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參考章節使用簡稱，例如綜線CH3代表廖亦德著:「綜合線性代數」第3章.
題型代表廖亦德著:「線性代數題型剖析」書中的題型.
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1. (10%)　【交大87資工】

Let $A = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 1 & 1 & 3 & 2 \\ 2 & 3 & 7 & 3 \\ 4 & 5 & 13 & 7 \end{bmatrix}$

(a) (2%) Find the dimension of and a basis for the nullspace of A .

(b) (1%) Find the dimension of and a basis for the row space of A .

(c) (2%) Find the dimension of and a basis for the column space of A .

(d) (2%) If B is a nonsingular 3×3 matrix, what is the dimension of the nullspace of BA ?　Why?

(e) (3%) Let $b=(8, 3, 11, 17)^T$　Is b in the column space of A ? How many solutions will the linear system $Ax=b$ have?

【勘誤】本題(d)小題B應是4×4矩陣.

【分析】本題(a)(b)(c)(d)屬於題型06C. 請參閱綜線CH6範例23a,CH6範例24a.
本題(e)屬於題型03A. 請參閱綜線CH3範例7.

【解】 先對A做列運算:　　(CH3演算法4a)

$$A=\begin{bmatrix}1&2&4&1\\1&1&3&2\\2&3&7&3\\4&5&13&7\end{bmatrix}\sim\begin{bmatrix}1&2&4&1\\0&-1&-1&1\\0&-1&-1&1\\0&-3&-3&3\end{bmatrix}\sim\ldots\sim\begin{bmatrix}1&0&2&3\\0&1&1&-1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

(a)

$$\ker A=\left\{\begin{bmatrix}-2t-3s\\-t+s\\t\\s\end{bmatrix}\middle|\, s,t\text{爲純量}\right\}=\left\{t\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix}+s\begin{bmatrix}-3\\1\\0\\1\end{bmatrix}\middle|\, s,t\text{爲純量}\right\}$$

dim(ker$A$)=2. 可取ker$A$的基底爲 $\left\{\begin{bmatrix}-2\\-1\\1\\0\end{bmatrix},\begin{bmatrix}-3\\1\\0\\1\end{bmatrix}\right\}$.

(b) dim(RSP(A))=2, 可取RSP(A)的基底爲

$\{[\,1\quad 0\quad 2\quad 3\,],[\,0\quad 1\quad 1\quad -1\,]\}$. (CH6定理23)

(c) A經列運算後pivatal column爲第1,2行,

∴ dim(CSPA)=2. 可取CSP(A)的基底爲 $\left\{\begin{bmatrix}1\\1\\2\\4\end{bmatrix},\begin{bmatrix}2\\1\\3\\5\end{bmatrix}\right\}$. (CH6定理24)

(d) dim(ker(BA))=BA的寬度–rank(BA) (CH8定理8)

=A的寬度–rank(A) (CH8定理16)

=dim(ker$A$) (CH8定理8)

=2

(e) 對分隔矩陣[A | b]做列運算化爲梯形: (此與先前的列運算可一起算)

$$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 8 \\ 1 & 1 & 3 & 2 & 3 \\ 2 & 3 & 7 & 3 & 11 \\ 4 & 5 & 13 & 7 & 17 \end{array}\right] \sim \dots \sim \left[\begin{array}{cccc|c} 1 & 2 & 4 & 1 & 8 \\ 0 & -1 & -1 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

∴ 方程式$Ax=b$ 有無限多組解. (CH3定理10)

∴ b在A的column space之內. (CH5定理17,參閱CH3定理11a)

2. (7%)　【交大87資工】

The sets $E=[1+t, t+t^2, 1+t^2]$ and $F=[1, 1+t, 1+t+t^2]$ are both ordered bases for the vector space P_3 (P_3 denote the set of polynomials of degree less than 3).

(a) (2%) What is the coordinates of $7+5t+9t^2$ with respect to the basis F?

(b) (5%) Find the transition matrix from E to F

【分析】本題屬於題型06D. 請參閱綜線CH6定義28, CH6定理33.

