ . ...(C)

(A)式微分得 $100x^{99}=q_1(x)(1-x)+2ax+b$

以$x=1$代入，得

$100=2a+b$. ....(D)

由(B)(C)(D)解得 $a=-101+2^{100}$, $b=302-2^{101}$, $c=2^{100}-200$.

(A)式以A代入，得

$A^{100}=q(A)\mathrm{ch}(A)+aA^2+bA+cI=aA^2+bA+cI$

$$=a\begin{bmatrix}4 & 0 & -3\\ 3 & 1 & -3\\ 0 & 0 & 1\end{bmatrix}+b\begin{bmatrix}2 & 0 & -1\\ 1 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}+c\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}4a+2b+c & 0 & -3a-b\\ 3a+b & a+b+c & -3a-b\\ 0 & 0 & a+b+c\end{bmatrix}$$

$$= \begin{bmatrix} 2^{100} & 0 & 1-2^{100} \\ -1+2^{100} & 1 & 1-2^{100} \\ 0 & 0 & 1 \end{bmatrix}$$

#

【另解】本題可對角化又有重根，利用最小多項式代替特徵多項式可簡化計算.

請參閱題型16D自解.

【另解】也可先對A做對角化再求算，請參閱題型16B

$$\text{令} \quad P=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}, D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{則} \ A=PDP^{-1}.$$

$$A^{100}=PD^{100}P^{-1}=P\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^{100} \end{bmatrix}P^{-1}=.... \ =\text{答案同上} \ .$$

１６C**08** 【 中正83資工[3] 】

Let $p(\lambda)$ be the characteristic polynomial of an $n\times n$ matrix A and $q(\lambda)$ be a second polynomial.

(a) Show that if $Ax_1=\lambda_1 x_1$, then $q(A)x_1=q(\lambda_1)x_1$. (5%)

(b) Show that if $p(\lambda)$ and $q(\lambda)$ are relatively prime (i.e., have no common noconstant factor), then $q(A)$ is nonsingular. (5%)

(c) Show that if λ_1 is a common zero of $p(\lambda)$ and $q(\lambda)$, then $q(A)$ is singular. (5%)

【解】(a) $A^{k+1}x_1=A^kAx_1=A^k\lambda_1 x_1=\lambda_1 A^k x_1$

而$Ax_1=\lambda_1 x_1$, ∴ 依上式歸納得知$\forall m=1,2,...,$ $A^m x_1=\lambda_1^m x_1$

令$q(\lambda)=a_m\lambda^m+a_{m-1}\lambda^{m-1}+...+a_1\lambda+a_0$

$$q(A)x_1=a_mA^mx_1+a_{m-1}A^{m-1}x_1+ \ ... \ +a_1Ax_1+a_0Ix_1$$
$$=a_m\lambda_1^m x_1+a_{m-1}\lambda_1^{m-1}x_1+ \ ... \ +a_1\lambda x_1+a_0x_1$$
$$=q(\lambda_1)x_1$$

(b) ∵ $p(\lambda)$與$q(\lambda)$互質,

∴ ∃ 多項式$r(\lambda)$, $s(\lambda)$使得 $r(\lambda)p(\lambda)+s(\lambda)q(\lambda)=1$ (高中數學定理)

∴ $r(A)p(A)+s(A)q(A)=I$ (綜線CH2定理16)

而 $p(A)=O$, (綜線CH16定理18:Caylay-Hamilton)

∴ $s(A)q(A)=I$

∴ $q(A)$可逆. (綜線CH3定理19)

(c) λ_1為特徵多項式$p(\lambda)$的根, ∴ λ_1為A的特徵值.

∴ $\exists x_1 \neq o$, 使得$Ax_1=\lambda_1 x_1$.

$q(A)x_1=q(\lambda_1)x_1=0x_1=o$. (由(a))

∴ $\ker(q(A))\neq\{o\}$.

∴ $q(A)$不可逆. (綜線CH8定理17)

１６Ｃ**09** 【 中正81資工[3] 】

(a) Describe the Cayley-Hamilton theorem.

(b) Let $A=\begin{bmatrix}1&1&0\\0&2&1\\0&0&3\end{bmatrix}$. Compute $A^8+5A^6+3A^2+4A$ by the use of the Cayley-Hamilton theorem.

【解】(a) Cayley-Hamilton 定理: (綜線CH16定理18)

對任意矩陣A, 設 $\mathrm{ch}(x)=\det(A-xI)$, 則 $\mathrm{ch}(A)=O$

(b) $\det(A-xI)=(1-x)(2-x)(3-x) =-(x^3-6x^2+11x-6)$

令$x^8+5x^6+3x^2+4x=q(x)(x^3-6x^2+11x-6)+ax^2+bx+c$.

分別以$x=1, 2, 3$代入上式得

$a+b+c=13$, $4a+2b+c=596$, $9a+3b+c=10245$.

可解出 $a=4533$, $b=-13016$, $c=8496$.

(註 : 餘式也可利用長除法求算, 商式為 $x^5+6x^4+30x^3+120x^2+426x+1416$)

∴ $A^8+5A^6+3A^2+4A$

$= q(A)\cdot O + aA^2 + bA + cI$ (綜線CH16定理18)

$$=4533\begin{bmatrix}1&3&1\\0&4&5\\0&0&9\end{bmatrix}-13016\begin{bmatrix}1&1&0\\0&2&1\\0&0&3\end{bmatrix}+8496\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}13&583&4533\\0&596&9649\\0&0&10250\end{bmatrix}$$

１６Ｃ**１０** 【 中正80資工[1] 】

(a) Describe the Cayley-Hamilton theorem. (5%)

(b) Let $A=\begin{bmatrix}1&-1&4\\3&2&-1\\2&1&-1\end{bmatrix}$. Use the Cayley-Hamilton theorem to compute

$A^6-2A^5-5A^4+3A^3+6A^2+15A-18I$ and A^{-1}, where I is the 3×3 identity matrix. (5%)

【解】(a) 同上題.

(b) 令$ch(x)=\det(A-xI)$, 則

$$ch(x)=\begin{vmatrix}1-x&-1&4\\3&2-x&-1\\2&1&-1-x\end{vmatrix}=....=-x^3+2x^2+5x-6$$

令 $f(x)=x^6-2x^5-5x^4+3x^3+6x^2+15x-18$

以長除法計算:

$$\begin{array}{r|l}
1\ -2\ -5+3+6+15-18 & -1+2+5-6 \\
1\ -2\ -5+6 & \hline \\
\hline
 & -1+0+0+3 \\
-3+6+15-18 & \\
-3+6+15-18 & \\
\hline
0 &
\end{array}$$

$\therefore\ f(x)=q(x)\text{ch}(x)$

$\therefore\ f(A)=q(A)\text{ch}(A)$ (綜線CH2定理16)

$=q(A)\cdot O$ (綜線CH16定理18)

$=O$

即 $A^6-2A^5-5A^4+3A^3+6A^2+15A-18I=O$.

另由 Cayley-Hamilton 定理得 $-A^3+2A^2+5A-6I=O$

$\therefore\ 6I=-A^3+2A^2+5A$

$\therefore\ A^{-1}=\dfrac{1}{6}(-A^2+2A+5I)$ (兩邊乘$(1/6)A^{-1}$)

$$=\frac{1}{6}\left(-\begin{bmatrix}6&1&1\\7&0&11\\3&-1&8\end{bmatrix}+\begin{bmatrix}2&-2&8\\6&4&-2\\4&2&-2\end{bmatrix}+\begin{bmatrix}5&0&0\\0&5&0\\0&0&5\end{bmatrix}\right)$$

$$=\frac{1}{6}\begin{bmatrix}1&-3&7\\-1&9&-13\\1&3&-5\end{bmatrix}\quad\#$$

16C11【中正79資工[4]】

Let $A=\begin{bmatrix}1&0&2\\0&-1&1\\0&1&0\end{bmatrix}$.Compute $A^6-5A^5-4A^4+3A^2-2A+I$, where I is the identity matrix.

