線性代數解析--交大89資料所

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以下1至6題是填充題, 只需給答案, 不必寫出任何計算式子.

1. (8%) 【填充題】【交大89資料】

matrix $A=\begin{bmatrix} -2 & -1 & -2 & 1 \\ 2 & 1 & 2 & 4 \\ 6 & 3 & 6 & 9 \\ -2 & -1 & -2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 & 4 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Find (a) det$A$,

(b) rank(A)

(c) a basis of the row space of A

(d) a basis of the column space of A

(e) a basis of the image space of the linear transform A

(f) a basis of the null space of the linear transform A

【勘誤】此題若將等號右邊相乘,乘積的第四列應該是 [–4, 1, –3, –5]

【分析】本題(a)屬於題型04A. (b)屬於題型08B. (c)(d)(e)(f)屬於題型06C.

【解】設所列出的三個A的因子依序為B, C, D.

(a) $\det D=\begin{vmatrix} 2 & 1 & 2 & 4 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 2 \end{vmatrix}=2\cdot 3\cdot 0\cdot 2=0$

∴ $\det(A)=\det B\det C\det D=0$

(b) 由列運算得知 rank$B$=4, rank$C$=4, rank$D$=3.　　(綜線CH6定理23)

此時得知B, C可逆,　　(綜線CH8定理17)

∴ rank(A)=rank(BCD)=rank(CD)　　(綜線CH8定理16)

=rank(D)=3

(c) 經列運算, $A=BCD\sim CD\sim D$　　(綜線CH3定理17)

$$\sim\begin{bmatrix} 2 & 1 & 2 & 4 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ 可取 {[2, 1, 2, 4], [0, 3, 1, 3], [0, 0, 0, 5] } 爲 row space的基底.

(綜線CH6定理23)

(d) 由(c)的計算發現pivot在第1,2,4行,

∴ 可取 { $[-2, 2, 6, -2]^T$, $[-1, 1, 3, -1]^T$, $[1, 4, 9, 1]^T$ }爲column space的基底.

(綜線CH6定理24)

(e) 將A視爲左乘A的線性映射, image space即column space,

∴ 可取 { $[-2, 2, 6, -2]^T$, $[-1, 1, 3, -1]^T$, $[1, 4, 9, 1]^T$ }爲image space的基底.

(f) 接(c)繼續計算,

$$\sim\begin{bmatrix} 1 & 1/2 & 1 & 2 \\ 0 & 1 & 1/3 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & 0 & 5/6 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

null space = { $[-5t/6, -t/3, t, 0]^T$ | t為任意數 }　　(綜線CHCH3範例7)

= { $(t/6)[-5, -2, 6, 0]^T$ | t為任意數 }

∴ 可取 { $[-5, -2, 6, 0]^T$ }為 null space的基底.

2. (5%)　【填充題】【交大89資料】

$$\text{matrix } A=\begin{bmatrix} -4 & -12 & 11 & -1 \\ 2 & 6 & -4 & 2 \\ 1 & 4 & -3 & 4 \\ -4 & -12 & 11 & -1 \end{bmatrix}$$

$$=\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ -4 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find (a) det$A$　(b) det(A^T)　(c) det(A^{-1})　(d) det(adj$A$)　(e) (adj$A$)$^{-1}$

【分析】本題屬於題型04A.

【解】設所列出的三個A的因子依序為B, C, D.

$$\text{(a) det}B=\begin{vmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{vmatrix}=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$　(第一列降階展開)

$= ... = -1$

$$\text{det}C=\begin{vmatrix} 2 & 0 & 0 & 0 \\ 3 & 5 & 0 & 0 \\ -4 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}=2\cdot 5\cdot 3\cdot 1=30$$

$$\det D=\begin{vmatrix}1&3&-2&1\\0&1&-2&1\\0&0&1&1\\0&0&0&1\end{vmatrix}=1\cdot1\cdot1\cdot1=1$$

$\therefore\ \det(A)=\det(B)\det(C)\det(D)=-1\cdot30\cdot1=-30$

(b) $\det(A^{T})=\det(A)=-30$

(c) $\det(A^{-1})=(\det A)^{-1}=-1/30$

(d) $\because\ A\cdot\text{adj}(A)=(\det A)I_4$　　(綜線CH4定理17①)

