交通大學88工業工程所

科目: 微積分與線性代數

I. 微積分部份: (50%)

II. 線性代數部份: (50%)

7. (7%) 【交大88工工】

Find the value of a determinant $A=\begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix}$

【分析】本題屬於題型04B. 請參閱綜線CH4範例12.

【解】所求行列式爲1134. (細節略)

8. (10%) 【交大88工工】

Space R and and W are orthogonal, or RW=0. If

$$R=\begin{bmatrix} -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

find the rank of W. Also show how to find a basis of W.

【分析】本題屬於題型03A. 請參閱綜線CH3範例7.

本題題目將space, 矩陣的列空間, 矩陣的行空間混爲一談. 考生須自行辨正, 勉強求解.

【解】

原矩陣經列運算化爲 $\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

此矩陣之齊次方程式之解爲 (綜線CH3範例7)

$x_1=t_3,\ x_2=t_3-t_8+t_9,\ x_3=t_3,\ x_4=t_5+t_8-t_9,\ x_5=t_5,\ x_6=t_7+t_8-t_9,\ x_7=t_7,\ x_8=t_8,\ x_9=t_9.$

W中之向量爲 $[\ t_3, t_3-t_8+t_9, t_3, t_5+t_8-t_9, t_5, t_7+t_8-t_9, t_7, t_8, t_9\ ]$

$= t_3[\ 1, 1, 1, 0, 0, 0, 0, 0, 0\ ]+t_5[\ 0, 0, 0, 1, 1, 0, 0, 0, 0\ ]+t_7[\ 0, 0, 0, 0, 0, 1, 1, 0, 0\ ]$

$+t_8[\ 0, -1, 0, 1, 0, 1, 0, 1, 0\ ]+t_9[\ 0, 1, 0, -1, 0, -1, 0, 0, 1\ ]$

W之rank爲5,

W之基底可取爲 $\{[\ 1, 1, 1, 0, 0, 0, 0, 0, 0\ ], [\ 0, 0, 0, 1, 1, 0, 0, 0, 0\ ],$

$[\ 0, 0, 0, 0, 0, 1, 1, 0, 0\ ], [\ 0, -1, 0, 1, 0, 1, 0, 1, 0\ ], [\ 0, 1, 0, -1, 0, -1, 0, 0, 1\ ]\ \}$

9.(11%) 【交大88工工】

Determine whether or not the given set is linearly independent. If the set is linearly dependent, express one of its vectors as a linear combination of the others.

(a). $\{x+2x^2+3x^3, -x+2x^2-x^3, x+x^3\}$ in P_3

(b). $\{x-\cos x, 3x+x^2+\cos x+2\sin x, -2x+2\cos x\}$ in C^∞.

(c). $\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 4 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right)$ in $M^{2,2}$

【分析】本題屬於題型06A. 請參閱綜線CH6範例11,12.

【解】(a) 設 $a(x+2x^2+3x^3)+b(-x+2x^2-x^3)+c(x+x^3)=0$, 試解$a,b,c$.

比較係數得: $a-b+c=0,\ 2a+2b=0,\ 3a-b+c=0$

解得$a=b=c=0$, 所以是linearly independent.

(b) 本題觀察即知 $(-2x+\cos x)=-2(x-\cos x)$, 所以不是linearly independent.

(c) 設 $a\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}+b\begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix}+c\begin{bmatrix}4 & -4\\ 0 & 1\end{bmatrix}+d\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}=O$

整理得 $2a+b+4c+d=0$, $-2b-4c+d=0$, $a+c+d=0$

經列運算發現有無限多組解, 所以不是linearly independent.

10. (10%) 【交大88工工】

Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be defined by $T(x, y, z)=(2x-y, x+y+z, 2y-z)$ for each $x=(x, y, z)$ in $\mathbb{R}^3$. Show that T is invertible, and compute T^{-1}.

【分析】本題屬於題型08F. 請參閱綜線CH8範例30.

【解】$\forall\ (p, q, r)\in\mathbb{R}^3$, 試解 $(p, q, r)=T(x, y, z)=(2x-y, x+y+z, 2y-z)$,

即 $2x-y=p$, $x+y+z=q$, $2y-z=r$.

經列運算可解得 $x=(3p+q+r)/7$, $y=(-p+2q+2r)/7$, $z=(-2p+4q-3r)/7$

由以上得知T為onto.

又因T的定義域與對應域同維度,

$\therefore T$為one-to-one. (綜線CH8定理11)

$\therefore T$為可逆.

由前述計算可知$T^{-1}(p, q, r)=(\ (3p+q+r)/7, (-p+2q+2r)/7, (-2p+4q-3r)/7\ )$

11.(12%) 【交大88工工】

Let $A=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}$. Find an orthogonal matrix P such that $P^{-1}AP$ is diagonal,

and use this result to compute A^3.

【分析】本題屬於題型13C. 請參閱綜線CH13範例16.

【解】(細節略)

$$P=\begin{bmatrix} 1/2 & -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} \\ 1/2 & 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} \\ 1/2 & 0 & 2/\sqrt{6} & -1/\sqrt{12} \\ 1/2 & 0 & 0 & 3/\sqrt{12} \end{bmatrix}, \quad P^{-1}AP=\text{diag}(4,0,0,0)$$

$A^3=(P\text{diag}(4,0,0,0)P^{-1})=P\text{diag}(64,0,0,0)P^{-1}$

$=64[1/2, 1/2, 1/2, 1/2]^T[1/2, 1/2, 1/2, 1/2]$

$=16[1, 1, 1, 1]^T[1, 1, 1, 1]=16A$.

【討論】A是rank爲1的矩陣, 本題求A^3時, 用下法較簡單: (綜線CH5定理16b)

$A=[1, 1, 1, 1]^T[1, 1, 1, 1]$,

$A^3=[1, 1, 1, 1]^T[1, 1, 1, 1][1, 1, 1, 1]^T[1, 1, 1, 1][1, 1, 1, 1]^T[1, 1, 1, 1]$

$=[1, 1, 1, 1]^T[4][4][1, 1, 1, 1]=16A$