線性代數解析--交大89資工所 廖亦德 解

Linear Algebra (50%)

1. (7%) 【交大89資工】

Let $A=\begin{bmatrix} 1 & -2 & 1 & 1 & 2 \\ -1 & 3 & 0 & 2 & -2 \\ 0 & 1 & 1 & 3 & 4 \\ 1 & 2 & 5 & 13 & 5 \end{bmatrix}$

Find a basis for the row sapace, a basis for the column space, and a basis for the nullspace.

【分析】本題屬於題型06C. 請參閱綜線CH6範例24a.

【解】A經列運算化爲

$$\begin{bmatrix} 1 & 0 & 3 & 7 & 0 \\ 0 & 1 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

row space之基底可取爲

$\{ [1 \ 0 \ 3 \ 7 \ 0], [0 \ 1 \ 1 \ 3 \ 0], [0 \ 0 \ 0 \ 0 \ 1] \}$ (綜線CH6定理23)

pivot在第1, 2, 5行,

∴ column space之基底可取爲

$\{ [1 \ -1 \ 0 \ 1]^T, [-2 \ 3 \ 1 \ 2]^T, [2 \ -2 \ 4 \ 5]^T \}$ (綜線CH6定理24)

解線性方程式$Ax=o$得 $x_1=-3s-7t$, $x_2=-s-3t$, $x_3=s$, $x_4=t$, $x_5=0$ (綜線CH3範例7)

∴ null space = $\{ [-3s-7t, -s-3t, s, t, 0]^T \mid s, t$爲任意數 $\}$

$= \{ s[-3, -1, 1, 0, 0]^T + t[-7, -3, 0, 1, 0]^T \mid s, t$爲任意數 $\}$

∴ null space之基底可取爲 $\{ [-3, -1, 1, 0, 0]^T, [-7, -3, 0, 1, 0]^T \}$

2. (10%) 【交大89資工】

Let L be the linear operator mapping $\mathbf{R}^3$ into $\mathbf{R}^3$ defined by $L(x)=Ax$, where

$$A=\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix} \text{ and let } v_1=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_3=\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Find the transition matrix V corresponding to a change-of-basis from $[v_1, v_2, v_3]$ to $[e_1, e_2, e_3]$ and use it to determine the matrix B representing L with respect to $[v_1, v_2, v_3]$.

【分析】本題屬於題型06D. 請參閱綜線CH6定理33.

本題未指明$[e_1, e_2, e_3]$, 依習慣爲標準基底$e_1=[1\ 0\ 0]^T$, $e_1=[0\ 1\ 0]^T$,$e_1=[0\ 0\ 1\ ]^T$.

由$[v_1, v_2, v_3]$ 到 $[e_1, e_2, e_3]$的座標變換需以$[e_1, e_2, e_3]$描述$[v_1, v_2, v_3]$.

【解】

所求$V=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$

$B=V^{-1}AV$ (綜線CH7定理19)

$= \ldots = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

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3. (8%) 【交大89資工】

Determine whether or not the following are subspaces of R^3.

(a) $\{(x_1, x_2, x_3)^T | x_1+x_3=1\}$

(b) $\{(x_1, x_2, x_3)^T | x_1=x_2=x_3\}$

(c) $\{(x_1, x_2, x_3)^T | x_3=x_1+x_2\}$

(d) $\{(x_1, x_2, x_3)^T | x_3^2=x_1^2+x_2^2\}$

【分析】本題屬於題型05B. 請參閱綜線CH5範例12,13.

【解】(a) No, (b) Yes, (c) Yes, (d) No

4. (7%) 【交大89資工】

Given the inner product space C[0, 1] with inner product

$$\langle f, g \rangle = \int_0^1 f(x)\, g(x)\, dx$$

Find the best least squares approximation to $2+x^{1/2}$ on [0, 1] by a function from the subspace S spanned by 1 and $4x-2$.

【分析】本題屬於題型09B. 請參閱綜線CH9範例14.

least square approximation 是指尋找 $p(x)\in S$, 使

$\| 2+x^{1/2}-p(x) \| = \min\{ \| 2+x^{1/2}- q(x) \| \mid q(x)\in S \}$

依綜線CH9定理13得知應求$2+x^{1/2}$對S的正投影.

【解】先對1, $4x-2$做內積發現兩函數正交$\{1, 4x-2\}$, (綜線CH9定義4)

求算積分得 $\langle 2+x^{1/2}, 1\rangle=8/3$, $\langle 1,1\rangle=1$,

$\langle 2+x^{1/2}, 4x-2\rangle=4/15$, $\langle 4x-2, 4x-2\rangle=4/3$

正投影= $((8/3)/1)\cdot 1 + ((4/15)/(4/3))(4x-2)$ (綜線CH9定理12)

$=.(34/15)+(4/5)x$ #

5. (8%) 【交大89資工】

Let A and B be $n\times n$ matrices, and B be similar to A.

(a) Show that A and B heve the same eigenvalues.

(b) Let x be an eigenvector of A belonging to an eigenvalue λ and y be an eigenvector of B belonging to λ. State the relation between x and y, and prove it.

【分析】本題屬於題型16A.

【解】(a) 請參閱綜線CH12定理8a.

(b) 設$B=P^{-1}AP$.

由題意, $Ax=\lambda x$, $By=\lambda y$.

$P^{-1}APy=\lambda y$, $APy=\lambda Py$.

∴ Py與x同在$\ker(A-\lambda I)$之中.

6. (10%) 【交大89資工】

$$\text{Given } A=\begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}$$

Find an orthonormal basis and a diagonal matrix D such that the linear transformation L defined by $L(v)=Av$ is represented by D with respect to that basis (used in both domain and range).

【分析】本題屬於題型13C. 請參閱綜線CH9範例16, 範例16a.

【解】$\det(A-xI)= \ldots = -x^2(x-6)$

解$(A-6I)v=o$, 得6的eigenvector $[-2, -1, 1]^T$

解$(A-0I)v=o$, 得0的eigenvector $[-1, 2, 0]^T$, $[1, 0, 2]^T$

經Gram-Schmidt process將$[-1, 2, 0]^T$, $[1, 0, 2]^T$調整為$[-1, 2, 0]^T$, $[4/5, 2/5, 10/5]^T$.

此三向量單位化後即得出正交單位基底:

$\{ [-2/6^{1/2}, -1/6^{1/2}, 1/6^{1/2}]^T, [1/5^{1/2}, 2/5^{1/2}, 0]^T, [4/120^{1/2}, 2/120^{1/2}, 10/120^{1/2}]^T \}$

$$\text{在此基底下, 矩陣表示為} D=\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$