線性代數解析—中正88資工所

廖亦德 解

[1]. (10%) 【中正88資工】

(a) Show that $W=\left\{ \begin{bmatrix} b & a \\ -a & b \end{bmatrix} \mid a, b\in R \right\}$ is a subspace of $R^{2\times 2}$ and also find a basis of W.

(b) Let $S=\{(x_1, x_2, ..., x_n)^T \mid x_1+x_2+...+x_n=0\} \subseteq R^n$, Find a basis of the orthogonal complement of S

【分析】本題(a)屬於題型06C. 請參閱綜線CH6範例23

本題(b)屬於題型11C. 請參閱綜線CH11範例24

【解】(a) 所求為

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

需證明生成及獨立 (略). (綜線CH6定義1,CH6定義9)

(b) $S=\ker[\,1, 1, ..., 1\,]$

$S^{\perp}=(\ker[\,1, 1, ..., 1\,])^{\perp}=(\mathrm{RSP}([\,1, 1, ..., 1\,]))^T$ (綜線CH11定理23)

$=\mathrm{CSP}([\,1, 1, ..., 1\,]^T)$ (綜線CH5定理16a)

$\therefore$ $\{[\,1, 1, ..., 1\,]^T\}$是$S^{\perp}$的基底

(b) [幾何解法]

S是R^n上的超平面，以$[1, 1, ..., 1]^T$為法向量, (綜線CH1定義19)

$S^{\perp}=\{k[1, 1, ..., 1]^T \mid k\in R\}$，以$\{[\,1, 1, ..., 1\,]^T\}$為基底 (綜線CH1定義20)

[2]. (10%)　　【中正88資工】

Consider the following augmented matrix of a linear system in R^n :

$$\left[\begin{array}{ccc|c} 1 & a & 3 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & a & a+3 \end{array}\right]$$

Detrmine all the possible value of a for the following cases:

(a) This linear system has infinite solutions. (4%)

(b) This linear system has no solutions. (3%)

(c) This linear system has a unique solution. (3%)

【分析】本題屬於題型03C. 請參閱綜線CH3範例9

【解】經列運算(細節略):

$$\left[\begin{array}{ccc|c} 1 & a & 3 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 3 & a & a+3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & a-2 & 1 & -1 \\ 1 & 2 & 2 & 3 \\ 0 & 1 & a-2 & a \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & -(a-3)(a-1) & -(a-1)^2 \\ 1 & 0 & -2a+6 & 3 \\ 0 & 1 & a-2 & a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2a+6 & 3 \\ 0 & 1 & a-2 & a \\ 0 & 0 & -(a-3)(a-1) & -(a-1)^2 \end{array}\right]$$

(a) 當a=1時，此方程組有無限多組解.　　(綜線CH3定理10)

(b) 當a=3時，此方程組無解.　　(綜線CH3定理10)

(c) 當$a \notin \{1,3\}$時，此方程組恰有一解.　　(綜線CH3定理10)

【補充說明】

當a=1時，繼續求解如下:

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -2a+6 & 3 \\ 0 & 1 & a-2 & a \\ 0 & 0 & -(a-3)(a-1) & -(a-1)^2 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 4 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

通解爲 $x_1=3-4t$, $x_2=1+t$, $x_3=t$, t爲任意常數.

當$a\notin\{1,3\}$時, 繼續求解如下:

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & -2a+6 & 3 \\ 0 & 1 & a-2 & a \\ 0 & 0 & -(a-3)(a-1) & -(a-1)^2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -2a+6 & 3 \\ 0 & 1 & a-2 & a \\ 0 & 0 & 1 & (a-1)/(a-3) \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2/(a-3) \\ 0 & 0 & 1 & (a-1)/(a-3) \end{array}\right]$$

解爲 $x_1=1$, $x_2=-2/(a-3)$, $x_3=(a-1)/(a-3)$.

[3]. (10%) 【中正88資工】

Given $B_1 = [\boldsymbol{v}_1, \boldsymbol{v}_2, ..., \boldsymbol{v}_n]$ to be an ordered basis of vector space V. Let $B_2=[\boldsymbol{v}_1+\boldsymbol{v}_2, \boldsymbol{v}_2+\boldsymbol{v}_3, ..., \boldsymbol{v}_{n-1}+\boldsymbol{v}_n, \boldsymbol{v}_n+\boldsymbol{v}_1]$. Show that if n is *odd* then B_2 is also an ordered basis of V, otherwise, B_2 is not an ordered basis of V (that is, n is even).

