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1 (25%) 2.(25%) 【離散】

3 (25%) 【元智87電資】

Let (C^2, R, +, ・) be the vector space, where $C^2=\{(a,b) \mid a,b\in C\}$ is the set of vectors; R is the set of scalars; "+" is defined by : $(a,b)+(c,d)=(a+c,b+d)$ $\forall\ (a,b), (c,d)\in C^2$;
"・" is defined by: $\alpha\cdot(a,b)=(\alpha a, \alpha b)$ $\forall\ (a,b)\in C^2$.

(1) Prove or disprove that $\{(1, 0), (0, 1)\}$ is a basis for (C^2, R, +, ・).

(2) Prove or disprove that $(1+i,1-i)$, $(1,1)$, $(i,0)$ are dependent in (C^2, R, +, ・).

【分析】本題屬於題型06A及題型06B. 請參閱綜線CH6範例3a, CH6範例13.

【解】(1) Disprove:

$(i, 0)\in C^2$, 但 $\forall\ h, k\in R$, $h(1,0)+k(0, 1) \neq (i, 0)$

(2) Disprove: 此三向量獨立, 證明如下:

設$a, b, c\in R$, 使 $a(1+i,1-i)+b(1,1)+c(i,0) = (0, 0)$.

比較分量得 $a(1+i)+b+ci=0$, $a(1-i)+b=0$.

比較實部虛部得 $a+b=0, a+c=0$, $a+b=0, a=0$.

$\therefore$ a, b, c皆為0.

4 (25%) 【元智87電資】

Let V be a 4-dimensional vector space over R and $T : V \longrightarrow V$ be a linear mapping whose characteristic polynomial is $f_T(x)=(x-\lambda)^4$, $\lambda\in R$.. Suppose x_1, x_2 are vectors in V such that $(T-\lambda I)^2(x_2)=o$, $(T-\lambda I)^2(x_1)=o$ and $(T-\lambda I)(x_1)$, $(T-\lambda I)(x_2)$ are linearly independent. Prove that $\{x_1, x_2, (T-\lambda I)(x_1), (T-\lambda I)(x_2)\}$ is a basis of V.

【分析】本題屬於題型14B. 本題爲綜線CH14定理11的活用題, 對一般同學來說算是難題.

【解】先證此集合爲獨立集.

設 $ax_1+bx_2+c(T-\lambda I)(x_1)+d(T-\lambda I)(x_2)=o$.

欲證各係數皆爲0:

上式左右兩邊都經$(T-\lambda I)$映射得:

$a(T-\lambda I)(x_1)+b(T-\lambda I)(x_2)+c\cdot o+d\cdot o=o$

而已知 $(T-\lambda I)(x_1)$, $(T-\lambda I)(x_2)$獨立,

∴ $a=b=0$.

代回原式得 $c(T-\lambda I)(x_1)+d(T-\lambda I)(x_2)=o$.

再由已知$(T-\lambda I)(x_1)$, $(T-\lambda I)(x_2)$獨立得知 c=d=0.

此集合爲4維空間中內含4向量的獨立集,

∴ 此爲基底. (綜線CH6定理22)