逢甲大學88工業工程所

科目: 線性代數

1. (25%)　【逢甲88工工】

Consider the linear operator $T:\mathbb{R}^2\rightarrow\mathbb{R}^2$ with $T(x, y)=(2x, x+y)$

(a) Find matrix A, the matrix representation of transformation, with respect to the basis $B=\{(1, 0), (1, 2)\}$.　(7%)

(b) Use similarity transformation to determine the matrix A' with respect to the basis $B'=\{(-2, 3), (1, -1)\}$.　(10%)

(c) If $u_{B'}=(2, 6)^t$, use transition matrix to find u_B.　(8%)

【分析】本題屬於題型07C.

(c)小題的$u_{B'}$是指向量u的B'座標, u_B是指向量u的B座標.

【解】(a) 設 $T(1,0)=a(1,0)+b(1,2)$, $T(1,2)=c(1,0)+d(1,2)$,

代入公式並解聯立方程式得 $a=3/2$, $b=1/2$, $c=1/2$, $d=3/2$.

$$\therefore\ A=\begin{bmatrix}3/2 & 1/2\\ 1/2 & 3/2\end{bmatrix}$$ (綜線CH7定義9)

(b) $(-2,3)=(-7/2)(1,0)+(3/2)(1,2)$,　$(1,-1)=(3/2)(1,0)+(-1/2)(1,2)$

$\therefore$ 以B描述B'之矩陣為$P=\begin{bmatrix}-7/2 & 3/2\\ 3/2 & -1/2\end{bmatrix}$.　(綜線CH6定理33)

$\therefore\ A'=P^{-1}AP=\ldots$　(綜線CH7定理19)

$$=\begin{bmatrix}-3 & 2\\ -10 & 6\end{bmatrix}$$

(c) 所求爲 $\begin{bmatrix} -7/2 & 3/2 \\ 3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ (綜線CH6定理33)

2. (25%) 【逢甲88工工】

Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ with $T(u)=Au$, in which

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

(a) Determine the kernel of T. (5%)

(b) Find a basis for the kernel of T. (5%)

(c) Determine the range of T. (5%)

(d) Find a basis for the range of T. (5%)

(e) Show whether the transformation is one-to-one. (5%)

【分析】本題屬於題型08A.

【解】(a)

A經列運算化爲 $\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$\text{Ker}T=\{[x, y, z]^T \mid x+5z=0, y-z=0\}=\{[-5t, t, t]^T \mid t\in\mathbb{R}\}$

(綜線CH8定理5a,CH5範例21)

(b) Ker$T$的基底取爲 $\{[-5, 1, 1]^T\}$

(c) A經行運算化爲 $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$.

$\text{Im}T=\text{CSP}A=\{[s, t, s+t]^T \mid s, t\in\mathbb{R}\}$ (綜線CH8定理5a, CH5範例18)

(d) ImT的基底取為 $\{[1, 0, 1]^T, [0, 1, 1]^T\}$.

(e) ∵ Ker$T\neq\{o\}$, ∴ T不是one-to-one. (綜線CH8定理7)

3. 【逢甲88工工】

Diagonize matrix A and give the similarity transformation, in which

$$A=\begin{bmatrix} 2 & 0 & 0 \\ -2 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix} \quad (25\%)$$

【分析】本題屬於題型12C. 請參閱綜線CH12範例17.

【解】(細節略)

$$P=\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix},\quad P^{-1}AP=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. (25%) 【逢甲88工工】

Let vector v=(3, 2, 1) be a vector in $\mathbb{R}^3$ and W be the subspace of $\mathbb{R}^3$ consisting of all vectors of the form $(a, b, a+b)$

(a) Decompose v into the sum of a vector that lies in W and a vector orthogonal to W. (15%)

(b) Find the distance of the point v from subspace W. (10%)

【分析】本題屬於題型11B.

【解】(a) $(a, b, a+b)=a(1, 0, 1)+b(0,1,1)$, W=span{(1, 0, 1),(0,1,1)}

$W^{\perp}$=span{(1, 0, 1)×(0,1,1)}=span{(–1,–1,1)} (綜線CH1定義11)

設 $v=p+q$, $p\in W$, $q\in W^{\perp}$.

$$q = v\text{對}W^{\perp}\text{的正投影} = \frac{(3, 2, 1)\cdot(-1,-1,1)}{(-1,-1,1)\cdot(-1,-1,1)}(-1,-1,1)$$ (綜線CH9定理12)

$= (4/3, 4/3, -4/3)$

$p = v-q = (5/3, 2/3, 7/3)$

(b) $d(v,W)= \| q \| = 4\sqrt{3}/3$

5. (15%) 【逢甲88工工】

Let A be an $m\times n$ matrix. Show that the linear system $AX=b$ has a solution for every $m\times 1$ matrix b if and only if rank(A)=m.

