台北科技大學88電腦通訊與控制研究所(乙丁組)

1. (25%) 【北科88電通】

Determine if the inverse of the square matrix A exists in the following conditions.

State your reason; or you won't get any credit.

(ⅰ) $A=\begin{bmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ & & \vdots & & \\ 1 & \cdots & 0 & 0 & 0 \end{bmatrix}$ (ⅱ) $A=\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ & & \vdots & & \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}$

(ⅲ) $A=\left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & 1 \\ 0 & 1 & \cdots & 0 & \vdots \\ & & & & \vdots \\ 0 & 0 & \cdots & 1 & 1 \\ \hline 1 & 1 & \cdots & 1 & 1 \end{array}\right]$ (ⅳ) $A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 3 & 5 \end{bmatrix}^{100}$

(ⅴ) $A=I-B$. Where $B^m=O$ (zero matrix) for some positive integer m.

【分析】本題(ⅰ--ⅳ)屬於題型04B. 請參閱綜線CH4範例9.

本題(ⅴ)屬於題型02B. 請參閱綜線CH2範例21.

【解】(ⅰ) A^{-1}存在.

$\because$ $\det(A)=\pm 1\neq 0$, $\therefore$ A可逆. (綜線CH4定理17)

(ⅱ) A^{-1}存在.

$$\det(A)=\begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ & & \vdots & & \\ 1 & 1 & 1 & \cdots & 0 \end{vmatrix}$$

(各列加入第一列)

$$=\begin{vmatrix} n-1 & n-1 & n-1 & \dots & n-1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ & & \dots\dots\dots & & \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}=(n-1)\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ & & \dots\dots\dots & & \\ 1 & 1 & 1 & \dots & 0 \end{vmatrix}$$

(第一列的–1倍加入各列)

$$=(n-1)\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ & & \dots\dots\dots & & \\ 0 & 0 & 0 & \dots & -1 \end{vmatrix}=(-1)^{n-1}(n-1)\neq 0$$

∴ A可逆.

(iii) A^{-1}存在.

$$\det A=\left|\begin{array}{cccc|c} 1 & 0 & \dots & 0 & 1 \\ 0 & 1 & \dots & 0 & . \\ & & & & . \\ & & & & . \\ 0 & 0 & \dots & 1 & 1 \\ \hline 1 & 1 & \dots & 1 & 1 \end{array}\right|$$

(最後一列的–1倍加入各列)

$$\det A=\left|\begin{array}{cccc|c} 0 & -1 & \dots & -1 & 0 \\ -1 & 0 & \dots & -1 & . \\ & & & & . \\ & & & & . \\ -1 & -1 & \dots & 0 & 0 \\ \hline 1 & 1 & \dots & 1 & 1 \end{array}\right|=\begin{vmatrix} 0 & -1 & \dots & -1 \\ -1 & 0 & \dots & -1 \\ & & & \\ -1 & -1 & \dots & 0 \end{vmatrix}$$

$$=\pm\begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ & & & \\ 1 & 1 & \dots & 0 \end{vmatrix}=\pm(n-2)\neq 0$$

∴ A可逆.

(iv) A^{-1}不存在.

$$\det(A)=\det\left(\begin{bmatrix}1&2&3\\0&1&2\\1&3&5\end{bmatrix}^{100}\right)=\left(\det\begin{bmatrix}1&2&3\\0&1&2\\1&3&5\end{bmatrix}\right)^{100}$$

$$=\left(\det\begin{bmatrix}1&2&3\\0&1&2\\0&1&2\end{bmatrix}\right)^{100}=0^{100}=0$$

∴ A不可逆.

(ⅴ) A^{-1}存在.

令$C=I+B+B^2+...+B^{m-1}$. 則

$AC=(I-B)(I+B+B^2+...+B^{m-1})=I-B^m=I.$

$CA=(I+B+B^2+...+B^{m-1})(I-B)=I-B^m=I.$

∴ A可逆且以C爲逆矩陣.

2. (15%) 【北科88電通】

Evaluate the determinant of the following $n\times n$ matrix.

$$A=\begin{bmatrix}2&1&&&\\1&2&1&&\\&1&2&1&\\&&&\ddots&\\&&&1&2\end{bmatrix}$$

【分析】本題屬於題型04B. 請參閱綜線CH4範例13.

本題與元智84工工[1]同數據. (綜線CH4習題13.3)

【解】(細節略)

$\det(A_{1\times1})=2$, $\det(A_{2\times2})=3$,

$n\geq3$時之一般式爲 $\det(A_{n\times n})=2\det(A_{(n-1)\times(n-1)})-\det(A_{(n-2)\times(n-2)})$.