【解】(a) 令 $7+5t+9t^2 = a \cdot 1 + b \cdot (1+t) + c \cdot (1+t+t^2)$

比較係數得 $a+b+c=7$, $b+c=5$, $c=9$.

解得 $a=2$, $b=-4$, $c=9$.

∴ 所求 $[\,7+5t+9t^2\,]_F = [\,2 \quad -4 \quad 9\,]^T$. (CH6定義28)

(b) 依(a)之法解得

$[1+t]_F = [0 \quad 1 \quad 0\,]^T$, $[\,t+t^2]_F = [-1 \quad 0 \quad 1]^T$, $[1+t^2]_F = [1 \quad -1 \quad 1]^T$.

∴ 所求爲 $\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ (CH6定理33)

3. (8%) 【交大87資工】

Let $A=\begin{bmatrix}1&2\\1&3\end{bmatrix}$, $B=\begin{bmatrix}1&2\\2&4\end{bmatrix}$, $C=\begin{bmatrix}-1&-2\\-3&-5\end{bmatrix}$, $D=\begin{bmatrix}-1&-2\\0&-2\end{bmatrix}$

(a) (4%) Test matrices A,B,C and D for linear independence or dependence.

(b) (4%) Find the dimension of and a basis for the space spanned by A,B,C and D.

【分析】本題(a)屬於題型06A. 本題(b)屬於題型06C.

【解】(a) 試解方程式 $x_1A+x_2B+x_3C+x_4D=O$.

比較各對應位置可得:

$x_1+x_2-x_3-x_4=0$, $2x_1+2x_2-2x_3-2x_4=0$, $x_1+2x_2-3x_3-0x_4=0$, $3x_1+4x_2-5x_3-2x_4=0$.

由列運算: (CH3演算法4a)

$$\begin{bmatrix}1&1&-1&-1\\2&2&-2&-2\\1&2&-3&0\\3&4&-5&-2\end{bmatrix}\sim\ldots\sim\begin{bmatrix}1&1&-1&-1\\0&1&-2&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

∴ 有無限多組解. (CH3定理10)

∴ A,B,C,D為linear dependent. (CH6定義9)

(b) 將(a)中的矩陣記為M. 依同構原則, 將A,B,C,D視為M的四個行向量. 則:

dim(span{A,B,C,D})=dim(Column space of M)

= rank$M$ = 2 (CH8定理13, CH6定理23)

並可取相對於前兩行的 { A, B }為span{A,B,C,D}的基底. (CH6定理24)

4. (8%) 【交大87資工】

Given the five data points: (x, y)=(–2, 3), (–1, 5), (0, 5), (1, 4), and (2, 3), find the linear function $y=\beta_0+\beta_1x$ which best fits the data in the least square sense. In the measurement, the last three data points are less reliable. The fitting errors of the three points are supposed to be weighted half as much as the first two data points in evaluating the least square errors.

【分析】本題屬於題型09E. 但估量誤差時引入較複雜的權重. 這就不能只背公式, 而是需要了解整個理論推導過程.

【解法1】(微積分解法)

令$Q=(\beta_0-2\beta_1-3)^2+(\beta_0-\beta_1-5)^2+\left(\frac{\beta_0+0\beta_1-5}{2}\right)^2+\left(\frac{\beta_0+\beta_1-4}{2}\right)^2+\left(\frac{\beta_0+2\beta_1-3}{2}\right)^2$

欲求極小值, 令 $\partial Q/\partial\beta_0=0$, $\partial Q/\partial\beta_1=0$. 即

$2(\beta_0-2\beta_1-3)+2(\beta_0-\beta_1-5)+(1/2)(\beta_0+0\beta_1-5)+(1/2)(\beta_0+\beta_1-4)+(1/2)(\beta_0+2\beta_1-3)=0$

$2(\beta_0-2\beta_1-3)(-2)+(\beta_0-\beta_1-5)(-1)+\quad 0\quad+(1/2)(\beta_0+\beta_1-4)+(1/2)(\beta_0+\beta_1-4)(2)=0$

整理得: $11\beta_0-9\beta_1-44=0$, $-9\beta_0+25\beta_1+34=0$

解得 $\beta_0=397/97$, $\beta_1=11/97$.