【分析】本題屬於題型03B.

【解】 $\forall m\times 1$矩陣b, $AX=b$有解

$\Longleftrightarrow$ $\forall m\times 1$矩陣b, $b\in \text{CSP}A$ (綜線CH3定理11a)

$\Longleftrightarrow$ $\text{CSP}A=\mathbb{R}^{m\times 1}$

$\Longleftrightarrow$ $\dim\text{CSP}A=m$ (向左推用到CH6定理22a)

$\Longleftrightarrow$ $\text{rank}A=m$ (綜線CH8定義13)

6. (35%) 【逢甲88工工】

In $\mathbb{R}^4$, let W be the subset of all vectors $v=(a_1, a_2, a_3, a_4)$ that satisfy $a_4-a_3=a_2-a_1$.

(a) Show that W is a subspace of $\mathbb{R}^4$. (10%)

(b) Show that $S=\{(1, 0, 0, -1), (0, 1, 0, 1), (1, 1, 1, 1), (0, 0, 1, 1)\}$ spans W. (10%)

(c) Find a subset of S that is a basis for W. (10%)

(d) Express $v=(0, 4, 2, 6)$ as a linear combination of the basis obtained in part (c). (5%)

【分析】本題(a)屬於題型05B. 本題(b)屬於題型05C. 本題(c)屬於題型06C.
本題(d)屬於題型03A

【解】(a) $W\neq\varnothing$, 再求證封閉性即可: (綜線CH5定理11)

$\forall\ (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4)\in W$, $\forall$scalar h, k

$h(a_1, a_2, a_3, a_4)+k(b_1, b_2, b_3, b_4)=(ha_1+kb_1, ha_2+kb_2, ha_3+kb_3, ha_4+kb_4)$

$(ha_4+kb_4)-(ha_3+kb_3)=h(a_4-a_3)+k(b_4-b_3)=h(a_2-a_1)+k(b_2-b_1)=(ha_2+kb_2)-(ha_1+kb_1)$

$\therefore\ h(a_1, a_2, a_3, a_4)+k(b_1, b_2, b_3, b_4)\in W$.

(a) [另解] $W=\{(a_1, a_2, a_3, a_4) \mid a_4-a_3=a_2-a_1\}=\{(a_1, a_2, a_3, a_4) \mid a_1-a_2-a_3+a_4=0\}$

$=\ker[\ 1, -1, -1, 1]$

$\therefore$ W是$\mathbb{R}^4$的子空間. (綜線CH8定理4)

(b) S中各向量都合於$a_4-a_3=a_2-a_1$的條件, ∴ $S\subseteq W$.

$\forall(a_1, a_2, a_3, a_4)\in W$,

$(a_1, a_2, a_3, a_4)=(a_2, a_2, a_4, a_4)=a_2(1,1,0,0)+a_4(0,0,1,1)$

$=a_2(1,0,0,-1)+a_2(0,1,0,1)+a_4(0,0,1,1)$

∴ span$S=W$.

(b) [另解] dim$W$=4–rank[1, –1, –1, 1]=4–1=3.

S中各向量都合於$a_4-a_3=a_2-a_1$的條件, ∴ span$S$是W的子空間.

$$\begin{bmatrix}1&0&0&-1\\0&1&0&1\\1&1&1&1\\0&0&1&1\end{bmatrix}\text{經列運算化爲}\begin{bmatrix}1&0&0&-1\\0&1&0&1\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

∴ dim(span$S$)=3=dim$W$. (綜線CH6定理23)

∴ span$S=W$. (綜線CH6定理22a)

(c) 將S的各向量取轉置並排成矩陣, 再做列運算:

$$\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&1&1\\-1&1&1&1\end{bmatrix}\text{經列運算化爲}\begin{bmatrix}1&0&1&0\\0&1&1&0\\0&0&1&1\\0&0&0&0\end{bmatrix}$$

∵ pivot在第1,2,3行,

∴ 原矩陣的第1,2,3行可當行空間的基底. (綜線CH6定理24)

∴ {(1, 0, 0, –1), (0, 1, 0, 1), (1, 1, 1, 1)}爲W之基底.

(d) 設(0,4,2,6)=a(1, 0, 0, –1)+b(0, 1, 0, 1)+c(1, 1, 1, 1)

比較各向量得 $0=a+c$, $4=b+c$, $2=c$, $6=-a+b+c$.

解得 $a=-2$, $b=2$, $c=2$.

∴ (0,4,2,6)= –2(1, 0, 0, –1)+2(0, 1, 0, 1)+2(1, 1, 1, 1)