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參考章節使用簡稱，例如綜線CH3代表廖亦德著:「綜合線性代數」第3章.
題型代表廖亦德著:「線性代數題型剖析」書中的題型.
***/

1. (50%) 【中央88資工】

True or false, with reason if true and counter example if false.

(01) If the entries of matrix A are integers, and det(A) is 1 or -1, then the entries of A^{-1} are integers. (Here, det(A) denotes the determinant of matrix A.) (5%)

(02) If the entries of A and A^{-1} are all integers, then det(A) is 1 or -1. (5%)

(03) Suppose V is a vector space of dimension 7 and W is a subspace of dimension 4. Then, every basis for W can be extended to a basis for V by adding three more vectors, (5%)

(04) and every basis for V can be reduced to a basis for W by removing three vectors. (5%)

(05) Every invertible matrix can be diagonalized. (5%)

(06) Exchanging the rows of a 2×2 matrix reverses the signs of its eigenvalues. (5%)

(07) If vectors x and y are orthogonal, and P is a projection matrix, then Px and Py are orthogonal. (5%)

(08) For any two matrices A and B with the same size, rank(A+B)≤rank(A)+rank(B) . (5%)

(09) For any two squares matrices A and B, AB and BA have the same set of eigenvalues. (5%)

(10) If A is a nonzero square matrix and A^3 is the zero matrix, then it is possible that $A-I$ is singular. (Here, I represents the identity matrix.) (5%)

【分析】本題(01)(02)屬於題型04A. 請參閱綜線CH4定理6, CH4定理17. 這兩小題較難,
須靈活運用公式才能證明
本題(03)(04)屬於題型06B.

本題(05)(06)屬於題型12B.

本題(07)屬於題型17C.

本題(08)屬於題型08C.

本題(09)屬於題型16F.

本題(10)屬於題型02B. 請參閱綜線CH2範例21.

本小題雖爲False, 但題目的要求錯誤, 應是證明而非舉反例.

【解】(01) True.

$A^{-1}=(\det A)^{-1}\mathrm{adj}(A)$,

因A是整數矩陣, 所以A的任何子行列式也都是整數,

所以adj(A)也是整數矩陣. 再由於det(A)是±1, 所以A^{-1}是整數矩陣.

(02) True.

由$AA^{-1}=I$, 取行列式得 $\det(A)\det(A^{-1})=1$

因A與A^{-1}都是整數矩陣, 所以det(A)與det(A^{-1})都是整數.

所以det(A)必須是±1.

(03) True.

W的basis是獨立集, 在V內看也是獨立集, (CH6定義8)

所以可擴張爲V的basis. (CH6定理21)

V的basis皆內含7個向量, 而原先W的basis內含4個向量, (CH6定理19)

所以是加三個向量.

(04) False.

例如$V=\mathbb{R}^7$, $W=\{(x_1,x_2,x_3,x_4,0,0,0) \mid x_1,x_2,x_3,x_4\in\mathbb{R}\}$,

$B=\{$ (1,0,0,0,0,0,1), (0,1,0,0,0,0,1), (0,0,1,0,0,0,1), (0,0,0,1,0,0,1), (0,0,0,0,1,0,1), (0,0,0,0,0,1,1), (0,0,0,0,0,0,1) $\}$

B爲V的basis, 但$B\cap W=\varnothing$, 所以不能刪成W的basis.

(05) False.

例如 $\begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$ 可逆. (CH4定理17),

但不可對角化(它已是Jordan form).

(06) False.

例如 $\begin{bmatrix} 2 & 0 \\ k & 3 \end{bmatrix}$ 的eigenvalue爲2, 3. (CH13定理8③)

列對調後 $\begin{bmatrix} k & 3 \\ 2 & 0 \end{bmatrix}$ 特徵多項式爲x^2-kx-6

取k=5, 則特徵多項式=$(x-6)(x-1)$, eigenvalue爲6, –1.

(07) False.

例如 $P=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, 爲正投影矩陣, (綜線附錄D範例7, CH11定理20)

$x=[\,1 \quad 1\,]^T$, $y=[\,-1 \quad 1\,]^T$, 則x, y正交,

但 $Px=[\,1 \quad 0\,]^T$, $Py=[\,-1\ 0\,]^T$不正交

(08) True.

此爲定理, 請參閱綜線CH8定理16自行證明.

(09) True.

此爲定理, 請參閱綜線CH16定理1自行證明.

(10) False. $A-I$必定可逆, 證明如下:

$(A-I)(-A^2-A-I)=(A-I)(-A^2-A-I)=-A^3+I=I$,

$\therefore$ $(A-I)$可逆且$(A-I)^{-1}=(-A^2-A-I)$.

2. (20%) 【中央88資工】

Let A be a 3×3 matrix that represents a rotation in $\mathbb{R}^3$.