【解】ch$(x)=\det(A-xI)=....=-x^3+2x-1$

令$f(x)=x^6-5x^5-4x^4+3x^2-2x+1$,

由長除法:

1	–5	–4	+0	+3	–2	+1	–1 +0 +2 –1
1	+0	–2	+1				–1 +5 +2 +11
	–5	–2	–1	+3			
	–5	+0	+10	–5			
		–2	–11	+8	–2		
		–2	+0	+4	–2		
			–11	+4	+0	+1	
			–11	+0	+22	–11	
				4	–22	+12	

$\therefore\ f(x)=q(x)\text{ch}(x)+(4x^2-22x+12)$

$\therefore\ f(A)=q(A)\text{ch}(A)+(4A^2-22A+12I)=4A^2-22A+12I$

$$=4\begin{bmatrix}1&2&2\\0&2&-1\\0&-1&1\end{bmatrix}-22\begin{bmatrix}1&0&2\\0&-1&1\\0&1&0\end{bmatrix}+12\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-6&8&-36\\0&42&-26\\0&-26&16\end{bmatrix}$$

16C**12** 【 清大79資科[1](1) 】

For each of the following statements,explain whether it is true or false

(1) If A is a square matrix satisfying $A^5+A^2+A+I=O$, where I is the identity matrix, A is invertible.

【解】(1) True. 證明如下:

由 $A^5+A^2+A+I=O$ 移項得 $I=-A^5-A^2-A$.

分別由左右提出A, 得 $I=A(-A^4-A-I)$ 及 $I=(-A^4-A-I)A$.

$\therefore$ A可逆. (綜線CH2定義10③)

１６Ｃ13 【 台大82資工[4] 】

If $A=\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}\in M_{2\times 2}(\mathbb{R})$, find A^n for any position integer n.

【勘誤】題中position爲筆誤或打字錯誤，應更正爲positive.

【解】 A的特徵多項式ch$(x)=\det(A-xI)=x^2-4x-5=(x-5)(x+1)$

令 $x^n=q(x)\text{ch}(x)+ax+b$.

分別以$x=5, -1$代入，得 $5^n=5a+b$, $(-1)^n=-a+b$.

可解得 $a=\dfrac{1}{6}(5^n-(-1))^n$, $b=\dfrac{1}{6}(5^n+\ 5\cdot(-1)^n)$.

$A^n=q(A)\text{ch}(A)+aA+bI=q(A)\cdot O+aA+bI$

$$=\frac{5^n-(-1)^n}{6}\begin{bmatrix}1 & 4\\ 2 & 3\end{bmatrix}+\frac{5^n+5(-1)^n}{6}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\frac{1}{3}\begin{bmatrix}5^n+2(-1)^n & 2\cdot 5^n-2(-1)^n\\ 5^n-(-1)^n & 2\cdot 5^n+(-1)^n\end{bmatrix}$$

#

１６Ｃ14 【 元智82工工[6] 】

Evaluate $A^{99}+3A^{31}+A$ for $A=\begin{bmatrix}0 & 3\\ 1 & 2\end{bmatrix}$.

【解】 $\det(A-xI)=x^2-2x-3=(x-3)(x+1)$

令 $f(x)=x^{99}+3x^{31}+x=q(x)(x-3)(x+1)+ax+b$

分別以 $x=3, 1$ 代入上式得 $f(3)=3a+b, 5=a+b$

解得 $a=(f(3)-5)/2$, $b=(15-f(3))/2$

由 Cayley-Hamilton定理, (綜線CH16定理18)

$$f(A)=q(A)\cdot O+aA+bI=aA+bI$$

$$=a\begin{bmatrix}0&3\\1&2\end{bmatrix}+b\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}b&3a\\a&2a+b\end{bmatrix}$$

$$=\ ....\ =\frac{1}{2}\begin{bmatrix}15-f(3)&3f(3)-15\\f(3)-5&f(3)+5\end{bmatrix}$$

其中 $f(3)=3^{99}+3\cdot3^{31}+3=3^{99}+3^{32}+3$.

１６C１５ 【 台大80資工[9] 】

[True or False Problem]

Suppose $T: F^3 \longrightarrow F^3$ satisfying $T(x_1, x_2, x_3)=(3x_1, x_1-x_2, 2x_1+x_2+x_3)$.

Then $(T^4-3T^3-T^2+3T)=O$ (zero transformation).

【解】 True, 證明如下:

$$\begin{cases}T(1,0,0)=(3,1,2)=3(1,0,0)+1(0,1,0)+2(0,0,1)\\ T(0,1,0)=(0,-1,1)=\quad -1(0,1,0)+1(0,0,1)\\ T(0,0,1)=(0,0,1)=\quad 1(0,0,1)\end{cases}$$

$$\therefore\ [T]=\begin{bmatrix}3&0&0\\1&-1&0\\2&1&1\end{bmatrix}$$ (綜線CH7定義9)

$\det(T-xI)=-(x-3)(x+1)(x-1)=-(x^3-3x^2-x+3)$

$\therefore\ T^3-3T^2-T+3I=O$ (綜線CH16定理18)

$\therefore\ T^4-3T^2-T+3T=T\circ(T^3-3T^2-T+3I)$ (綜線CH8定理36)

$\qquad =T\circ O=O$

１６C１６ 【 清大78工工[5](i) 】

If there exists a nonsingular matrix P such that $P^{-1}AP=B$, then B is said to be similar to A.

Then,

(ⅰ) Suppose that f is a polynomial. Prove that if A is similar to a diagonal matrix and f is the characteristic polynomial of A, then $f(A)=O$. (10%)

【分析】Cayley-Hamilton定理的證明相當複雜, 所以考到證明時常常會添加條件以降低難度. 例如台大75年資訊系考的是3x3矩陣的Cayley-Hamilton定理. 而這一題是加上 "A可對角化" 的條件.

【解】

設存在可逆矩陣P使得$A=P^{-1}DP$, 其中$D=\begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$,

則對任意正整數k,

$A^k=(P^{-1}DP)(P^{-1}DP)\ ...\ (P^{-1}DP)=P^{-1}D^kP$

$f(x)=\det(A-xI)=\det(P^{-1}DP-xP^{-1}IP)=\det(P^{-1}(D-xI)P)=\det(D-xI)$

$$=\begin{vmatrix} d_1-x & & & 0 \\ & d_2-x & & \\ & & \ddots & \\ 0 & & & d_n-x \end{vmatrix}=(d_1-x)(d_2-x)\cdots(d_n-x)$$

$\therefore\ f(d_i)=0,\quad i=1, 2, ..., n$.

令 $f(x)=\sum_{i=0}^{n}a_ix^i$, 則 $f(A)=f(P^{-1}DP)=P^{-1}(\sum_{i=0}^{n}a_iD^i)P$ (綜線CH6定理2)

$$=P^{-1}\left(\sum_{i=0}^{n}a_i\begin{bmatrix} d_1^i & & & 0 \\ & d_2^i & & \\ & & \ddots & \\ 0 & & & d_n^i \end{bmatrix}\right)P=P^{-1}\begin{bmatrix} f(d_1) & & & 0 \\ & f(d_2) & & \\ & & \ddots & \\ 0 & & & f(d_n) \end{bmatrix}P$$

$= P^{-1}OP = O$

１６Ｃ**１７** 【 台大75資工[12] 】

Let A be a 3×3 matrix and $p_A(x)$ the characteristic polynomial of A, show that $p_A(A) = O$

【解】請參閱綜線CH16範例18a.

１６Ｃ**１８** 【 台大80資工[7] 】

[True or False Problem]

Given $A \in M_{n\times n}(F)$ and a smooth function $f(t)$ which has Taylor series expansion, then there always exists a polynomial $g(t)$ with degree at most $(n-1)$ such that $f(A) = g(A)$.

