交通大學87資科所

以下所有線性代數的問題, 都是填充題. 只需寫答案, 不必寫出任何計算過程.

1.(4%) 【交大87資科】

Find a P^TLU decomposition for the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 5 \end{bmatrix}$$

【分析】本題屬於題型03E.

【解】$P=I$, L=原矩陣, $U=I$.

2.(4%) 【交大87資科】

Find a QR decomposition for the following matrix.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

【分析】本題屬於題型09D.

【解】(細節略)

$$Q=\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix},\quad R=\begin{bmatrix} \sqrt{5} & 3\sqrt{5} \end{bmatrix}$$

3.(4%) 【交大87資料】

Find the Jordan form $A=QJQ^{-1}$ for the following matrix.

$$\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}$$

【分析】本題雖說是要化成Jordan form，但其實是可對角化的題目，屬於題型12C.

為簡化計算，可先對2A做，做完再做調整.

【解】(細節略)

$$Q=\begin{bmatrix} 1 & -1/2 \\ 1 & 1 \end{bmatrix},\quad J=\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

4.(8%) 【交大87資料】

Let C$[-\pi,\pi]$ be the real vector space of all real-valued continuous functions on the interval $[-\pi,\pi]$ with the inner product:

$$(f(x),g(x))=\int_{-\pi}^{\pi} f(x)g(x)dx$$

(A) Give the Schwarz inequality with respect to this inner product.

(B) Find the angle between 1 and x^2.

(C) Consider the subspace V_0=span$\{1,x,x^2\}$, find an orthonormal basis for V_0.

(D) Find the orthogonal projection of sin x onto the subspace V_0.

【分析】本題(A)屬於題型09A，請參閱綜線CH9定理9.

本題(B)屬於題型09A，請參閱綜線CH9定義4.

本題(C)屬於題型09C, 請參閱綜線CH9範例18.

本題(D)屬於題型09B, 請參閱綜線CH9定理12.

本題(C)的計算過程會用到integration-by-part:

$$\int x\sin x\, dx = -\int x\, d\cos x = -[\, x\cos x - \int \cos x\, dx\,]$$

另外, 知道下列事實可增快計算速度:

若$g(x)$為奇函數, $h(x)$為偶函數, 則

$$\int_{-\pi}^{\pi} g(x)\, dx = 0, \qquad \int_{-\pi}^{\pi} h(x)\, dx = 2\int_{0}^{\pi} h(x)\, dx$$

【解】

(A) $\left(\int_{-\pi}^{\pi} f(x)g(x)dx\right)^2 \le \int_{-\pi}^{\pi} (f(x))^2 dx \int_{-\pi}^{\pi} (g(x))^2 dx$

(B) $\langle 1, x^2 \rangle = \dots = (2/3)\pi^3$, $\langle 1,1 \rangle = \dots = 2\pi$, $\langle x^2, x^2 \rangle = \dots = (2/5)\pi^5$.

$\theta = \cos^{-1}(\langle 1, x^2 \rangle / (\langle 1,1 \rangle^{1/2} \langle x^2, x^2 \rangle^{1/2})) = \cos^{-1}(5^{1/2}/3)$

(C) 以Gram-Schmidt process由 $1, x, x^2$ 導出W的正交基底:

$f_1(x)=1, \qquad \langle f_1, f_1 \rangle = \langle 1, 1 \rangle = \int_{-\pi}^{\pi} 1\, dx = 2\pi$

$\langle x, f_1 \rangle = \langle x, 1 \rangle = \int_{-\pi}^{\pi} x\, dx = 0$

$f_2(x) = x - (\langle x, f_1 \rangle / \langle f_1, f_1 \rangle) f_1 = x. \qquad \langle f_2, f_2 \rangle = \int_{-\pi}^{\pi} x^2\, dx = 2\pi^3/3$

$\langle x^2, f_1 \rangle = \int_{-\pi}^{\pi} x^2\, dx = 2\pi^3/3\,. \qquad \langle x^2, f_2 \rangle = \int_{-\pi}^{\pi} x^3\, dx = 0$

$f_3(x) = x^2 - (\langle x^2, f_1 \rangle / \langle f_1, f_1 \rangle) f_1 - (\langle x^2, f_2 \rangle / \langle f_2, f_2 \rangle) f_2 = x^2 - \pi^2/3\,.$

$\langle f_3, f_3 \rangle = \int_{-\pi}^{\pi} (x^2 - (\pi^2/3))^2 dx = 8\pi^5/45$