【分析】本題較難, 屬於題型06B, 相關考古題爲06B13--06B16. 請參閱綜線CH6定理27c

【解】設B_2中的向量依序命名爲$\boldsymbol{w}_1, \boldsymbol{w}_2, ..., \boldsymbol{w}_n$. 並設$n\times n$矩陣$P_n=[\,p_{ij}\,]$, 使

$$\boldsymbol{w}_j=\sum_{i=1}^{n} p_{ij}\boldsymbol{v}_i\,, \quad j=1, 2, ..., n\,.$$

則 $P_n= \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 1 & \cdots & 0 \\ & & \cdots\cdots & & & \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{bmatrix}$

對所有的n求算$\det(P_n)$:

先計算 n=5, 6的情形.

$$\det(P_5) = \begin{vmatrix} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

(對首行降階)

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$ (綜線CH4定理11)

$$= 1 - \begin{vmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$ (綜線CH4定理4a)

(對首列降階)

$$= 1 + \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 + 1 = 2$$

$$\det(P_6) = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

(對首行降階)

$$= \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= 1 - \begin{vmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{vmatrix}$$

(對首列降階)

$$= 1 - \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 - 1 = 0$$

一般情形下依此類推得知, n爲奇數(odd)時$\det(P_n)=2$, n爲偶數(even)時$\det(P_n)=0$.

當 n爲奇數時, P_n可逆, (綜線CH4定理17)

$\therefore$ B_2仍爲獨立集 (綜線CH4定理27c③)

而B_2內含n個向量, $\therefore$ 仍爲基底 (綜線CH6定理22)

當 n爲偶數時, P_n不可逆, (綜線CH4定理17)

∴ B_2不爲獨立集 (綜線CH4定理27c③)

∴ B_2不是基底 (綜線CH4定理27c③)

[4]. (10%) 【中正88資工】

(a) Let A be a singular matrix. Show that adj(A) is also singular. (5%)

(b) Let A be a square matrix with entries in C (complex number). If $AA^*=O$, show that $A=O$, where $A^*=\overline{A}^T$ and O is the zero matrix. (5%)

【分析】本題(a)屬於題型04A. 請參閱綜線CH4定理17

本題(b)屬於題型02A. 只用到矩陣乘法及複數的性質.

【解】(a) 已知A不可逆, ∴ det$A$=0. (綜線CH4定理17)

用矛盾證法證明adj(A)不可逆如下:

假設adj(A)爲可逆, 則

$A = A\ \mathrm{adj}(A)\ (\mathrm{adj}(A))^{-1}$

$= (\det A)I\ (\mathrm{adj}(A))^{-1}$ (綜線CH4定理17)

$= 0\ I\ (\mathrm{adj}(A))^{-1} = O.$

於是由定義得知 adj(A)=O (綜線CH4定義16,CH4定義10)

但O不可逆, 導致矛盾.

(b) 令$A=[a_{ij}]$, $A^*=[\ b_{jk}]$, $AA^*=[\ c_{ik}\ ]=O$

$\forall\ i,\ 0 = c_{ii} = \sum_j a_{ij}b_{ji} = \sum_j a_{ij}\,\overline{a}_{ij} = \sum_J |\ a_{ij}|^2$

∴ $a_{i1}=a_{i2}=....=a_{in}=0$

[5]. (10%) 【中正88資工】

As we known, any real symmetric matrix can be diagonalized and all its eigenvalues are real. Use this hint to show that there exists a real number k such that $A+kI$ is positive where A is real symmetrix, I is the identity matrix.

[Note: A matrix is called positive if it is symmetrix and all its eigenvalues are positive.]

【分析】本題屬於題型13D.

實數對稱矩陣必可正交對角化, 請參閱綜線CH13定理15

正定矩陣的定義與特性請參閱綜線CH10定義19及CH13定理17c.

【解】將A對角化: 令 $A=P\text{diag}(\lambda_{1,\ldots,}\lambda_n)P^{-1}$. (綜線CH12定義15)

於是 $A+kI=P\,\text{diag}(\lambda_{1,\ldots,}\lambda_n)P^{-1}+kPIP^{-1}$

$=P\,(\text{diag}(\lambda_{1,\ldots,}\lambda_n)+kI\,)P^{-1}$

$=P\,\text{diag}(\lambda_1+k_{,\ldots,}\lambda_n+k)\,P^{-1}$

取$k=\max\{|\lambda_1|_{,\ldots,}|\lambda_n|\}+1$, 則$A+kI$的eigenvalue $\lambda_1+k_{,\ldots,}\lambda_n+k$ 全是正數.

而$A+kI$仍為對稱, $\therefore$ A為正定.