交通大學88資科所

參考章節使用簡稱，例如綜線CH3代表廖亦德著:「綜合線性代數」第3章.
題型代表廖亦德著:「線性代數題型剖析」書中的題型.
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1. (10%) 【交大88資科】

Let $A=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}2&4&6\\0&0&4\\0&0&2\end{bmatrix}$

(a) Find a basis for the row space of A. (3%)

(b) Find an orthonormal basis for the row space of A. (4%)

(c) Find a basis for the column space of A. (3%)

【分析】本題屬於題型06C.

【解】(a) 經列運算, (綜線CH3定理17)

$$A\sim\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}\begin{bmatrix}2&4&6\\0&0&4\\0&0&2\end{bmatrix}\sim\begin{bmatrix}2&4&6\\0&0&4\\0&0&2\end{bmatrix}\sim\begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}$$

$\therefore$ row space of A 的基底可取爲

$\{[1\quad 2\quad 0],\ [0\quad 0\quad 1]\}$ (綜線CH6定理23)

(b) $\because$ $[1\quad 2\quad 0]\cdot[0\quad 0\quad 1]=0$,

$\therefore$ 只須再做單位化

$\|[1 \quad 2 \quad 0]\| = \sqrt{5}$, $\|[0 \quad 0 \quad 1]\| = 1$,

∴ row space of A 的orthonormal基底可取爲

$$\{ [1/\sqrt{5}, 2/\sqrt{5}, 0], [0 \quad 0 \quad 1] \}$$

(c)

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 0 & 0 & 4 \\ -4 & -8 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \\ 2 & 4 & 6 \\ -4 & -8 & -10 \end{bmatrix}$$

由(a), A經列運算後的pivot在第1,3行,

∴ 可取A的1,3行當做 column space的基底, 即

$$\left\{ \begin{bmatrix} 0 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ -10 \end{bmatrix} \right\}$$ (綜線CH6定理24)

【討論】(b)小題若不是剛好兩向量已正交, 就要先做Gram-Schmidt process, 然後才做單位化

2. (5%) 【交大88資科】

Let $b_1=[1 \quad 1 \quad 0]^T$, $b_2=[1 \quad 0 \quad 1]^T$, $b_3=[0 \quad 1 \quad 1]^T$, and let T be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1b_1+x_2b_2+(x_1+x_2)b_3$. Find the matrix representation of T with respect to the ordered bases $\{e_1; e_2\}$ and $\{b_1; b_2; b_3\}$.

【分析】本題屬於題型07B.

本題未說明e_1, e_2是什麼，依慣例視爲標準基底，即e_1= $[1 \quad 0]^T$, e_2=$[0 \quad 1]^T$

【解】 依所給之公式，

$T(e_1)=T([1 \quad 0]^T) = b_1+b_3$

$T(e_2)=T([0 \quad 1]^T) = b_2+b_3$

∴ 所求爲 $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ (綜線CH7定義9)

【討論】可能有人會將b_1, b_2, b_3代入，求出T的公式再計算. 但這樣反而繞了一大圈.

3. (10%) 【交大88資料】

(a) Let $A=\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Are A and B similar ? Justify your answer. (5%)

(b) Let $C=\begin{bmatrix} 3 & 4 \\ 1 & 3 \end{bmatrix}$ and $D=\begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}$. Are C and D similar ? Justify your answer. (5%)

【分析】本題屬於題型07D.

【解】 (a) tr(A)=1+4=5, tr(B)=2+2=4,

∵ tr(A)≠tr(B), ∴ A, B不similar. (綜線CH7定理22)

(b) $\det(C-xI) = \ldots = x^2-6x+5=(x-5)(x-1)$

∴ C可對角化爲 diag(5, 1). (綜線CH12定理23)

$\det(D-xI) = \ldots = x^2-6x+5=(x-5)(x-1)$

∴ D可對角化爲 diag(5, 1).

∴ C, D 爲 similar. (綜線CH15定理14)

【討論】若不是剛好能對角化，就要求出它的Jordan form(不必求Jordan basis)，才能判定是否相似.

4. (15%) 【交大88資料】

Let $A=\begin{bmatrix}1 & 2\\ 2 & 1\\ 1 & -1\end{bmatrix}$ and $y=\begin{bmatrix}2\\ 3\\ 0\end{bmatrix}$. The orthogonal projection of y onto the column space of A is the vector u in the column space of A such that $y-u$ is orthogonal to all vactors in the column space of A.

(a) Find the orthogonal projection of y onto the column space of A. (10%)

(b) Solve the least-squares problem $Ax \approx y$. (5%)

【分析】本題(a)屬於題型09B, .本題(b)屬於題型09E.
由(b)可得出(a), 所以先解(b).

【解】 (b) 本題解它的normal equation $A^TAx=A^Ty$, 即

$$\begin{bmatrix}6 & 3\\ 3 & 6\end{bmatrix}x=\begin{bmatrix}8\\ 7\end{bmatrix},$$ (綜線CH9定理21a)

依列運算解出 $x=\begin{bmatrix}1\\ 2/3\end{bmatrix}$

(a) $u=Ax$ (綜線CH9定理21a,CH9定理13)

$$=\begin{bmatrix}1 & 2\\ 2 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1\\ 2/3\end{bmatrix}=\begin{bmatrix}7/3\\ 8/3\\ 1/3\end{bmatrix}$$

5. (10%) 【交大88資料】

For nonzero column vector w in $\mathbb{R}^n$, the $p\times p$ Householder matrix H_w is defined as

$$H_w = I_p - \left(\frac{2}{w^T w}\right) w w^T$$

where I_p is the $p \times p$ identity matrix. For each x, $H_w x$ equals the reflection of x about the subspace of all v orthogonal to w.

(a) Find the 2×2 matrix H_w for $w=[1 \quad 2]^T$. (5%)

(b) Find a vector w in $\mathbb{R}^2$ such that $H_w[3 \quad 4]^T=5[1 \quad 0]^T$. (5%)

【分析】本題屬於題型01A.

【解】 (a) $w^T w=[1 \quad 2][1 \quad 2]^T=5$

$$ww^T=[1 \quad 2]^T[1 \quad 2]=\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$H_w = I-(2/5)ww^T=\begin{bmatrix} 3/5 & -4/5 \\ -4/5 & -3/5 \end{bmatrix}$$

(b) $[\,3 \quad 4\,]^T - H_w[\,3 \quad 4\,]^T = [\,3 \quad 4\,]^T - 5[\,1 \quad 0\,]^T = [\,-2 \quad 4\,]^T$

由H_w的幾何意義，取$w=[\,-2 \quad 4\,]^T$ 即可.