大同工學院87資工所

1 (8%) 【大同87資工】

A and B are invertible (non-singular). What are the inverses of the following matrices if they are invertible ? (a) A^2 (b)$A+B$ (4%)

Show that $A^{-1}(A+B)B^{-1}=A^{-1}+B^{-1}$. (4%)

【分析】本題屬於題型02A. (b)請參閱綜線CH2習題12.3.

【解】第一部份:

(a) $(A^2)^{-1}=(AA)^{-1}=(A^{-1})(A^{-1})=(A^{-1})^2$.

(b) 由本題第二部份 $A^{-1}(A+B)B^{-1}=A^{-1}+B^{-1}$

因A, B, $A+B$皆可逆, 所以$(A^{-1}+B^{-1})$也可逆. (CH2定理12)

左乘A並右乘B得 $(A+B)=A(A^{-1}+B^{-1})B$.

$\therefore$ $(A+B)^{-1}=B^{-1}(A^{-1}+B^{-1})^{-1}A^{-1}$

以下證第二部份:

$A^{-1}(A+B)B^{-1}=(A^{-1}A+A^{-1}B)B^{-1}=(I+A^{-1}B)B^{-1}=IB^{-1}+A^{-1}BB^{-1}=B^{-1}+A^{-1}=A^{-1}+B^{-1}$

【討論】通常是將$(A^{-1}+B^{-1})^{-1}$表示成$B(A+B)^{-1}A$.

本題將$(A+B)^{-1}$表示成$B^{-1}(A^{-1}+B^{-1})^{-1}A^{-1}$是化簡爲繁, 並不合理.

2 (10%) 【大同87資工】

Compute the condition number of the following matrix (6%)

$$A = \begin{bmatrix} 1 & 2 \\ 1.0001 & 2 \end{bmatrix}$$

In addition to large condition number, what are the characteristics of an ill-conditioned system in numerial computation ? (4%)

【分析】本題屬於題型17A. 請參閱綜線附錄B定義25, 附錄B定義30.

【解】令r=0.0001,

$$A^{H}A=\begin{bmatrix}1 & 1+r\\ 2 & 2\end{bmatrix}\begin{bmatrix}1 & 2\\ 1+r & 2\end{bmatrix}=\begin{bmatrix}2+2r+r^2 & 4+2r\\ 4+2r & 8\end{bmatrix}$$

$\det(A^{H}A-xI)=x^2-(10+2r+r^2)x+4r^2=(x-\mu_1)(x-\mu_2)$. $\mu_1\ge\mu_2$.

$D=((10+2r+r^2)^2-16r^2)^{1/2}=(\ 100+40r+8r^2+4r^3+r^4)^{1/2}$

$\mu_1=((10+2r+r^2)+D)/2$, $\mu_2=((10+2r+r^2)-D)/2$

$\mu_1/\mu_2=(\ (10+2r+r^2)+D\ )\ /\ (\ (10+2r+r^2)-D\ )$

$=(\ (10+2r+r^2)+D\ )^2\ /\ (\ (10+2r+r^2)^2-D^2\ )$ (分母有理化)

$=(\ (10+2r+r^2)+D\ )^2\ /\ (\ 16r^2\ )$

$\text{cond}(A)=(\mu_1/\mu_2)^{1/2}$ (綜線附錄B定理27)

$=(\ (10+2r+r^2)+D\ )\ /\ (\ 4r\ )$

$\fallingdotseq(\ (10+2r)+(10+2\ r)\ /\ (\ 4r\ )$ (相加時忽視r^2項)

$=5r^{-1}+1=50001$

因cond(A)的數值大，計算時的誤差會很大.

3 (8%) 【大同87資工】

Find the rank and the four eigenvalues of the following checkboard matrix:

$$A=\begin{bmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{bmatrix}$$

Which eigenvectors correspond to the non-zero eigenvalues ?

【分析】本題屬於題型12C. rank的部份見題型08B

所謂checkboard是說它像西洋棋的棋盤，並無其它意義.

【解】經列運算化成梯形後有兩個非零列，∴ rank$A$=2

$\det(A-xI)=...=x^2(x-2)(x+2)$，∴ eigenvalue爲0,0,2,–2

解 $(A-0I)v=o$ 可得0的eigenvectors爲 $[-s\quad -t\quad s\quad t]^T$， s,t不全爲零.

【討論】本題之對角化爲 $P^{-1}AP=\text{diag}(0,0,2,-2)$，而

$$P=\begin{bmatrix} -1 & 0 & 1 & -1 \\ 0 & -1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

4 (8%) 【大同87資工】

Find the line of best fit (in the least square sense) for the data (0,0), (1,1),(3,2) and (4,5). You are required to solve this problem with matrix manipulation.

【分析】本題屬於題型09E.

【解】設所求爲 $y=a+bx$, 即 $y=[1 \quad x][a \quad b]^T$,

希望 $0=[1 \quad 0][a \quad b]^T$, $1=[1 \quad 1][a \quad b]^T$, $2=[1 \quad 3][a \quad b]^T$, $5=[1 \quad 4][a \quad b]^T$

即

$$\begin{bmatrix} 0 \\ 1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

normal equation爲

即

$$\begin{bmatrix} 8 \\ 27 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

解得 $a=-1/5$, $b=11/10$

∴ 所求爲 $y=(1/10)(-2+11x)$

5 (16%) 【大同87資工】

In computer graphics, rotation about x-, y-, and z-axis are performed by pre-multiplying the following transformation matrices with the coordinate of the points respectively.

$$R_x=\begin{bmatrix}1&0&0\\0&\cos\theta&-\sin\theta\\0&\sin\theta&\cos\theta\end{bmatrix},\ R_y=\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{bmatrix},\ R_z=\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$$

Verify that these rotational matrices are orthogonal (5%)

A line segment is rotated by rotating its two end points. While orthogonal matrices are length-preserving, explain why after a series of rotations, $x'=Rx$, $R\equiv R_n \ldots R_1$, the length of the line segment might change. (4%)

Verify that the following rotational matrix is not exactly orthogonal. Apply the Gram-Schmidt orthogonalization to restore the orthogonal condition.

$$R=\begin{bmatrix}0.5&0&-0.8\\0&1&0\\0.8&0&0.5\end{bmatrix}$$

(7%)

【分析】本題屬於題型17D. 請參閱綜線附錄D範例23.

【解】讀者請自驗 $R_x^TR_x=I, R_y^TR_y=I, R_z^TR_z=I$.

線段端點經左乘orthogonal matrix. 在數値計算時, 線段長度可能因誤差而與先前的值不同.

讀者請自驗 $R^TR\neq I$.

R的三個行已經兩兩正交, 再各自除以其長度即為正交單位集

經調整後, $R=\begin{bmatrix}0.5618&0&-0.7016\\0&1&0\\0.7016&0&0.5618\end{bmatrix}$