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1. (20%) 【北科87電通】

Answer the following statements as true (T) or false (F).

(a) Let L: $R^n \to R^m$ be a linear transformation. Then there exists a unique $m\times n$ real matrix A such that $L(x)=Ax$ for x in R^n.

(b) For a linear system of three equations and four unknown variables, we should obtain multiple solutions.

(c) Let W_1 and W_2 be subspaces of a vector space V. Then $W_1 \cup W_2$ is also a subspace of V.

(d) The zero vector is a linear combination of any non-empty set of vectors.

(e) An $n\times n$ real matrix A is singular if and only if 0 is an eigenvalue of A.

(f) If v_1, v_2, ... and v_p are vectors in a nonzero finite-dimensional vector space V, and $S=\{v_1, v_2, ...v_p\}$. If S is a linear independent set, then S is a basis for V.

(g) The rank of a matrix equals the number of nonzero rows.

(h) If a linear system $Ax=b$ has more than one solution, then so does the system $Ax=o$.

(i) If A is an orthogonal matrix, then A is symmetrix.

(j) The sum of two eigenvectors of a matrix A is also an eigenvector of A.

【分析】本題(a)屬於題型07B, 請參閱綜線CH7範例5, CH7定理6.

本題(b)屬於題型03B, 請參閱綜線CH3定理10.

本題(c)屬於題型05B, 請參閱綜線CH5定理24.

本題(d)屬於題型05A, 請參閱綜線CH5定義9.

本題(e)屬於題型12A, 請參閱綜線CH14定理2b或CH16定理12.

本題(f)屬於題型06B, 請參閱綜線CH6定義16.

本題(g)屬於題型08B, 請參閱綜線CH6定理23

本題(h)屬於題型03B, 請參閱綜線CH3定理11a

本題(i)屬於題型02A, 請參閱綜線附錄D範例22.

本題(j)屬於題型12A, 請參閱綜線CH12定理3.

【解】(a) True.

設$\mathbb{R}^n$的標準基底爲$\{e_1, e_2, ..., e_n\}$，則A的第j行爲$T(e_j)$. (綜線CH7定理6)

(b) False.

例如下列方程組無解:

$x_1+2x_2+3x_3+4x_4=5$,

$2x_1+4x_2+6x_3+8x_4=10$,

$3x_1+6x_2+9x_3+12x_4=16$.

(c) False.

例如$V=\mathbb{R}^2$, $W_1=X$軸, $W_2=Y$軸.

(d) True.

由集合中任取有限多個向量，並各自配上純量0，得出之線性組合即爲零向量.

(e) True.

A singular(不可逆) $\Longleftrightarrow$ det$A$=0 $\Longleftrightarrow$ det($A-0I$)=0 $\Longleftrightarrow$ 0爲A的eigenvalue.

(f) False.

例如$V=\mathbb{R}^3$, $p=2$, $v_1=(1,0,0)$, $v_2=(0,1,0)$. S獨立但不是V的基底.

[討論] 若添加dim$V=p$，則本題爲True. (綜線CH6定理22)

(g) False.

例如 $\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ 有兩個非零列，但rank爲1.

(h) True.

若$Av_1=b$, $Av_2=b$, $v_1\neq v_2$.

則 $A(v_1-v_2)=o$, $Ao=o$，且$v_1-v_2\neq o$.

(i) False.

例如 $\begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix}$ orthongonal，但不symmetric.

(綜線CH2定義25)

(j) False. 例如

$$A=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix},\ v_1=\begin{bmatrix}1\\ 0\end{bmatrix},\ v_2=\begin{bmatrix}0\\ 1\end{bmatrix},$$

則 v_1, v_2都是A的eigenvector, 但v_1+v_2不是A的eigenvector. (綜線CH12定義1)

2. (15%) 【北科87電通】

A real symmetric matrix is called positive definite if all of its eigenvalues are positive. Prove that a real symmetric matrix A is positive definite if and only if there exists a nonsingular matrix P such that $P^TP=A$.

【分析】本題屬於題型13D, 請參閱綜線CH13定理18.