所求之orthonormal basis為$\{ (2\pi)^{-1/2}, (2\pi^3/3)^{-1/2}x, (8\pi^5/45)^{-1/2}(x^2 - \pi^2/3) \}$

(D) $\langle \sin x, f_1\rangle = \int_{-\pi}^{\pi} \sin x\, dx = 0$,　$\langle \sin x, f_2\rangle = \int_{-\pi}^{\pi} x\sin x\, dx = 2\pi$

$\langle \sin x, f_3\rangle = \int_{-\pi}^{\pi} (x^2-\pi^2/3)\sin x\, dx = 0$

∴ 所求之projection爲:

$(\langle \sin x, f_1 \rangle / \langle f_1, f_1 \rangle) f_1 + (\langle \sin x, f_2 \rangle / \langle f_2, f_2 \rangle) f_2 + (\langle \sin x, f_3 \rangle / \langle f_3, f_3 \rangle) f_3$

$= ((2\pi)/(2\pi^3/3))x = (3/\pi^2)x$.

5.(5%)　【交大87資料】

(A) Let T be the linear transformation mapping R^2 into R^2, which rotates each vector by an angle θ in the counter clockwise direction, Find the matrix representation of T with respect to the standard basis.

(B) Let $B_1=\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} ; \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, $B_2=\left\{ \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} ; \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix} \right\}$

be two bases in R^2.　Find the change of basis matrix form B_1 to B_2.

【分析】本題(A)屬於題型17C. 請參閱綜線附錄D範例22.

本題(B)屬於題型06D. 請參閱綜線CH6定理33.

"the change of basis matrix"應是"the change-of-basis matrix", 是指座標變換矩陣.

【解】(A)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

(B) 爲要合於 $[v]_{B_2} = P[v]_{B_1}$, 須以B_2描述B_1.　(綜線CH6定理33)

令

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} + b \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = c \begin{bmatrix} 1/2 \\ \sqrt{3}/2 \end{bmatrix} + d \begin{bmatrix} -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$$

可解得 $a=1/2, b=-\sqrt{3}/2$， $c=\sqrt{3}/2$ ， $d=1/2$

所求為 $\begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}$ #

6. (5%) 【交大87資科】

How many inversions of the sequence 1,3,4,7,8,2,6,9,5 are there ?

【分析】本題為冷僻題，請參閱綜線CH4定義1④及CH4範例2解法二.

【解】 $f(1)=1, f(2)=3, f(3)=4, f(4)=7, f(5)=8, f(6)=2, f(7)=6, f(8)=9, f(9)=5.$

以畫圖法觀察，每個交點代表有一對(i, j)，使 $i<j$，且 $f(i)>f(j)$

可看出逆序(2,6), (3,6), (4,6), (5,6),

(4,9), (4,7), (7,9), (5,7), (5,9),

(8,9)

共有10組

7. (5%) 【交大87資科】

Calculate the rank of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & 3 & 6 \\ 1 & 2 & 3 & 14 & 32 \\ 4 & 5 & 6 & 32 & 77 \end{bmatrix}$$

【分析】本題屬於題型08B.

【解】(細節略)　rank爲3

8. (5%)　【交大87資料】

Let $A=\begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 3 & 1 & -2 \\ 3 & -2 & 4 \\ -3 & 5 & 1 \end{bmatrix}$. Compute AB–BA.

【分析】本題屬於題型02A, 爲基本矩陣運算.

【解】由矩陣乘法及減法算出

$$AB-BA=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 2 & -4 & 0 \end{bmatrix}$$

9.(5%)　【交大87資料】

Let $\begin{bmatrix} 1 & 3 & -5 & 7 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$, Find a_{12}.

【分析】本題屬於題型03D. 做一半就可知道答案.

【解】(細節略)　–3

10.(5%) 【交大87資科】

Find the coordinate of (1, 2, 1, 1) if the basis is E_1=(1, 1, 1, 1), E_2=(1, 1,–1,–1), E_3=(1,–1, 1,–1), E_4=(1,–1,–1, 1).

【分析】本題屬於題型06D.

【解】 設$(1,2,1,1)=x_1E_1+x_2E_2+x_3E_3+x_4E_4$.

解方程式可得 $x_1=5/4$, $x_2=1/4$, $x_3=-1/4$, $x_4=-1/4$.

∴ 所求爲 $[\ 5/4 \quad 1/4 \quad -1/4 \quad -1/4\ ]^T$