【分析】(1) 對smooth function $f(t) = \sum_{k=0}^{\infty} a_n t^n$，可定義$f(A) = \sum_{k=0}^{\infty} a_n A^n$.

但這個矩陣級數在$M_{n\times n}(F)$中未必收斂.

(2) 設$W = \text{gen}\{I, A, A^2, A^3, ...\}$.

即使(1)中的矩陣級數收斂，也並不表示$f(A) \in W$.

因爲W是由"<u>有限多個</u>A的乘冪的線性組合"所組成. (綜線CH6定義1要訣1).

要記得無限級數並不是線性組合.

【解】False. 反例如下:

取$F = \mathbb{Q}$(有理數系), $n = 2$, $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$,

則 $f(A) = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} e^2 & 0 \\ 0 & e \end{bmatrix} \notin M_{n\times n}(\mathbb{Q})$

對任意$n-1$次多項式$g(t) = a + bt$, $a, b \in F$,

$g(A) = aI + bA \in M_{n\times n}(F)$

$\therefore\ g(A) \neq f(A)$

【討論】若題目將F改爲$\mathbb{R}$，則本題答案爲True.

證明須用到一些topology的知識:

令$f(t)=\lim\limits_{m\to\infty} s(t)$，其中$s_m(t)=\sum\limits_{k=0}^{m}\dfrac{f^{(k)}(0)}{k!}t^k$.

則$f(A)=\lim\limits_{m\to\infty} s_m(A)$ (定義)

$ch(t)=\det(A-tI)$為n次多項式. (綜線CH12定義8要訣(6))

以$s_m(t)$除以$ch(t)$，可得:

$s_m(t)=q_m(t)ch(t)+r_m(t)$，其中$\deg r_m(t)\le n-1$，或$r_m(t)=0$

由Cayley-Hamilton定理:

$s_m(A)=q_m(A)O+r_m(A)=r_m(A)$

$\therefore\ f(A)=\lim\limits_{m\to\infty} r_m(A)$

$\therefore$ 矩陣序列$\langle r_m(A)\rangle$是Cauchy sequence.

令$W=\text{gen}\{I, A, A^2, ..., A^{n-1}\}$， (為$M_{n\times n}(\mathbb{R})$的子空間)

則$\langle r_m(A)\rangle$也是W中的Cauchy sequence.

由完備性(completeness), $\exists\, L\in W$ 使得 $L=\lim\limits_{m\to\infty} r_m(A)$

由$M_{n\times n}(\mathbb{R})$中極限的唯一性得知 $L=f(A)$.

$\therefore\ f(A)\in W$

$\therefore\ \exists\ a_i$使得 $f(A)=a_0I+a_1A+a_2A^2+....+a_{n-1}A^{n-1}$.

令 $g(t)=a_0+a_1t+a_2t^2+....+a_{n-1}t^{n-1}$,

則得 $f(A)=g(A)$.

題型16D: 最小多項式

１６D01 【 中正84資工[4](b) 】

(b) Let A be a matrix with characteristic polynomial $\lambda^2(\lambda-1)^4(\lambda-2)^2$ and minimal polynomial $\lambda^2(\lambda-1)^2(\lambda-2)$. Find the possible Jordan form of A .

【解】(b) 令 $K_\lambda = \bigcup \mathrm{Ker}(A-\lambda I)^j$. (綜線CH15定義3)

$\dim K_0 = 2$, A在K_0的index為2 (綜線CH15定理6, CH16定理24)

∴ K_0對A的循環分解的不變集為 2 . (綜線CH14範例13a要訣)

$\dim K_1 = 4$, A在K_1的index為2 (綜線CH15定理6, CH16定理24)

∴ K_1 對$A-I$的循環分解的不變集為2,2或2,1,1. (綜線CH14範例13a要訣)

$\dim K_2 = 2$, A在K_2的index為1 (綜線CH15定理6, CH16定理24)

∴ K_2對$A-2I$的循環分解的不變集為1,1. (綜線CH14範例13a要訣)

A的Jordan form有下列兩種可能形. (綜線CH15範例8a)

$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&2&0\\ 0&0&0&0&0&0&0&2 \end{bmatrix} \qquad \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0\\ 0&0&1&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&2&0\\ 0&0&0&0&0&0&0&2 \end{bmatrix}$$

１６D02 【 台大80資工[12] 】

[True or False Problem]

Let T and U be linear operators on a finite dimensional vector space V. If their characteristic polynomials are the same to be $(-1)^5(t-1)^3(t-2)^2$ and their minimal polynomials are $(t-1)(t-2)$ then T and U have the same Jordan canonical form.

【解】True, 解說如下:

由特徵多項式 $(-1)^5(t-1)^3(t-2)^2$ 可知Jordan form為

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ 0 & b & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & c & 2 \end{bmatrix}$$

之形, 其中 $a, b, c \in \{0, 1\}$. (綜線CH16定理8)

由最小多項式$(t-1)^2(t-2)$

可知 $c=0, a=1, b=0$ (綜線CH16定理24)

∴ T與U的Jordan form相同.

１６D**03** 【 成大81資工丙[3](c) 】

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}$$

(c) What is the minimal polynomial for the matrix A ? (7%)

【解】(c) 特徵多項式 $f(t)=\det(A-xI)=-(t^3-4t^2-15t+8)$

(以 $t=1, -1, 2, -2, 4, -4, 8, -8$ 代入皆不爲零)

$f'(t)=-(3t^2-8t-15)$

由輾轉相除法可知 f 與 f' 無公因式,

∴ $f(t)$ 無重根

∴ A 的 minimal polynomial 爲 $t^3-4t^2-15t+8$ (綜線CH16定理23)

１６D**04** 【 台大78資工[2] 】

Let C be the 3×3 matrix $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$

(a) What is the characteristic polynomial of C. (5%)

(b) What is the minimal polynomial of C. (5%)

(c) evaluate C^{100} . (10%)

【解】(a) 設C的characteristic polynomial爲ch(x), 則

$$\mathrm{ch}(x)=\begin{vmatrix} 5-x & -6 & -6 \\ -1 & 4-x & 2 \\ 3 & -6 & -4-x \end{vmatrix}=\begin{vmatrix} 5-x & 0 & -6 \\ -1 & 2-x & 2 \\ 3 & x-2 & -4-x \end{vmatrix}$$

$$= \ldots = -(x-2)^2(x-1)$$

(b) 令C的minimal polynomial爲$m(x)$,

則 $m(x)=(x-1)(x-2)^d$, $d=1$ 或 2 (綜線CH16定理23)

$(C-I)(C-2I)$

$$=\begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}\begin{bmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix}=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore\ m(x)=(x-1)(x-2)$

(c) 令$x^{100}=q(x)(x-1)(x-2)+ax+b$

以$x=1, 2$ 代入, 可得 $1=a+b$, $2^{100}=2a+b$

可解出 $a=2^{100}-1$, $b=-2^{100}+2$

$\therefore\ C^{100}=q(C)m(C)+(2^{100}-1)C+(-2^{100}+2)I=\ldots$

$$=2^{100}\begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix}+\begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \\ -3 & 6 & 6 \end{bmatrix}$$

16D05 【清大86工工[5]】

$$A=\begin{bmatrix}2&0&0&0\\0&1&0&1\\0&0&2&0\\0&1&0&1\end{bmatrix}$$

The eigenvalues of the matrix A are 2 and 0. Find A^{10}. (5%)

【解】通常須先求A的特徵多項式，但本題已給出特徵值，所以本段可省略.

$$\det(A-2I)=\begin{vmatrix}2-x&0&0&0\\0&1-x&0&1\\0&0&2-x&0\\0&1&0&1-x\end{vmatrix}=(2-x)\begin{vmatrix}1-x&0&1\\0&2-x&0\\1&0&1-x\end{vmatrix}$$

$$=(2-x)^2\begin{vmatrix}1-x&1\\1&1-x\end{vmatrix}=x(x-2)^3$$

∵ A為實數對稱矩陣，∴ A可對角化. (綜線CH13定理15)

∴ A的minimal polynomial $m(x)=x(x-2)$ (綜線CH16定理23,定理26)

令 $x^{10}=q(x)m(x)+ax+b$ (綜線CH16範例27)

以$x=0$及$x=2$代入，可解得 $a=2^9$, $b=0$

∴ $A^{10}=Q(A)m(A)+aA=Q(A)\cdot O+aA=2^9A$

$$=2^9\begin{bmatrix}2&0&0&0\\0&1&0&1\\0&0&2&0\\0&1&0&1\end{bmatrix}=\begin{bmatrix}1024&0&0&0\\0&512&0&512\\0&0&1024&0\\0&512&0&512\end{bmatrix}$$ #

１６D**０６** 【 元智85電資[3]& 】

Find A^5 if $A=\begin{bmatrix}4 & -2 & 1\\ 2 & 0 & 1\\ 2 & -2 & 3\end{bmatrix}$.