∴ $\det(A_{3\times3})=2\cdot\det(A_{2\times2})-\det(A_{1\times1})=2\cdot3-2=4.$

∴ $\det(A_{4\times4})=2\cdot\det(A_{3\times3})-\det(A_{2\times2})=2\cdot4-3=5.$

可用數學歸納法證明 $\det(A_{n\times n})=n+1$.

【討論】本題結構較簡單，可觀察得知一般公式. 對一般情形需解解遞迴方程式.

3. (15%)　【北科88電通】

Find the closed-form solution for the following matrix function

$$e^{At}=I+At+\frac{A^2}{2!}t^2+....$$

where

$$A=\begin{bmatrix}-5 & 3 & 1\\ -4 & 2 & 1\\ -4 & 3 & 0\end{bmatrix}$$

Hint: $\lambda=-1,-1,-1.$

【分析】本題屬於題型16B. 請參閱綜線CH16範例8.

【解】(細節略)

$$A+I=\begin{bmatrix}-4 & 3 & 1\\ -4 & 3 & 1\\ -4 & 3 & 1\end{bmatrix},\quad (A+I)^2=O\ .$$

$$e^{At}=\exp((A+I)t-It)=e^{-t}\cdot(I+(A+I)t+(A+I)^2t^2+...)=e^{-t}\cdot(I+(A+I)t)$$

$$=e^{-t}\left(\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}+t\begin{bmatrix}-4 & 3 & 1\\ -4 & 3 & 1\\ -4 & 3 & 1\end{bmatrix}\right)$$

$$= \begin{bmatrix} e^{-t}-4te^{-t} & 3te^{-t} & te^{-t} \\ -4te^{-t} & e^{-t}+3te^{-t} & te^{-t} \\ -4te^{-t} & 3te^{-t} & e^{-t}+te^{-t} \end{bmatrix}$$ #

4. (15%) 【北科88電通】

Find the coefficients of the polynomial $f(x)= a_0+a_1x+a_2x^2$ such that the following is minimal.

$$\int_{-1}^{1} [f(x) - \sin \pi x]^2 \, dx$$

Hint: Use Gram-Schmidt process and projection.

【分析】本題屬於題型09B, 並用到題型09C. 請參閱綜線CH9定理12及CH9範例14.

另請參閱交大87資料[4].

本題計算過程會用到integration-by-part:

$$\int_{-1}^{1} x\sin \pi x \, dx = (-1/\pi) \int_{-1}^{1} x \, d\cos \pi x = (-1/\pi) \left[x \cos \pi x \Big|_{-1}^{1} - \int_{-1}^{1} \cos \pi x \, dx \right]$$

另外, 知道下列事實可增快計算速度:

若$g(x)$為奇函數, $h(x)$為偶函數, 則

$$\int_{-1}^{1} g(x) \, dx = 0, \qquad \int_{-1}^{1} h(x) \, dx = 2\int_{0}^{1} h(x) \, dx$$

【解】考慮多項式空間的內積:

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x) \, dx$$

設W=span$\{1, x, x^2\}$, 本題所求為$\sin(\pi x)$對W的正投影. (綜線CH9定理13)

先以Gram-Schmidt process由 $1, x, x^2$ 導出W的正交基底:

$$f_1(x)=1, \qquad \langle f_1, f_1 \rangle = \langle 1, 1 \rangle = \int_{-1}^{1} 1 \, dx = 2$$

$\langle x, f_1\rangle = \langle x, 1\rangle = \int_{-1}^{1} x\,dx = 0$

$f_2(x) = x - (\langle x, f_1\rangle / \langle f_1, f_1\rangle) f_1 = x.$ $\quad \langle f_2, f_2\rangle = \int_{-1}^{1} x^2\,dx = 2/3$

$\langle x^2, f_1\rangle = \int_{-1}^{1} x^2\,dx = 2/3$ $\qquad \langle x^2, f_2\rangle = \int_{-1}^{1} x^3\,dx = 0$

$f_3(x) = x^2 - (\langle x^2, f_1\rangle / \langle f_1, f_1\rangle) f_1 - (\langle x^2, f_2\rangle / \langle f_2, f_2\rangle) f_2 = x^2 - 1/3$.