【解】[if-part] 請參閱綜線CH10定理26. 此處不再重複.

[only-if-part]

∵ A爲real symmetric

∴ ∃ real orthogonal matrix U, 使 $U^{-1}AU=\mathrm{diag}(\lambda_1,\dots,\lambda_n)$ (綜線CH13定理15)

而已知各λ_i皆爲正數, ∴ 可令σ_i爲λ_i的正平方根.

$A=U\mathrm{diag}(\lambda_1,\dots,\lambda_n)U^{-1}=U\mathrm{diag}(\sigma_1^2,\dots,\sigma_n^2)U^T=U\mathrm{diag}(\sigma_1,\dots,\sigma_n)\mathrm{diag}(\sigma_1,\dots,\sigma_n)U^T$

再取$P=\mathrm{diag}(\sigma_1,\dots,\sigma_n)U^T$即得$A=P^TP$.

3. (10%) 【北科87電通】

Let A be an $m\times n$ real matrix with rank(A)=n, and B be an $m\times 1$ real vector. Then derive and find a vector x in R^n to minimize the norm $\|B-Ax\|$. Express x by a matrix form in terms of A, B, A^T etc..

【分析】本題屬於題型09E.

【解】請參閱綜線CH9定理21a.

4. (10%) 【北科87電通】

If $A=\begin{bmatrix}-2 & 3\\ -6 & 7\end{bmatrix}$; $X(t)=\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$; $X''(t)=\begin{bmatrix}x_1''(t)\\ x_2''(t)\end{bmatrix}$; $G(t)=\begin{bmatrix}2+e^{3t}\\ 2+2e^{3t}\end{bmatrix}$;

then solve the system $X''(t)=AX(t)+G(t)$ by the matrix method.

【分析】本題屬於題型16E. 請參閱綜線CH16範例10, CH16範例10a.

【解】A經對角化化爲PDP^{-1}, 其中

$$P=\begin{bmatrix}1&1\\2&1\end{bmatrix},\quad \Lambda=\begin{bmatrix}4&0\\0&1\end{bmatrix}.$$

令$Z(t)=P^{-1}X(t)=[\,z_1(t), z_2(t)]^T$，　又令$H(t)=P^{-1}G(t)$.

原方程式左乘P^{-1}得 $P^{-1}(d^2/dx^2)X(t)=P^{-1}AX(t)+P^{-1}G(t)$,

即 $(d^2/dx^2)Z(t)=\Lambda Z(t)+H(t)$,

即

$$\begin{bmatrix}z_1''(t)\\z_2''(t)\end{bmatrix}=\begin{bmatrix}4&0\\0&1\end{bmatrix}\begin{bmatrix}z_1(t)\\z_2(t)\end{bmatrix}+\begin{bmatrix}e^{3t}\\2\end{bmatrix}$$

乘開得 $z_1''(t)-4z_1(t)=e^{3t}$,　　$z_2''(t)-z_2(t)=2$

分別解此二方程式得

$z_1(t)=C_1e^{2t}+C_2e^{-2t}+(1/5)e^{3t}$,　$z_2(t)=C_3e^{t}+C_4e^{-t}-2$.　(請自行參閱微分方程式的書)

$\therefore\ X(t)=PZ(t)=P\,[\,z_1(t), z_2(t)]^T= ....$

$=[\,C_1e^{2t}+C_2e^{-2t}+C_3e^{t}+C_4e^{-t}+(1/5)e^{3t}-2,\quad 2C_1e^{2t}+2C_2e^{-2t}+C_3e^{t}+C_4e^{-t}+(2/5)e^{3t}-2\,]^T$

即 $x_1(t)=C_1e^{2t}+C_2e^{-2t}+C_3e^{t}+C_4e^{-t}+(1/5)e^{3t}-2$

$x_2(t)=2C_1e^{2t}+2C_2e^{-2t}+C_3e^{t}+C_4e^{-t}+(2/5)e^{3t}-2$

(代回原方程式驗算無誤)

5. (15%)　【北科87電通】

Let A, B, C and D be $n\times n$ matrices with A invertible.