【解】$\det(A-xI)=(3-x)(2-x)^2$

$$A-2I=\begin{bmatrix}2 & -2 & 1\\ 2 & -2 & 1\\ 2 & -2 & 1\end{bmatrix}\sim\begin{bmatrix}2 & -2 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

∴ $\text{rank}(A-2I)=1$ (綜線CH6定理23)

∴ $\dim\ker(A-2I)=3-1=2$ (綜線CH8定理8)

∴ A的每個特徵值的代數重數都等於幾何重數, (綜線CH12定理19)

∴ A可對角化 (綜線CH12定理21)

∴ A的minimal polynomial爲$\min(x)=(x-3)(x-2)$ (綜線CH16定理26)

令 $x^5=q(x)(3-x)(2-x)+ax+b$

以$x=3,2$分別代入得 $243=3a+b$, $32=2a+b$

解得 $a=211, b=-390$.

$$\therefore\quad A^5=q(A)\cdot O+211A-390I=\begin{bmatrix}454 & -422 & 211\\ 422 & -390 & 211\\ 422 & -422 & 243\end{bmatrix}$$

題型16E: 特徵值的應用問題

１６Ｅ**01**【 交大84工工[1] 】

Let $A=\begin{bmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$, find a matrix B such that $B^2=A$.

【分析】本題考對角化的應用. 對角化之後就可求平方根矩陣. 依此法對3×3矩陣通常可求得2^3 個平方根矩陣, 但本題只要求任一個.

【解】 先做對角化, （細節略, 解法參閱題型12C）

$$令\ P=\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, D=\begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E=\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

則 $A=PDP^{-1}=PE^2P^{-1}=PEP^{-1}PEP^{-1}$

$$\therefore\ 可取\ B=PEP^{-1}=\ \ldots\ =\begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

１６Ｅ**02**【 成大85資工[5] 】

Let $y_0, y_1, y_2,\ldots$ be the sequence of the Fibonacci numbers where $y_0=0$, $y_1=1$ and $y_{n+1}=y_n+y_{n-1}$ for all $n\geq 2$. Let $z_n=y_{n-1}$ for $n\geq 1$. Then the Fibonacci sequence can be written as a first order recurrence system

$$\begin{cases} y_{n+1}=y_n+z_n \\ z_{n+1}=y_n \end{cases}$$

, with initial conditions $y_1=1$ and $z_1=0$. By setting $v_n=(y_n, z_n)^t$

and $A=\begin{bmatrix}1&1\\1&0\end{bmatrix}$, one obtain $v_{n+1}=Av_n$. Now, diagonalize A and obtain a formula for the $(n+1)$-th Fibonacci numbers y_n.

【勘誤】本題給的初值是y_0 , y_1 . 因此公式 $y_{n+1}=y_n+y_{n-1}$ 的範圍應是$n\geq 1$.

【解】由 $v_{n+1}=Av_n$ ，得知 $v_{n+1}=A^n v_1$.

依題意先對A做對角化:

$\det(A-xI)=x^2-x-1=(x-p)(x-q)$,

其中 $p=\dfrac{1+\sqrt{5}}{2}$, $q=\dfrac{1-\sqrt{5}}{2}$.

$A-pI$ (利用$p^2-p-1=0$的條件!)

$$=\begin{bmatrix}1-p&1\\1&-p\end{bmatrix}\sim\begin{bmatrix}0&1+(-p)(p-1)\\1&-p\end{bmatrix}=\begin{bmatrix}0&0\\1&-p\end{bmatrix}\sim\begin{bmatrix}1&-p\\0&0\end{bmatrix}$$

∴ 取得p的特徵向量 $\begin{bmatrix}p\\1\end{bmatrix}$. 同法可取得$q$的特徵向量 $\begin{bmatrix}q\\1\end{bmatrix}$.

令 $S=\begin{bmatrix}p&q\\1&1\end{bmatrix}$, 則 $S^{-1}AS=D=\begin{bmatrix}p&0\\0&q\end{bmatrix}$.

$v_{n+1}=A^n v_1=(SDS^{-1})^n v_1=SD^nS^{-1}v_1$

$$=\begin{bmatrix}p&q\\1&1\end{bmatrix}\begin{bmatrix}p^n&0\\0&q^n\end{bmatrix}\left(\frac{1}{p-q}\begin{bmatrix}1&-q\\-1&p\end{bmatrix}\right)\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{p-q}\begin{bmatrix}p^{n+1}-q^{n+1}\\p^n-q^n\end{bmatrix}$$

$$\therefore\ y_n = z_{n+1} = \frac{1}{p-q}(p^n - q^n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

１６E03 【 中正79資工[1] 】

Let { S_k}, $k=1, 2, ...$ be a sequence of numbers such that $S_0=0$, $S_1=2$, and $S_{k+2}=S_{k+1}+S_k$. Find S_k .

【解】 1° $\begin{bmatrix} S_{k+1} \\ S_{k+2} \end{bmatrix} = \begin{bmatrix} S_{k+1} \\ S_{k+1}+S_k \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} S_k \\ S_{k+1} \end{bmatrix}$

令 $v_k = \begin{bmatrix} S_k \\ S_{k+1} \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

則上式化爲 $v_{k+1} = Av_k$.

$\therefore\ v_k = Av_{k-1} = A^2 v_{k-2} = ... = A^k v_0$

2° $\mathrm{ch}(x) = \det(A - xI) = = x^2 - x - 1$

令 $\mathrm{ch}(x)=0$, 可解得eigenvalues λ_1, λ_2, 其中

$$\lambda_1 = \frac{1+\sqrt{5}}{2},\quad \lambda_2 = \frac{1-\sqrt{5}}{2}.$$

令 $x^k = q(x)\mathrm{ch}(x) + ax + b$,

以 λ_1, λ_2 代入得 $\lambda_1^k = a\lambda_1 + b$, $\lambda_2^k = a\lambda_2 + b$

可解得 $a = \dfrac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}$, $b = \dfrac{\lambda_1\lambda_2^k - \lambda_2\lambda_1^k}{\lambda_1 - \lambda_2}$

由Cayley-Hamilton定理,

$A^k = q(A)\mathrm{ch}(A) + aA + bI = q(A)O + aA + bI$

$$=a\begin{bmatrix}0&1\\1&1\end{bmatrix}+b\begin{bmatrix}1&0\\0&1\end{bmatrix}=\begin{bmatrix}b&a\\a&a+b\end{bmatrix}$$

3°

$$\therefore\ v_k=A^k v_0=\begin{bmatrix}b&a\\a&a+b\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix}=2\begin{bmatrix}a\\a+b\end{bmatrix}$$

$$\therefore\ S_k=2a=2\ \frac{\lambda_1^k-\lambda_2^k}{\lambda_1-\lambda_2}=\frac{2}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

【討論】解答中的第2°步驟(求A^k)也可用對角化的方法計算(如CH16範例5), 但計算上較複雜.