$\langle f_3, f_3\rangle = \int_{-1}^{1} (x^2 - (1/3))^2 dx = 8/45$

$\langle \sin\pi x, f_1\rangle = \int_{-1}^{1} \sin\pi x\,dx = 0$

$\langle \sin\pi x, f_2\rangle = \int_{-1}^{1} x\sin\pi x\,dx = 2/\pi$

$\langle \sin\pi x, f_3\rangle = \int_{-1}^{1} (x^2 - 1/3)\sin\pi x\,dx = 0$

∴ 所求之函數爲:

$f(x) = (\langle \sin\pi x, f_1\rangle / \langle f_1, f_1\rangle) f_1 + (\langle \sin\pi x, f_2\rangle / \langle f_2, f_2\rangle) f_2$

$+ (\langle \sin\pi x, f_3\rangle / \langle f_3, f_3\rangle) f_3$

$= ((2/\pi)/(2/3))x = (3/\pi)x$.

5. (15%) 【北科88電通】

Find the eigenvalues of the following matrix

$$A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

【分析】本題屬於題型12A. 請參閱綜線CH12範例14.

【解】(細節略) eigenvalues爲 0, 0, 6, –2.

【說明】利用行列式運算降階求特徵值:

det($A-xI$)

$$= \begin{vmatrix} 1-x & 1 & 2 & 2 \\ 1 & 1-x & 2 & 2 \\ 2 & 2 & 1-x & 1 \\ 2 & 2 & 1 & 1-x \end{vmatrix} \begin{matrix} (-1) \\ \hookleftarrow \\ (-1) \\ \hookleftarrow \end{matrix} = \begin{vmatrix} 1-x & 1 & 2 & 2 \\ x & -x & 0 & 0 \\ 2 & 2 & 1-x & 1 \\ 0 & 0 & x & -x \end{vmatrix}$$

接下來將第一行加入第二行，就可對第二列降階取出因式 x.

6. (15%) 【北科88電通】

Suppose that A, an $n\times n$ complex matrix, is such that $AA^*=A^*A$ where A^* is the Hermitian adjoint of A. Prove that if $A^2x = o$ for $x \in \mathbb{C}^{(n)}$, then $Ax = o$.

【分析】本題屬於題型13D. 本題稍難，單式對角化後還要經過多步推導才證得.

【解】$\because$ A爲normal,

$\therefore$ $\exists$ unitary matrix U, 使$U^{-1}AU$=diag($\lambda_1, \lambda_2, \dots, \lambda_n$)=$D$. (CH13定理15)

$\therefore$ $A=UDU^{-1}=UDU^{H}$

由 $A^2x=o$ 得 $UDU^HUDU^Hx=o$, $\therefore$ $UDDU^Hx=o$.

再左乘U^H得 $D^2U^Hx=o$.

令$U^Hx=[z_1, z_2, \dots, z_n]^T$. 上式展開得 $\lambda_1^2z_1=0, \lambda_2^2z_2=0, \dots, \lambda_n^2z_n=0$.

對各個k=1,2,...,n,

若$\lambda_k\neq 0$, 則可除以λ_k 而得 $\lambda_kz_k=0$.

若$\lambda_k=0$, 仍然 $\lambda_kz_k=0$.

$\therefore$ $DU^Hx=o$.

$\therefore$ $UDU^Hx=o$, 即得$Ax=o$.

3. 【另解】.

$$P=\begin{bmatrix}0&1&0\\0&1&1\\1&1&-3\end{bmatrix},.\ J=\begin{bmatrix}-1&0&0\\1&-1&0\\0&0&-1\end{bmatrix}$$

則 $A=PJP^{-1}$

$e^{tA}=\exp(tPJP^{-1})=P\exp(tJ)P^{-1}$

$\exp(tJ)=\exp(t(S_2\oplus S_1)-tI_3)=\exp(t(S_2-I_2))\oplus\exp(t(S_1-I_1))$

$$\exp(t(S_2-I_2))=e^{-t}\exp(tS_2)=e^{-t}(I+tS_2)=e^{-t}\begin{bmatrix}1&0\\t&1\end{bmatrix}$$

$\exp(t(S_1-I_1))=\exp(-tI_1)=e^{-t}$

$$\exp(tJ)=\begin{bmatrix}e^{-t}&0&0\\te^{-t}&e^{-t}&0\\0&0&e^{-t}\end{bmatrix}=e^{-t}I_3+te^{-t}(S_2\oplus S_1)$$

$e^{tA}=e^{-t}I_3+te^{-t}P(S_2\oplus S_1)P^{-1}=$

$$=e^{-t}\left(\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}+t\begin{bmatrix}-4&3&1\\-4&3&1\\-4&3&1\end{bmatrix}\right)$$

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