(a) Describe a method which can find the axis and the angle of the rotation represented by A. (8%)

(b) Consider a rotation that takes vectors (x_1, x_2, x_3) into vector (x_2, x_3, x_1). Find the matrix that represents this transformation. (7%)

(c) Apply the method in (a) to the matrix you get in (b), and find the axis and the angle of the rotation given in (b). (5%)

【分析】本題屬於題型17C. 請參閱綜線附錄D定理28.

【解】將向量 (x_1, x_2, x_3) 以行矩陣 $[x_1, x_2, x_3]^T$ 表示.

(a) A必有eigenvalue 1, 解eigenvalue 1的eigenvector即爲旋轉軸.

由 $\cos(\theta)= (\mathrm{tr}(A)-1)/2$ 解出θ即爲旋轉角.　　(綜線附錄D定理28)

(b) $A\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}, A\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}, A\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\1\\0\end{bmatrix},$

$$\therefore\ A=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$$

(c) $\det(A-xI)=-x^3+1=-(x-1)(x^2+x+1)$

$$A-1I=\begin{bmatrix}-1&1&0\\0&-1&1\\1&0&-1\end{bmatrix}\sim\begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\end{bmatrix}$$

解得eigenvector $k\,[1, 1, 1]^T$, 此即旋轉軸.

$\cos(\theta)=(\mathrm{tr}(A)-1)/2=-1/2$,

$\therefore\ \theta=2\pi/3$.

3. (20%)　【中央88資工】

Let S be the subspace of $\mathbb{R}^4$ containing all vectors (x_1, x_2, x_3, x_4) with $x_1+x_2+x_3+x_4=0$ and $x_1+2x_2+3x_3+4x_4=0$

(a) Find two bases for the space S and the space $S^\perp$ (the space containing all vectors orthogonal to S) respectively. (10%)

(b) Find the projection of vector (0, 1, 2, 7) onto the space $S^\perp$.(10%)

【分析】本題(a)屬於題型11C. 請參閱綜線CH11定理23.

本題(b)屬於題型09B. 請參閱綜線CH9定理22.

【解】將向量 (x_1, x_2, x_3, x_4) 以行矩陣 $[x_1, x_2, x_3, x_4]^T$ 表示.

(a) 1° 令 $A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$, 則 $S=\ker A$. (the null space of A)

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$

解得$S=\{ [s+2t, -2s-3t, s, t]^T \mid s, t\in \mathbb{R} \}$

$=\{ s[1, -2, 1, 0]^T + t [2, -3, 0, 1] \mid s, t\in \mathbb{R} \}$

S之basis可取爲$\{ (1, -2, 1, 0), (2, -3, 0, 1) \}$

2° $S^\perp=(\ker A)^\perp=\mathrm{CSP}(A^T)$, (CSP代表column space), (CH11定理23)

S之basis可取爲$\{ (1, 1, 1, 1), (1, 2, 3, 4) \}$

(b) 令$B=A^T$, $B(B^TB)^{-1}B^Tv= ... = [-4/5, 7/5, 18/5, 29/5]^T$ (CH9定理22)

所求之投影爲 $(-4/5, 7/5, 18/5, 29/5)$

4. (10%) 【中央88資工】

Find the intersection $V\cap W$ and the sum $V+W$ if

(a) V= null space of a matrix A and W = row space of A. (5%)

(b) V= the set of symmetric 3×3 matrices and

W= the set of upper triangular 3×3 matrices. (5%)

【分析】本題(a)屬於題型11C. 請參閱綜線CH11定理23.

本來row space中都是列向量, null space中都是行向量, 兩者不能求交集.

但有的書(G. Strang)不區分列向量與行向量. 此時的row space of A其實是$(\mathrm{RSP}A)^T$, 也就是$\mathrm{CSP}(A^T)$. (綜線CH5定義16要訣)

本題(b)屬於題型05C. 請參閱綜線CH5範例26, 26a.

【解】(a) ∵ V, W互爲orthogonal complement,

∴ $V\cap W = \{ o \}$,

$V+W=\mathbb{R}^n$, n是A的寬度.

(b) $V\cap W$ ={ A | A is a symmetric upper triangle 3×3 matrix }

={ A | A is a 3×3 diagonal matrix }

$\dim(V+W)=\dim V+\dim W-\dim(V\cap W)=6+6-3=9=\dim(\mathbb{R}^{3\times3})$

$\therefore\ V+W=\mathbb{R}^{3\times3}$

【討論】(b)中 $V+W=\mathbb{R}^{3\times3}$ 的部份也可由下式推出.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & I \end{bmatrix} = \begin{bmatrix} a & d & g \\ d & e & h \\ g & h & i \end{bmatrix} + \begin{bmatrix} 0 & b-d & c-g \\ 0 & 0 & f-h \\ 0 & 0 & 0 \end{bmatrix}$$