(a) Find matrices X and Y to satisfy the following matrices equation

$$\begin{bmatrix}A&B\\C&D\end{bmatrix}=\begin{bmatrix}I&O\\X&I\end{bmatrix}\begin{bmatrix}A&B\\O&Y\end{bmatrix}$$

and then show that

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A)\det(D-CA^{-1}B)$$

(b) Under what condition, then we can say that

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD-CB)$$

【分析】本題屬於題型04E. 請參閱綜線CH4定理21

【解】(a) 由矩陣塊狀乘法乘開, 原式即:

$A=A$, $B=B$, $C=XA$, $D=XB+Y$,

此等價於 $X=CA^{-1}$, $Y=D-CA^{-1}B$.

由原矩陣式兩邊取行列式得

$$\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det\begin{bmatrix} I & O \\ X & I \end{bmatrix} \det\begin{bmatrix} A & B \\ O & Y \end{bmatrix} \quad \text{(綜線CH4定理6)}$$

$=\det I \det I \det A \det Y$ (綜線CH4定理20)

$= \det A \det Y$

$= \det A \det(D-CA^{-1}B)$

(b) 只須$AC=CA$, 則上式可繼續化簡如下:

$= \det(A(D-CA^{-1}B))$ (綜線CH4定理6)

$= \det(AD-ACA^{-1}B) = \det(AD-CAA^{-1}B) = \det(AD-CB)$

6. (15%)　【北科87電通】

$$\text{If } A = \begin{bmatrix} -1 & 4 & -2 \\ -3 & 4 & 0 \\ -3 & 1 & 3 \end{bmatrix}$$

(a) Find the matrices P and D, let $A=PDP^{-1}$.

(b) Let $B=5I-3A+A^2-A^3$, show that B is also diagonalizable and find the eigenvalues of B.

【分析】本題(a)屬於題型12C. 本題(b)屬於題型16A.

本題計算過份煩雜, 此種考題以在最短耗時下求取最高得分為原則.

【解】

(a) $\det(A-xI)=-(x-3)(x^2-3x+14)=(x-3)(x-r_1)(x-r_2)$, $r_1, r_2=(3\pm\sqrt{47}\,i)/2$

由$(A-3I)v=o$可解得3的eigenvector $u=[-1\quad -3\quad 4]^T$.

由$(A-r_1I)v=o$可解得r_1的eigenvector v_1. (實際計算略)

由$(A-r_2I)v=o$可解得r_2的eigenvector v_2. (實際計算略)

將u, v_1, v_2排成矩陣P, 並令$D=\text{diag}(3, r_1, r_2)$, 則$A=PDP^{-1}$.

(b) $B=5I-3A+A^2-A^3=5I-3(PDP^{-1})+(PDP^{-1})^2-(PDP^{-1})^3$

$=5(PIP^{-1})-3(PDP^{-1})+(PD^2P^{-1})-(PD^3P^{-1})$

$=P(5I-3D+D^2-D^3)P^{-1}$

$=P\,(\,5\text{diag}(1,1,1)-3\text{diag}(3, r_1, r_2)+\text{diag}(3, r_1, r_2)^2-\text{diag}(3, r_1, r_2)^3\,)\,P^{-1}$

$=P\,(\,5\text{diag}(1,1,1)-3\text{diag}(3, r_1, r_2)+\text{diag}(3^2, r_1^2, r_2^2)-\text{diag}(3^3, r_1^3, r_2^3)\,)\,P^{-1}$

$=P\,(\,\text{diag}(5-3\cdot 3+3^2-3^3,\ 5-3r_1+r_1^2-r_1^3,\ 5-3r_2+r_2^2-r_2^3)\,)\,P^{-1}$

∴ B可對角化, 且其eigenvalues為

-22, $5-3r_1+r_1^2-r_1^3$, $5-3r_2+r_2^2-r_2^3$

7. (15%) 【北科87電通】

The singular value decomposition of any m by n matrix A is defined as:

$A=U\Sigma V^T$=(orthogonal matrix) (diagonal matrix)(orthogonal matrix).