16E04 【 台大77資工[6] 】

Let F_k , $k=$ 0,1,2,... be a sequence of numbers such that $F_0=1$, $F_1=3$ and $F_{k+2}=F_{k+1}+F_k$, find F_{100}.

【解】同上題,

$$令\ v_k=\begin{bmatrix}F_k\\F_{k+1}\end{bmatrix},\quad A=\begin{bmatrix}0&1\\1&1\end{bmatrix}.$$

$$v_k=A^k v_0=\begin{bmatrix}b&a\\a&a+b\end{bmatrix}\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}3a+b\\4a+3b\end{bmatrix}$$

$F_k=3a+b=$ ... 讀者自行化簡

16E05 【 元智85工工乙[3] 】

Consider the following difference equation: $F_{k+2}=F_{k+1}+F_k$, where $F_0=F_1=1$, $k\geq 0$ and k integer.

Show that $F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$

【勘誤】題目的初值給錯了. 應是$F_0=0, F_1=1$，或是 $F_1=F_2=1$.

若要用本題的初值, 公式應是

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

【分析】此種遞迴關係的初值問題可用離散數學的方法解, 也可如綜線CH16範例13 用矩陣法求解. 但本題已給出答案, 所以用數學歸納法證明就可以了.

【解】以數學歸納法證明:

先以$k=0$及$k=1$代入檢驗,

$$F_0=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^0-\left(\frac{1-\sqrt{5}}{2}\right)^0\right]=\frac{1}{\sqrt{5}}(1-1)=0$$

$$F_1=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)\right]=\frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right)=1$$

$$\left(\begin{array}{l}\left(\frac{1\pm\sqrt{5}}{2}\right)^2=\frac{1\pm2\sqrt{5}+5}{4}=\frac{3\pm\sqrt{5}}{2}\\[2ex] F_2=\frac{1}{\sqrt{5}}\left[\left(\frac{3+\sqrt{5}}{2}\right)-\left(\frac{3-\sqrt{5}}{2}\right)\right]=\frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right)=1\end{array}\right)$$

設此公式對k及$k+1$已成立, 對$k+2$：

$F_{k+1}+F_k$

$$=\frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{k}-(\frac{1-\sqrt{5}}{2})^{k}\right]+\frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{k+1}-(\frac{1-\sqrt{5}}{2})^{k+1}\right]$$

$$=\frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{k}(1+\frac{1+\sqrt{5}}{2})-(\frac{1-\sqrt{5}}{2})^{k}(1+\frac{1-\sqrt{5}}{2})\right]$$

$$=\frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{k}(\frac{3+\sqrt{5}}{2})-(\frac{1-\sqrt{5}}{2})^{k}(\frac{3-\sqrt{5}}{2})\right]$$

$$=\frac{1}{\sqrt{5}}\left[(\frac{1+\sqrt{5}}{2})^{k+2}-(\frac{1-\sqrt{5}}{2})^{k+2}\right]=F_{k+2}$$

１６E**06** 【 元智84工工X[4] 】

考慮 $\begin{cases} r_{n+1}=4r_n-t_n \\ t_{n+1}=2r_n+t_n \end{cases}$, 其中 $r_0=100,\ t_0=10,\ n=0,1,2,\ldots,$ 試求 $\lim_{n\to\infty}\left(\frac{r_n}{t_n}\right)$

【解】

$$\begin{bmatrix} r_{n+1} \\ t_{n+1} \end{bmatrix}=\begin{bmatrix} 4r_n-t_n \\ 2r_n+t_n \end{bmatrix}=\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} r_n \\ t_n \end{bmatrix}$$

$$\therefore \begin{bmatrix} r_n \\ t_n \end{bmatrix}=\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}^{n}\begin{bmatrix} r_0 \\ t_0 \end{bmatrix}$$

經對角化, 可得

$$\begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$$

$$\therefore \begin{bmatrix} r_n \\ t_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 100 \\ 10 \end{bmatrix}$$

$$= \ldots\ldots = \begin{bmatrix} -90 \cdot 2^n + 190 \cdot 3^n \\ -180 \cdot 2^n + 190 \cdot 3^n \end{bmatrix}$$

$$\lim_{n\to\infty} \left(\frac{r_n}{t_n} \right) = \lim_{n\to\infty} \left(\frac{-90 \cdot 2^n + 190 \cdot 3^n}{-180 \cdot 2^n + 190 \cdot 3^n} \right) = \lim_{n\to\infty} \left(\frac{-90 \cdot (2/3)^n + 190}{-180 \cdot (2/3)^n + 190} \right)$$

$$= 1$$

16E07【中正84資工[3]】

Show that if A is diagonalizable, with all eigenvalues less than 1 in magnitude, then A^k tends to the zero matrix as $k \to \infty$.

【解】$A = P \cdot \text{diag}(\lambda_1, ..., \lambda_n) \cdot P^{-1}$.

$A^k = P \cdot \text{diag}(\lambda_1, ... , \lambda_n)^k \cdot P^{-1}$

$= P \cdot \text{diag}(\lambda_1^k, ... , \lambda_n^k) \cdot P^{-1}$ (綜線CH16定理3)

$\because \ |\lambda_i| < 1, \ \therefore \ \lim \lambda_i = 0$, $i = 1,2,...,n$.

$\therefore \ \lim \text{diag}(\lambda_1^k, ... , \lambda_n^k) = \text{diag}(0, ..., 0) = O$

$\therefore \ \lim A^k = P \cdot \lim \text{diag}(\lambda_1^k, ... , \lambda_n^k) \cdot P^{-1} = POP^{-1} = O$.

16E08【中正82資工[5]】

Let $u_{k+1} = Au_k = \begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix} u_k$ and $u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(a) Compute $u_k = S\Lambda^k S^{-1} u_0$ for any a and b.

(b) Under what condition on a and b does u_k approach a finite limit as $k \longrightarrow \infty$, and what is the limit ? (10%)

【解】(a)

$$A=\begin{bmatrix} a & b \\ 1-a & 1-b \end{bmatrix}$$，先將A對角化爲$S\Lambda S^{-1}$：

$\det(A-xI)=(x-1)(x-(a-b))$

$$A-I=\begin{bmatrix} a-1 & b \\ 1-a & -b \end{bmatrix} \sim \begin{bmatrix} a-1 & b \\ 0 & 0 \end{bmatrix}$$

以下分三種情況加以討論:

ⅰ. 若$a-b\neq 1$: (此時必可對角化, 見綜線CH12定理23)

則 $a-1\neq b$， 取特徵值1的特徵向量$\begin{bmatrix} b \\ 1-a \end{bmatrix}$

$$A-(a-b)I=\begin{bmatrix} b & b \\ 1-a & 1-a \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

取特徵值$a-b$的特徵向量$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

令 $S=\begin{bmatrix} b & 1 \\ 1-a & -1 \end{bmatrix}$, $\Lambda=\begin{bmatrix} 1 & 0 \\ 0 & a-b \end{bmatrix}$, 則$A=S\Lambda S^{-1}$.

由 $u_{k+1}=Au_k$ 可推得

$$u_k=A^k u_0=S\Lambda^k S^{-1} u_0=S\begin{bmatrix} 1 & 0 \\ 0 & (a-b)^k \end{bmatrix} S^{-1}u_0$$

ⅱ. 若$a-b=1$，且$b=0$, (即 $a=1, b=0$)

此時 $A=I$， 取$\Lambda=I, S=I$, 則 $A=S\Lambda S^{-1}$.

$$u_k = S\Lambda^k S^{-1} u_0 = Iu_0 = u_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

ⅲ. 若$a-b=1$，且$b\neq 0$,

則 $a-1=b\neq 0$，此時 $A-I \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$,

dim ker$(A-I)$ = 2–rank$(A-I)=2-1=1$, (綜線CH8定理8)

而特徵值 1 是 2 重根， ∴ A不可對角化. (綜線CH12定理21)

取$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. 再解$(A-I)x=v_1$，可得特解 $v_2 = \begin{bmatrix} 1/b \\ 0 \end{bmatrix}$.

