逢甲大學87資工所

1.2.3.4.5 (離散數學)(各10%)

6. (20%) 【逢甲87資工】

True or False (2% each problem)

Let A and B be two distinct $n\times n$ matrices unless other explicit declarations.

(1) If $AB=I$ then $BA=I$

(2) If B is nonsingular then AB must be nonsingular.

(3) The (2,3) entry of A^{-1} is 3/4 where $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

(4) $\det(A+B)=\det(A)+\det(B)$

(5) If $\det(A)$ is zero, then the column vectors of A are linearly independent.

(6) Let $W=\{(a, 2) \mid a \text{ is a real}\}$ with addition and scalar multiplication defined in the usual way. The set W with the operations of addition and scalar multiplication is a vector space.

(7) If $\det(A)=2$ and $\det(B)=3$, then $\det(2AB)=12$.

(8) Any matrix has an LU-factorization where L is lower triangular and U is unit upper triangular.

(9) $\dim N(A^T) = \dim N(A)$

(10) It is impossible for a matrix to have the vector (3, 1, 2) in its row space and $(2, 1, 1)^T$ in its nullspace.

【分析】本題(1)屬於題型02A. 請參閱綜線CH3定理19, CH8定理17.
本題(2)屬於題型02A. 請參閱綜線CH2定理12.

本題(3)屬於題型03D.

本題(4)屬於題型04A. 請參閱綜線CH4定理7.

本題(5)屬於題型08E. 請參閱綜線CH8定理17.

本題(6)屬於題型05B.

本題(7)屬於題型04A.

本題(8)屬於題型03E. 請參閱綜線CH3定理27.

若列運算過程做過列對調, 這矩陣就未必有LU分解. 最簡單的實例爲

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

本題(9)屬於題型08E. 請參閱綜線CH8定理8, CH8定理13. 注意本題是方陣.

本題(10)屬於題型11C. 請參閱綜線CH11定理23.

【解】(1) True. (此因兩矩陣都是$n\times n$) (綜線CH8定理17)

(2) False. 例如取$A=O$, $B=O$, 則B可逆, AB不可逆.

(3) False. 經列運算解, 應是 $-3/4$.

(4) False. 例如$A=B=I_2$, 則$\det(A+B)=\det(2I_2)=4$, 而$\det A+\det B=2\det I_2=2$

(5) False. 剛好講反了, $\det A=0$導致A的行線性相關. (CH6定理14)

(6) False. W不具封閉性,

例如$(0,2)\in W$, $(1,2)\in W$, 但$(0,2)+(1,2)=(1,4)\notin W$.

(7) False. $\det(2AB)=2^n\det A\det B=2^n\cdot 2\cdot 3$

(8) False. 例如

$$\begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix} = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

導致 $0 = a\cdot 1 + 0\cdot 0$, $1 = a\cdot d + 0\cdot 1$

此二式爲a,d的矛盾方程式.

(9) True. (本題用到方陣的條件)

$\dim N(A^T) = n-\text{rank}(A^T)= n-\text{rank}(A)= \dim N(A)$ (CH8定理8, CH8定理15)

(10) True.

$(3,1,2)(2,1,1)^T=6+1+2\neq 0.$ (CH10定理23)

假設$(3,1,2)\in RSP(A)$, $(2,1,1)^T\in Ker(A)$,

則可取u使$(3,1,2)=uA$,

於是$(3,1,2)(2,1,1)^T=uA(2,1,1)^T=uo=0$

7. (10%) 【逢甲87資工】

Given $A=\begin{bmatrix}1&2&4\\2&1&3\\1&0&2\end{bmatrix}$, $B=\begin{bmatrix}1&2&4\\2&1&3\\2&2&6\end{bmatrix}$, $C=\begin{bmatrix}1&2&4\\0&-1&-1\\2&2&6\end{bmatrix}$,

(a) Find an elementary matrix E such that $EA=B$. (5%)

(b) Is C row equivalent to A ? Explain. (5%)

【分析】本題(a)屬於題型03E. 本題(b)請參閱綜線CH3定理4d.

【解】(a) A將第一列的1倍加入第三列就得到B, 所以

$$E=\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix}$$

(b) No. 化到列簡化梯形時, 兩者不相等.

8. (10%) 【逢甲87資工】

Given $x_1=(1, 1, 1)^T$ and $x_2=(3, -1, 4)^T$.

(a) Do x_1 and x_2 span R^3 ? Explain. (5%)

(b) Find a third vector x_3 that will extend the set $\{x_1, x_2\}$ to a basis for R^3. (5%)

【分析】本題屬於題型06B. 本題(a)請參閱綜線CH6定理18, 本題(b)請參閱綜線CH6範例24c

【解】(a) No.

$\dim(\text{span}\{x_1, x_2\}) \leq \#\{x_1, x_2\} = 2$ (CH6定理18)

而$\dim(\mathbb{R})=3$

$\therefore$ span$\{ x_1, x_2 \}\neq\mathbb{R}$

(b) 經列運算,

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -4 & 1 \end{bmatrix}$$

$\therefore$ 可取$x_3=(0, 0, 1)^{T}$.

9. (10%) 【逢甲87資工】

Reduce matrix A to Jordan form, where $A=\begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}$.

【分析】本題屬於題型15B.

【解】 (細節略)

$\det(A-xI)= -(x-4)^2(x+2)$

解 $(A-4I)v=o$ 可得eigenvector $v_1=[1, -1, 1]^{T}$,

再解 $(A-4I)v=v_1$ 可得generized eigenvector $v_2=[1, 0, 0]^{T}$,

解 $(A+2I)v=o$ 可得eigenvector $v_3=[-1, 1, 1]^{T}$,

令 $P=\begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, 則 $P^{-1}AP=\begin{bmatrix} 4 & 0 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$