東華大學88資工所

科目: 離散數學與線性代數(各50%)

參考章節使用簡稱，例如綜線CH3代表廖亦德著:「綜合線性代數」第3章.
題型代表廖亦德著:「線性代數題型剖析」書中的題型.
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1. (9%), 2. (9%), 3. (10%), 4. (7%), 5.(5%), 6. (10%)　　離散數學

7.(14%)　　【東華88資工】

Suppose that the dat (t_1, y_1), $(t_2\ y_2)$, ..., (t_m, y_m) are plotted as points in the plane, there exists an linear relationship between y and t, say, $y=ct+d$, please find the constants c and d so that the line $y=ct+d$ best fit the data collected. (10%)

What is the error E ? (4%)

(The data collected is (1, 2), (2, 3), (3, 5) and (4, 7))

(hint: $E=\sum_{i=1}^{m}(y_i-ct_i-d)^2$)

【分析】本題屬於題型09E. 參閱綜線CH9範例21c

【解】欲求d, c,使 $y_i \approx d + c\, t_i$，即求下列方程式之最佳近似解:

$$\begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} d \\ c \end{bmatrix}$$

左乘係數矩陣之transpose後得出normal equation:

$$\begin{bmatrix} 17 \\ 51 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} d \\ c \end{bmatrix}$$

解得 d=0, c=17/10,

∴ 所求直線為 y=(17/10)t+0.

所求之誤差為 $\left\| \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 17/10 \end{bmatrix} \right\|^2 = 3/10$

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8. (12%)　【東華88資工】

Prove that there exist a linear transformation $T:\mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1, 1)=(1, 0, 2)$ and $T(2, 3)=(1, -1, 4)$. (8%)

What is $T(8, 11)$? (4%)

【分析】本題屬於題型07A. 參閱綜線CH7定理3, CH7範例4

【解】(a) 令 $(x, y)=\alpha(1, 1)+\beta(2, 3)$,

可解得 $\alpha=3x-2y$, $\beta=-x+y$.

由線性條件, $T(x, y)=\alpha T(1, 1)+\beta T(2, 3)$

$= (3x-2y)(1, 0, 2) + (-x+y)(1, -1, 4)$

$= (2x-y, x-y, 2x)$

可驗證T合於線性條件, 且 $T(1, 1)=(1, 0, 2)$, $T(2, 3)=(1, -1, 4)$.

(b) 由前述公式, $T(8, 11)=(5, -3, 16)$

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9. (8%)　【東華88資工】

Is the matrix $A=\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ similar to matrix $B=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$?

【分析】本題屬於題型07D. 參閱綜線CH15定理14.

【解】$\det(A-xI)=x^2-2x+1=(x-1)^2$. $\det(B-xI)=x^2-2x+1=(x-1)^2$.

$$A-I=\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}\sim\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix},\quad \dim(\mathrm{Ker}(A-I))=2-\mathrm{rank}(A-I)=1.$$

$$B-I=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\quad \dim(\mathrm{Ker}(B-I))=2-\mathrm{rank}(B-I)=1.$$

$\therefore$ A, B的Jordan form皆爲 $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$

$\therefore$ A,B爲similar.

10. (10%) 【東華88資工】

Let A be the 3×3 matrix $\begin{bmatrix}5 & -6 & -6\\ -1 & 4 & 2\\ 3 & -6 & -4\end{bmatrix}$

(1) What is the characteristic polynomial of A ? (6%)

(2) What is the minimal polynomial of A ? (4%)

【分析】本題屬於題型16D. 參閱綜線CH16定理26.

【解】(1) $\det(A-xI)=-x^3+5x^2-8x+4=-(x-1)(x-2)^2$.

(2) $$A-2I=\begin{bmatrix}3 & -6 & -6\\ -1 & 2 & 2\\ 3 & -6 & -6\end{bmatrix}\sim\begin{bmatrix}1 & -2 & -2\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\dim(\mathrm{Ker}(A-2I))=3-\mathrm{rank}(A-2I)=2$

$\therefore$ A可對角化 (CH12定理21)

∴ minimal polynomial 爲 $(x-1)(x-2)$ (CH16定理26)

11.(6%) 【東華88資工】

Let $u=[x_1, x_2, x_3]^T$, $v=[y_1, y_2, y_3]^T$ and

$f(u, v)=5x_1y_1-4x_1y_2+3x_1y_3+2x_2y_1+5x_2y_2-2x_3y_2+x_3y_3$

find a 3×3 real matrix A such that $f(u, v)=u^TAv$.

【分析】本題屬於題型10A. 參閱綜線CH10範例2a.

【解】

取 $A=\begin{bmatrix} 5 & -4 & 3 \\ 2 & 5 & 0 \\ 0 & -2 & 1 \end{bmatrix}$