令$S = \begin{bmatrix} 1/b & 1 \\ 0 & -1 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$，則 $A=S\Lambda S^{-1}$

$$u_k = S\Lambda^k S^{-1} u_0 = S \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^k S^{-1}u_0 = S \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} S^{-1}u_0$$

(b) 關於u_k 的極限, 則取決於 $\lim\limits_{k\to\infty} \Lambda^k$:

依前述討論, 情況共分四種:

(ⅰ–1)若 $a-b\neq 1$，且 $|a-b|<1$

$$\therefore\ (a-b)^k \longrightarrow 0 \qquad \therefore\ \Lambda^k \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore\ u_k \longrightarrow S \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} S^{-1}u_0 = \frac{-2}{a-b-1} \begin{bmatrix} b \\ 1-a \end{bmatrix}$$

(ⅰ–2)若 $a-b\neq 1$，且 $|a-b|\geq 1$

∴ $(a-b)^k$ 發散 ∴ Λ^k 發散

∴ u_k 的極限不存在.

(ⅱ) 若$a-b=1$，且$b=0$,（即 $a=1, b=0$）

此時 $u_k=\begin{bmatrix}1\\1\end{bmatrix}$，∴ u_k的極限為$\begin{bmatrix}1\\1\end{bmatrix}$

(ⅲ) 若$a-b=1$，且$b\neq 0$,

此時 $\Lambda^k=\begin{bmatrix}1 & 0\\k & 1\end{bmatrix}$ 發散

∴ u_k 的極限不存在.

１６E**09** 【 元智85電資[4] 】

Consider a preator-prey population consisting of the foxes and rabbits living in a certain area. There are F_i foxes and R_i rabbits in the ith month. Assume that the transition from each month to the next can be described by the following equation:

$$F_i=0.4F_{i-1}+0.3R_{i-1}$$
$$R_i=-rF_{i-1}+1.2R_{i-1}$$

where $r>0$ represents the average number of rabbits consumed monthly by each fox. Show that three possibilities for what may happen to the fox and rabbit population in the long term, i.e.,

set $r=0.4, 0.5, 0.325$, and find the difference.

【解】 $$\begin{bmatrix}F_i\\R_i\end{bmatrix}=\begin{bmatrix}0.4F_{i-1}+0.3R_{i-1}\\-rF_{i-1}+1.2R_{i-1}\end{bmatrix}=\begin{bmatrix}0.4 & 0.3\\-r & 1.2\end{bmatrix}\begin{bmatrix}F_{i-1}\\R_{i-1}\end{bmatrix}$$

令$A=\begin{bmatrix}0.4 & 0.3\\-r & 1.2\end{bmatrix}$,

則由 $\begin{bmatrix} F_i \\ R_i \end{bmatrix} = A \begin{bmatrix} F_{i-1} \\ R_{i-1} \end{bmatrix}$ 得知 $\begin{bmatrix} F_i \\ R_i \end{bmatrix} = A^i \begin{bmatrix} F_0 \\ R_0 \end{bmatrix}$

$\det(A-xI) = x^2 - 1.6x + 0.48 + 0.3r$

$\therefore\ x = 0.8 \pm \sqrt{0.16 - 0.3r}$

依題目要求, 分$r=0.4, 0.5, 0.325$三種情形加以討論:

(1) $r=0.4$時, A的特徵值為$0.8\pm\sqrt{0.04} = 1.0, 0.6$

(2) $r=0.5$時, A的特徵值為$0.8\pm\sqrt{0.01} = 0.9, 0.7$

(3) $r=0.325$時, A的特徵值為$0.8\pm\sqrt{0.0625} = 1.05, 0.55$

此三種情形A都具有完全相異的特徵值, 所以都可對角化.

即存在可逆矩陣P使$A=PDP^{-1}$, $D=\mathrm{diag}(\lambda_1, \lambda_2)$

$\therefore\ A^i = PD^iP^{-1} = P \cdot \mathrm{diag}(\lambda_1^i, \lambda_2^i) \cdot P^{-1}$

(1) 當$r=0.4$時, $\lim\lambda_1^i = 1$, $\lim\lambda_2^i = 0$, $\lim A^i = P \cdot \mathrm{diag}(1,0) \cdot P^{-1}$.
此情形表示F_i , R_i 將趨於穩定.

(2) 當$r=0.5$時, $\lim\lambda_1^i = 0$, $\lim\lambda_2^i = 0$, $\lim A^i = P \cdot O \cdot P^{-1} = O$.
此情形表示F_i , R_i 都將趨於零.

(3) 當$r=0.325$時, $\lim \lambda_1^i = \infty$,
此情形表示F_i , R_i 將大到失控. 自然界的生物當然不可能會變成無限多, 這表示此模型與實際情況不合. 須考慮其它被忽略的因素.

【討論】本題題意似乎只要求定性的結果, 而不是要算定量的值. (初值也沒有給定.) 通常解到如上即可. 若時間足夠, 可對$r=0.4$真的做對角化, 並求出 $\lim A^i$, 然後將$\lim F_i$及$\lim R_i$用F_0與R_0表出.

１６Ｅ**１０**【 交大82工工[11] 】

In a certain town 30 percent of the married women get divorced each year and 20 percent of the single women get married each year. There are 8000 married women and 2000 single women and the total population remains constant. Find the number of married women and single women after 5 years. What will be the long-range prospects if these percentages of marriages and divorces continue indefinitely into the future ?

【解】 請參閱綜合線性代數CH16範例14. 本題由於未規定經由對角化求A^n，所以應採用Cayley-Hamilton定理求算以節省時間.

１６E**１１** 【 交大80資科[3] 】

Assume that in a certain experiment, a certain number of foxes and chickens are raised in a certain environment, and their population are controlled in a certain way. Let F_i and C_i denote the numbers of foxes and chickens in the i-th time period, $i=0, 1, 2, \ldots$ Suppose those those numbers in the i-th and (i+1)-th time period are related in the in the following ways:

$$F_{i+1}=aF_i+C_i$$
$$C_{i+1}=-kF_i+bC_i$$

where $a+b=1.9$, $a-b=1$, and $0<k\leq 1/4$ is a control parameter. Given initial condition $F_0>0$, $C_0>0$, answer the following question (justify your answer).

(A) For what values of k, $\lim_{i\to\infty} F_i=\infty$?

(B) For what values of k, $\lim_{i\to\infty} F_i=0$?

(c) For what values of k, $\lim_{i\to\infty} F_i=$ a non-zero constant ?