∴ 所求為 $(397/97)+(11/97)x$

【解法2】 (線代解法)

$$\text{欲求}\beta_0,\beta_1\text{使}\ \left\| \begin{bmatrix}3\\5\\5\\4\\3\end{bmatrix} - \begin{bmatrix}1 & -2\\1 & -1\\1 & 0\\1 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix} \right\|^2 \text{儘量小}$$

將上式記為 $\|y-M\beta\|^2$.

上式中norm的取法為 $\|v\|^2=v^TWv$, $W=\mathrm{diag}(1, 1, 1/4, 1/4, 1/4)$.

為滿足要求, 必須在內積取為$\langle u,v\rangle = v^TWu$ 的條件下

使$M\beta$為y對column space of M的正投影,　　(CH9定理13)

∴ $\forall u, \langle y-M\beta, Mu\rangle=0$　　(CH9定義11)

即 $\forall u, u^TM^TW(y-M\beta)=0$

即 $M^TW(y-M\beta)=0$　　(CH9範例10)

即 $M^TWy=M^TWM\beta$　　(weighted normal equation)

乘開得

$$\begin{bmatrix} 11/4 & -9/4 \\ -9/4 & 25/4 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 11 \\ -17/2 \end{bmatrix},$$

解得 β_0=397/97,　β_1=11/97.

∴　所求爲 $(397/97)+(11/97)x$

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5. (8%)　【交大87資工】

Define the transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ by $T(x)=Ax$, where

$$A = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$$

Supposed $\mathscr{B}$ is the basis for $\mathbb{R}^2$ with respect to which the transformation A is a diagonal matrix. Find the diagonal matrix and the basis $\mathscr{B}$.

【勘誤】“ the transformation A ” 應更正爲 “ the transformation T ”

【分析】本題屬於題型12C.

【解】$\det(A-xI)=...=(x-3)(x-5)$.　　∴ eigenvalue爲3, 5

解$(A-3I)v=o$ 可得3的eigenvector $[\,1 \quad -2\,]^T$.

解$(A-5I)v=o$ 可得5的eigenvector $[\,1 \quad -1\,]^T$.

令 $\mathscr{B}=\{\,[\,1\ -2]^T, [\,1\ -1\,]^T\,\}$,　則　$[\,T\,]_{\mathscr{B}} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$　　(CH12定理16)

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6. (9%)　【交大87資工】

Let $T : P_2 \longrightarrow P_3$ be the transformation that maps a polynomial c_0+c_1t into the polynomial $c_1+(c_0+c_1)t+(c_0-c_1)t^2$.　Find the matrix for T with respect to the ordered bases $\{1+2t, 3+t\}$ and $\{1, 1+t, 1+t+t^2\}$

【分析】本題屬於題型07B. 請參閱綜線CH7定義9.

【解】由T的公式得 $T(1+2t)=2+3t-t^2$, $T(3+t)=1+4t+2t^2$.

設 $2+3t-t^2 = \alpha\cdot 1+\beta\square(1+t)+\gamma\cdot(1+t+t^2)$

$1+4t+2t^2=\delta\square 1+\varepsilon\square(1+t)+\zeta\square(1+t+t^2)$ (CH7定義9)

比較係數得 $\alpha+\beta+\gamma=2$, $\beta+\gamma=3$, $\gamma=-1$

$\delta+\varepsilon+\zeta=1$, $\varepsilon+\zeta=4$, $\zeta=2$.

解得 $\alpha=-1$, $\beta=4$, $\gamma=-1$, $\delta=-3$, $\varepsilon=2$, $\zeta=2$.

$\therefore$ 所求爲 $\begin{bmatrix} -1 & -3 \\ 4 & 2 \\ -1 & 2 \end{bmatrix}$.