[The columns of U (m by m) are eigenvectors of AA^T, and the columns of V (n by n) are eigenvectors of A^TA. The r ($r<m$ and $r<n$) singular values on the diagonal of Σ (m by n) are σ_1, σ_2, ...and σ_r]

Then, the pseudo-inverse of A is: $A^+=V\Sigma^+U^T$. The r diagonal values of Σ^+ are $1/\sigma_1$, $1/\sigma_2$, ..., and $1/\sigma_r$.

(a) Find the eigenvalues of AA^T and A^TA.

(b) If rank(A)=n, show that $A^+=(A^TA)^{-1}A^T$.

(c) Under what conditions, A^+ will be equal to A^{-1} ?

【分析】本題屬於題型13F. 請參閱綜線CH13定理28.

本題之定義漏列“各singular value皆正”的條件.

pseudo-inverse較冷僻, 但依所給的定義, 本題仍可利用普通定理解出.

【解】(a) $\Sigma\Sigma^{T}=\text{diag}(\sigma_1^2, \sigma_2^2, ..., \sigma_r^2, 0, 0, ..., 0)$ ($m\times m$矩陣)

$\Sigma^{T}\Sigma=\text{diag}(\sigma_1^2, \sigma_2^2, ..., \sigma_r^2, 0, 0, ..., 0)$ ($n\times n$矩陣)

$AA^{T}=(U\Sigma V^{T})(U\Sigma V^{T})^{T}=U\Sigma V^{T}V\Sigma^{T}U^{T}=U\Sigma\Sigma^{T}U^{T}=U\Sigma\Sigma^{T}U^{-1}$.

$\therefore$ AA^{T}的eigenvalues爲$\sigma_1^2, \sigma_2^2, ..., \sigma_r^2, 0, 0, ..., 0$ (共$m-r$個0)

$A^{T}A=(U\Sigma V^{T})^{T}(U\Sigma V^{T})=V\Sigma^{T}U^{T}U\Sigma V^{T}=V\Sigma^{T}\Sigma V^{T}=V\Sigma^{T}\Sigma V^{-1}$.

$\therefore$ $A^{T}A$的eigenvalues爲$\sigma_1^2, \sigma_2^2, ..., \sigma_r^2, 0, 0, ..., 0$ (共$n-r$個0)

(b) 當rank$A=n$時,

$A^{T}A=V\Sigma^{T}\Sigma V^{-1}=V\text{diag}(\sigma_1^2, \sigma_2^2, ..., \sigma_r^2)V^{-1}$, 爲可逆矩陣,

且 $(A^{T}A)^{-1}=V\text{diag}(\sigma_1^{-2}, \sigma_2^{-2}, ..., \sigma_r^{-2})V^{-1}$.

$(A^{T}A)^{-1}A^{T}=V\text{diag}(\sigma_1^{-2}, \sigma_2^{-2}, ..., \sigma_r^{-2})V^{-1}(U\Sigma V^{T})^{T}$

$=V\text{diag}(\sigma_1^{-2}, \sigma_2^{-2}, ..., \sigma_r^{-2})V^{T}V\Sigma^{T}U^{T}$

$=V\text{diag}(\sigma_1^{-2}, \sigma_2^{-2}, ..., \sigma_r^{-2})\Sigma^{T}U^{T}$

$=V\Sigma^{+}U^{T}=A^{+}$

(c) 當A可逆時, 才有所謂的A^{-1}.

此時$m=n=r$. (綜線CH8定理17)

$\Sigma=\text{diag}(\sigma_1, \sigma_2, ..., \sigma_r)$,

$\therefore$ $\Sigma^{-1}=\text{diag}(\sigma_1^{-1}, \sigma_2^{-1}, ..., \sigma_r^{-1})$

$A^{-1}=(U\Sigma V^{T})^{-1}=(U\Sigma V^{-1})^{-1}=V\Sigma^{-1}U^{-1}=V\Sigma^{+}U^{T}=A^{+}$