【解】1° (先將問題整理成矩陣型)

$$\begin{bmatrix} F_{i+1} \\ C_{i+1} \end{bmatrix}=\begin{bmatrix} a & 1 \\ -k & b \end{bmatrix}\begin{bmatrix} F_i \\ C_i \end{bmatrix} \text{. 令 } A=\begin{bmatrix} a & 1 \\ -k & b \end{bmatrix}\text{, 則}$$

$$\begin{bmatrix} F_n \\ C_n \end{bmatrix}=A\begin{bmatrix} F_{n-1} \\ C_{n-1} \end{bmatrix}=A^2\begin{bmatrix} F_{n-2} \\ C_{n-2} \end{bmatrix}=\ldots\ldots=A^n\begin{bmatrix} F_0 \\ C_0 \end{bmatrix}$$

2° (計算A^n)

$$\text{ch}(x)=\begin{vmatrix} a-x & 1 \\ -k & b-x \end{vmatrix}=x^2-(a+b)x+ab+k$$

判別式 $=(a+b)^2-4(ab+k)=(a-b)^2-4k=1-4k \geq 0$

$\therefore\ \mathrm{ch}(x)=(x-\lambda_1)(x-\lambda_2)$,

其中 $\lambda_1=\dfrac{1.9+\sqrt{1-4k}}{2}>0$, $\lambda_2=\dfrac{1.9-\sqrt{1-4k}}{2}>0$

令 $x^n=q(x)\mathrm{ch}(x)+\alpha_n x+\beta_n$

以 λ_1, λ_2 代入, 得 $\lambda_1^n=\alpha_n\lambda_1+\beta_n$, $\lambda_2^n=\alpha_n\lambda_2+\beta_n$

可解出 $\alpha_n=\dfrac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}=\dfrac{\lambda_1}{\lambda_1-\lambda_2}\left(1-\left(\dfrac{\lambda_2}{\lambda_1}\right)^n\right)$

$$\beta_n=\frac{\lambda_1\lambda_2^n-\lambda_2\lambda_1^n}{\lambda_1-\lambda_2}=\frac{\lambda_1^n\lambda_2}{\lambda_1-\lambda_2}\left(\left(\frac{\lambda_2}{\lambda_1}\right)^{n-1}-1\right)$$

令 $S_n=1-\left(\dfrac{\lambda_2}{\lambda_1}\right)^n$, 則 $\alpha_n=\dfrac{\lambda_1^n S_n}{\lambda_1-\lambda_2}$, $\beta_n=\dfrac{-\lambda_1^n\lambda_2 S_{n-1}}{\lambda_1-\lambda_2}$,

其中 S_n, S_{n-1} 在 n 大時都趨近1 .

$$x^n=q(x)\mathrm{ch}(x)+\frac{\lambda_1^n}{\lambda_1-\lambda_2}(S_n x-S_{n-1}\lambda_2)$$

$$\therefore\ A^n=q(A)\mathrm{ch}(A)+\frac{\lambda_1^n}{\lambda_1-\lambda_2}(S_nA-S_{n-1}\lambda_2 I)=\frac{\lambda_1^n}{\lambda_1-\lambda_2}(S_nA-S_{n-1}\lambda_2 I)$$

(綜線CH16定理18)

$$=\frac{\lambda_1^n}{\lambda_1-\lambda_2}\begin{bmatrix} aS_n-S_{n-1}\lambda_2 & S_n \\ -kS_n & bS_n-S_{n-1}\lambda_2 \end{bmatrix}$$

(註: 這種題目 A^n 也可用對角化的方法算, 但 A 未必可對角化.
即使可以, 在計算上也較複雜.)

3° (處理 F_n 的極限)

$$\begin{bmatrix} F_n \\ C_n \end{bmatrix}=A^n\begin{bmatrix} F_0 \\ C_0 \end{bmatrix}$$

$$=\frac{\lambda_1^{n}}{\lambda_1-\lambda_2}\begin{bmatrix} aS_n-S_{n-1}\lambda_2 & S_n \\ -kS_n & bS_n-S_{n-1}\lambda_2 \end{bmatrix}\begin{bmatrix} F_0 \\ C_0 \end{bmatrix}$$

$\therefore\ F_n=\dfrac{\lambda_1^{n}}{\lambda_1-\lambda_2}((aS_n-S_{n-1}\lambda_2)F_0+S_n C_0)$

$\because\ S_n, S_{n-1}$在n大時都趨近1

$\therefore\ ((aS_n-S_{n-1}\lambda_2)F_0+S_n C_0) \longrightarrow (a-\lambda_2)F_0+C_0$

$\therefore\ F_n$的極限如下:

(A) $F_n\longrightarrow\infty \iff \lambda_1^{n}\longrightarrow\infty \iff \lambda_1>1 \iff \dfrac{1.9+\sqrt{1-4k}}{2}>1$

$\iff \sqrt{1-4k}>0.1 \iff 1-4k>0.01 \iff k<0.2475$

(B) $F_n \longrightarrow 0 \iff \lambda_1^{n} \longrightarrow 0 \iff \lambda_1<1 \iff k<0.2475$

(C) $F_n\longrightarrow$ a non-zero constant $\iff \lambda_1^{n} \longrightarrow 1 \iff k=0.2475$

１６E１２【 大同85資工[5] 】

(a) Find the eigenvalues and the corresponding eigenspaces of

$$A=\begin{bmatrix} 2 & 0 & -6 \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{bmatrix} \qquad (4\%)$$

(b) Solve the initial value problem

$$\begin{cases} y_1'=2y_1-6y_3\ , \\ y_2'=\ y_1-3y_3\ , \\ y_3'=\ y_2-2y_3\ , \end{cases} \quad \text{with } y_1(0)=y_2(0)=y_3(0)=2. \qquad (3\%)$$

(c) Find A^5 . (3%)

【解】(a) (請參閱題型12C)

$$\det(A-xI)=\begin{vmatrix} 2-x & 0 & -6 \\ 1 & -x & -3 \\ 0 & 1 & -2-x \end{vmatrix} \begin{matrix}\leftarrow\!\urcorner \\ (-2) \\ \ \end{matrix} = \begin{vmatrix} -x & 2x & 0 \\ 1 & -x & -3 \\ 0 & 1 & -2-x \end{vmatrix}$$

$= \ ... = -x(x-1)(x+1)$

A的eigenvalue 爲 $0, 1, -1$.

解 $(A-0I)v=o$, 得出0的eigenspace$\{ [3t, 2t, t]^T \mid t$爲純量 $\}$

解 $(A-1I)v=o$, 得出1的eigenspace$\{ [6t, 3t, t]^T \mid t$爲純量 $\}$

解 $(A+I)v=o$, 得出–1的eigenspace$\{ [2t, t, t]^T \mid t$爲純量 $\}$

(b)

接(a), 令$P=\begin{bmatrix} 3 & 6 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $\Lambda=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, 則 $A=P\Lambda P^{-1}$

#

令 $v=[y_1, y_2, y_3]^T$

則原方程式可化爲 $v'=Av$, 即 $v'=P\Lambda P^{-1}v$

再令 $P^{-1}v=w=[z_1, z_2, z_3]^T$

則原方程式化爲 $w'=\Lambda w$, 即

$z_1'=0,\quad z_2'=z_2,\quad z_3'=-z$.

解得 $z_1=c_1,\quad z_2=c_2 e^t,\quad z_3=c_3e^{-t}$ .... (甲)

$Pw(0)=v(0)=[2 \quad 2 \quad 2]^T$

$$\left[\begin{array}{ccc|c} 3 & 6 & 2 & 2 \\ 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{array}\right] \sim \ .... \ \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

解得 $w(0)=[2, -1, 1]^T$,

即 $z_1(0)=2, z_2(0)=-1, z_3(0)=1$.

代入甲式, 求得 $c_1=2, c_2=-1, c_3=1$.

$\therefore$ 解得 $z_1=2, z_2=-e^t, z_3=e^{-t}$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = v = Pw = \begin{bmatrix} 3 & 6 & 2 \\ 2 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -e^t \\ e^{-t} \end{bmatrix}$$

$\therefore$ 解得 $\begin{cases} y_1 = 6 - 6e^t + 2e^{-t} \\ y_2 = 4 - 3e^t + e^{-t} \\ y_3 = 2 - e^t + e^{-t} \end{cases}$

(代回原方程式驗算無誤)

(c) (請參閱題型16B)

接(a)，$\Lambda^5 = \text{diag}(0, 1, -1)^5 = \text{diag}(0, 1, -1) = \Lambda$.

$\therefore\ A^5 = (P\Lambda P^{-1})^5 = P\Lambda^5 P^{-1} = P\Lambda P^{-1} = A$

１６Ｅ**１３** 【 交大85工工[5] 】

Solve the following system of equations.

$$\frac{dy_1}{dt} = 3y_1 + 4y_2 \quad ; \quad y_1(0) = 6$$

$$\frac{dy_2}{dt} = 3y_1 + 2y_2 \quad ; \quad y_2(0) = 1$$

【解】

令 $A = \begin{bmatrix} 3 & 4 \\ 3 & 2 \end{bmatrix}$ ， $v = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$

則原方程式可寫爲 $\frac{d}{dt} v = Av$.

