# 台灣大學88資工所



科目: 數學

1. (20%) 【台大88資工】

Let $P_2[x]$={ $a_0 + a_1x + a_2x^2$ | $a$ is a real number }, define a linear transformation $T$ from $P_2[x]$ to $P_2[x]$ by $T(f(x)) = f(x) + f'(x)$

(a) find eigenvalues of $T$.

(b) find a basis $\alpha$ of $P_2[x]$ such that $[T]_\alpha$ is a diagonalized

【分析】本題屬於題型12C. 請參閱綜線CH12範例12a.

本題不能對角化, (b)的文句也不完整, 顯然含有筆誤或打字錯誤.

【解】(a) $T(a_0 + a_1x + a_2x^2) = (a_0 + a_1x + a_2x^2) + (a_0 + a_1x + a_2x^2)'$

$= (a_0+a_1x+a_2x^2)+(a_1+2a_2x)=(a_0+a_1)+(a_1+2a_2)x+a_2x^2$

取$P_2[x]$的基底 $\beta = \{1, x, x^2\}$, 並令$A=[T]_\beta$ 則可求得

$$A=\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(CH7定義9)}$$

$\det(A-xI) = -(x-1)^3$, ∴ $T$的eigenvalues為1,1,1.

(b) $A-I=\begin{bmatrix}0&1&0\\0&0&2\\0&0&0\end{bmatrix}\sim\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$

$\dim(\ker(A-I)) = 3-\text{rank}(A-I) = 1 <$ multiplicity of 1

$\therefore$ $A$不能對角化.　(CH12定理21)

以下姑且求算其Jordan form:

$A$的eigenvector爲 $k[\,1\quad 0\quad 0\,]^T$

令$b_3=[\,1\quad 0\quad 0\,]^T$. 由 $(A-I)v=b_3$, 解得 $v=[\,t\quad 1\quad 0]^T$,

令$b_2=[\,0\quad 1\quad 0\,]^T$. 由 $(A-I)v=b_2$, 解得 $v=[\,t\quad 1\quad 1/2]^T$,

令$b_1=[\,0\quad 1\quad 1/2\,]^T$.

則 $(A-I)b_1=b_2$, $(A-I)b_2=b_3$, $(A-I)b_3=o$ ,

即 $Ab_1=b_1+b_2$, $Ab_2=b_2+b_3$, $Ab_3=b_3$.

( 爲使 $[f_1(x)]_\beta=b_1$ , $[f_2(x)]_\beta=b_2$ , $[f_3(x)]_\beta=b_3$ )

令 $f_1(x)=x+(1/2)x^2$, $f_2(x)=x$, $f_3(x)=1$,

並令 $\alpha=\{f_1(x),\ f_2(x),\ f_3(x)\}$

則 $T(f_1(x))=f_1(x)+f_2(x)$, $T(f_2(x))=f_2(x)+f_3(x)$, $T(f_3(x))=f_3(x)$

$\therefore$ $[T]_\alpha=\begin{bmatrix}1&0&0\\1&1&0\\0&1&1\end{bmatrix}$　(CH7定義9)

2. (10%)　【台大88資工】

Let $P_1[x]=\{a_0+a_1x \mid a_i \text{ is a real number}\}$, define the inner product on $P_2[x]$ as

$$\langle f, g\rangle=\int_0^1 f(t)\,g(t)\,dt.$$

Find the orthogonal projection of $h(x)=4+3x-2x^2$ on $P_1[x]$.

【分析】本題屬於題型09B. 請參閱綜線CH9定理12.

【解】由$P_1[x]$的基底$\{1, x\}$做Gram-Schmidt正交化,

可得正交基底{ 1, $x$–1/2}　　(CH9習題18.1)

$\langle 1, 1\rangle = \int_0^1 1\, dx = 1$,　　$\langle x-1/2, x-1/2\rangle = \int_0^1 (x-1/2)^2\, dx = 1/12$,

$\langle h(x), 1\rangle = \int_0^1 (4 + 3x - 2x^2)\, dx = 29/6$ ,

$\langle h(x), x-1/2\rangle = \int_0^1 (4 + 3x - 2x^2)(x-1/2) dx = 1/12$ ,

所求爲 $\dfrac{\langle h(x), 1\rangle}{\langle 1, 1\rangle}\, 1 + \dfrac{\langle h(x), x-1/2\rangle}{\langle x-1/2, x-1/2\rangle}\,(x-1/2)$

$= 29/6 + (x-1/2) = x+13/3.$　　#

【另解】設所求爲 $p(x)=a+bx$, 由投影之定義得知

$\langle h(x)-p(x), 1\rangle=0$,　　$\langle h(x)-p(x), x\rangle=0$,

即　$\langle p(x), 1\rangle=\langle h(x), 1\rangle$,　　$\langle p(x), x\rangle=\langle h(x), x\rangle$,

$\langle h(x), 1\rangle = \int_0^1 (4 + 3x - 2x^2)\, dx = 29/6$ ,

$\langle h(x), x\rangle = \int_0^1 (4x + 3x^2 - 2x^3)\, dx = 5/2$ ,

$\langle p(x), 1\rangle = \int_0^1 (a + bx)\, dx = a\int_0^1 1\, dx + b\int_0^1 x\, dx = a + (1/2)b$ ,

$\langle p(x), x\rangle = \int_0^1 (ax + bx^2)\, dx = a\int_0^1 x\, dx + b\int_0^1 x^2\, dx = (1/2)a + (1/3)b$ ,

由 $a+(1/2)b=29/6$, $(1/2)a+(1/3)b=5/2$ 可解得 $a=13/3$, $b=1$.

∴ 所求爲 $13/3+x$.

3. (20%)　【台大88資工】

(a) Define elementary matrices.

(b) Prove that every invertible matrix is a product of elementary matrices.

【分析】本題屬於題型03E.

【解】(a) 由單位矩陣經一步的基本列運算所得的矩陣稱爲elementary row matrix,
也稱爲elementary matrix. (綜線CH2定義13,定義3)

(b) 請參閱綜線CH3定理16.

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| 4. (30%) 離散數學 |
| 5. (20%) 離散數學 |