先將A對角化(細節略):

令 $P = \begin{bmatrix} 4 & 1 \\ 3 & -1 \end{bmatrix}$，則 $A = P \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} P^{-1}$

$$\therefore\ e^{tA}=P\begin{bmatrix} e^{6t} & 0 \\ 0 & e^{-t}\end{bmatrix}P^{-1}=\begin{bmatrix} 4 & 1 \\ 3 & -1\end{bmatrix}\begin{bmatrix} e^{6t} & 0 \\ 0 & e^{-t}\end{bmatrix}(-1/7)\begin{bmatrix} -1 & -1 \\ -3 & 4\end{bmatrix}$$

$$=(-1/7)\begin{bmatrix} 4e^{6t} & e^{-t} \\ 3e^{6t} & -e^{-t}\end{bmatrix}\begin{bmatrix} -1 & -1 \\ -3 & 4\end{bmatrix}$$ (右邊的矩陣不必急著乘入)

方程式之解爲 $v=e^{tA}c,\quad c\in\mathbb{R}^{2\times 1}$.

以$t=0$之初值代入: $\begin{bmatrix} 6 \\ 1\end{bmatrix}=e^{0A}c=Ic=c$

$$\therefore\ \begin{bmatrix} y_1 \\ y_2\end{bmatrix}=(-1/7)\begin{bmatrix} 4e^{6t} & e^{-t} \\ 3e^{6t} & -e^{-t}\end{bmatrix}\begin{bmatrix} -1 & -1 \\ -3 & 4\end{bmatrix}\begin{bmatrix} 6 \\ 1\end{bmatrix}$$

$$=\begin{bmatrix} 4e^{6t} & e^{-t} \\ 3e^{6t} & -e^{-t}\end{bmatrix}\begin{bmatrix} 1 \\ 2\end{bmatrix}=\begin{bmatrix} 4e^{6t}+2e^{-t} \\ 3e^{6t}-2e^{-t}\end{bmatrix}$$

方程式之解爲 $y_1=4e^{6t}+2e^{-t},\quad y_2=3e^{6t}-2e^{-t}$. #

１６Ｅ**１４**【 交大78資料[2] 】

$A=\begin{bmatrix} 5 & -4 \\ 2 & -1\end{bmatrix}$ Solve $\dfrac{dX}{dt}=AX$, where $X(0)=\begin{bmatrix} 1 \\ -1\end{bmatrix}$, using Matrix methods.

【解】先將A對角化(細節略):

令$P=\begin{bmatrix} 2 & 1 \\ 1 & 1\end{bmatrix}$, 則$P^{-1}AP=\begin{bmatrix} 3 & 0 \\ 0 & 1\end{bmatrix}$

令$Y=P^{-1}X=\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$

由 $\dfrac{dX}{dt}=AX$, 左乘P^{-1}得 $P^{-1}\dfrac{dX}{dt}=P^{-1}AX$,

$\therefore \dfrac{dP^{-1}X}{dt}=P^{-1}APP^{-1}X$, 即 $\dfrac{dY}{dt}=\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}Y$

$\therefore \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix}=\begin{bmatrix} 3y_1(t) \\ y_2(t) \end{bmatrix}$ $\therefore \begin{cases} y_1'(t)=3y_1(t) \\ y_2'(t)=y_2(t) \end{cases}$

$\therefore \begin{cases} y_1(t)=c_1e^{3t} \\ y_2(t)=c_2e^{t} \end{cases}$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix}=P^{-1}\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 2 \\ -3 \end{bmatrix}$$

$\therefore c_1=2$, $c_2=-3$.

$$\therefore \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}=\begin{bmatrix} 2e^{3t} \\ -3e^{t} \end{bmatrix}$$

$$\therefore X=PY=\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2e^{3t} \\ -3e^{t} \end{bmatrix}=\begin{bmatrix} 4e^{3t}-3e^{t} \\ 2e^{3t}-3e^{t} \end{bmatrix}$$

１６E１５【 交大82工工[10] 】

a. Solve the system of equations

$$y_1'=y_1+y_2$$
$$y_2'=4y_1-2y_2$$

where $y_1 = f_1(x)$ and $y_2 = f_2(x)$ are unknown functions to be determined, $y_1' = dy_1/dx$ and $y_2' = dy_2/dx$ are their derivatives, respectively. (8%)

b. Find the solution that satisfies the initial conditions $y_1(0) = 1, y_2(0) = 6$. (2%)

【解】(a) 原方程式為

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

令 $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$，先求做對角化(細節略):

令 $P = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}$，則 $P^{-1}AP = D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$

$$e^{xA} = \begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} e^{-3x} & 0 \\ 0 & e^{2x} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} e^{-3x} & e^{2x} \\ -4e^{-3x} & e^{2x} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}^{-1}$$

原方程式之解為

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^{-3x} & e^{2x} \\ -4e^{-3x} & e^{2x} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}^{-1}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-3x} & e^{2x} \\ -4e^{-3x} & e^{2x} \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$，k_1 , k_2 為任意常數.

即 $\begin{cases} y_1 = k_1e^{-3x} + k_2 e^{2x} \\ y_2 = -4k_1e^{-3x} + k_2e^{2x} \end{cases}$，$k_1 , k_2$ 為任意常數.

(在此可代回原式驗算)

(b) 以 $y_1(0) = 1$, $y_2(0) = 6$ 代入上式, 得 $1 = k_1 + k_2$, $6 = -4k_1 + k_2$

解得 $k_1=-1$, $k_2=2$

$$\therefore \begin{cases} y_1=-e^{-3x}+2e^{2x} \\ y_2=4e^{-3x}+2e^{2x} \end{cases}$$

【另解】原方程式為

$$\frac{d}{dx}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad \text{即} \quad \frac{d}{dx}y=Ay$$

先將A對角化, 求出

$$P=\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}, \text{使 } P^{-1}AP=\Lambda=\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$

令 $z=P^{-1}y=\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$, 則原式化為

$$\frac{d}{dx}P^{-1}y = P^{-1}APP^{-1}y, \quad \text{即} \quad \frac{d}{dx}z=\Lambda z \quad ,$$

$$\frac{d}{dx}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} -3z_1 \\ 2z_2 \end{bmatrix}$$

解出 $z_1=k_1e^{-3x}$, $z_2=k_2e^{2x}$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=P\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}=\begin{bmatrix} 1 & 1 \\ -4 & 1 \end{bmatrix}\begin{bmatrix} k_1e^{-3x} \\ k_2e^{2x} \end{bmatrix},$$

$$\text{即} \begin{cases} y_1=k_1e^{-3x}+k_2e^{2x} \\ y_2=-4k_1e^{-3x}+k_2e^{2x} \end{cases}$$

其它部份相同.

題型16F: 特徵值的特種問題

１６Ｆ**01**【 中央83資工[3](c) 】

True or False (Give a reason if true, and give a counterexample if false). Let A and B be two different $n\times n$ nonsingular matrices.

(c) AB and BA have the same eigenvalues. (3%)

【解】 (c) True. 此爲定理. (綜線CH16範例1)

１６Ｆ**02**【 大同82資工[二3] 】

If A,B are $n\times n$ matrices. Show that AB and BA have the same eigenvalues.

【解】 請參閱綜合線性代數CH16範例1.

１６Ｆ**03**【 交大80資工[5](b) 】

(b) (6%) Let A and B be $n\times n$ matrices. Show that AB and BA have the same eigenvalues.

【解】同上題.

１６Ｆ**04**【 清大76資科[5](b) 】

Assume A and B are two $n\times n$ square matrices, show that (b) AB, BA have the same set of eigenvalues. (10%)

【解】同上題.