兩邊取行列式得　$\det A\cdot\det(\text{adj}A)=(\det A)^4$

$\therefore\ \det(\text{adj}A)=(\det A)^3=-27000$

(e) $\because\ A\cdot\text{adj}(A)=(\det A)I_4$

兩邊取逆矩陣得 $(\text{adj}A)^{-1}A^{-1}=(\det A)^{-1}I.$

$\therefore\ (\text{adj}A)^{-1}=(\det A)^{-1}A=(-1/30)A.$

3. (3%)　【填充題】【交大89資料】

$$\text{matrix } A=\begin{bmatrix}0&0&1&0\\1&0&0&0\\0&0&0&1\\0&1&0&0\end{bmatrix}\begin{bmatrix}1&0&0&0\\-2&1&0&0\\-1&0&1&0\\3&0&1&1\end{bmatrix}\begin{bmatrix}3&2&3&4\\0&2&1&1\\0&4&2&5\\0&6&3&1\end{bmatrix}$$

Find　(a) $\det(A^{T})$　(b) rank(A)

【分析】本題(a)屬於題型04A, (b)屬於題型08B. .

【解】設所列出的三個A的因子依序為B, C, D.

(a) $\det B=-1$,

$\det C=1\cdot1\cdot1\cdot1=1$,

$$\det D=\begin{vmatrix}3&2&3&4\\0&2&1&1\\0&4&2&5\\0&6&3&1\end{vmatrix}=\begin{vmatrix}3&2&3&4\\0&2&1&1\\0&0&0&3\\0&0&0&-2\end{vmatrix}=\begin{vmatrix}3&2&3&4\\0&2&1&1\\0&0&0&3\\0&0&0&0\end{vmatrix}=0$$

∴ $\det(A^T)=\det(A)=\det B\det C\det D=0$

(b) 由(a)得知B,C皆可逆. (綜線CH4定理17)

∴ $\text{rank}(A)=\text{rank}(BCD)=\text{rank}(CD)=\text{rank}(D)=3$. (綜線CH8定理16,CH6定理23)

4. (3%) 【填充題】【交大89資料】

Let A and B be two matrices. Compare the following ranks and fill in the blank with =, >, <, ≥ or ≤

(a) rank(AB) ____ rank(A)

(b) rank(AB) ____ rank(B)

(c) A is nonsingular. rank(AB) _____ rank(A)

(d) A is nonsingular. rank(AB) _____ rank(B)

(e) rank(A) _____ rank(A^T)

【分析】本題屬於題型08C. 請參閱綜線CH8定理15,16.

【解】(a) $\text{rank}(AB)\le\text{rank}(A)$ (綜線CH8定理16①)

(b) $\text{rank}(AB)\le\text{rank}(B)$ (綜線CH8定理16①)

(c) $\text{rank}(AB)\le\text{rank}(A)$ (綜線CH8定理16①)

(d) A可逆時, $\text{rank}(AB)=\text{rank}(B)$ (綜線CH8定理16④)

(e) $\text{rank}(A)=\text{rank}(A^T)$ (綜線CH8定理15)

5. (3%) 【填充題】【交大89資料】

Find all possible matrices X for which $AX=o$, where A is

$$A=\begin{bmatrix} 1 & -2 & 3 \\ -2 & 5 & -6 \\ 2 & -3 & 6 \end{bmatrix}$$

【分析】本題屬於題型03A. 請參閱綜線CH3範例7.

【解】以列運算求解:

$$A=\begin{bmatrix} 1 & -2 & 3 \\ -2 & 5 & -6 \\ 2 & -3 & 6 \end{bmatrix} \sim \ldots \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

通解為 $X=[\,-3t, 0, t\,]^T$, 　t為任意數.

6. (3%)　　【填充題】【交大89資科】

Find all right-hand sides b for which $Ax=b$ has solutions, and find all solutions, where

$$A=\begin{bmatrix} 4 & -1 & 2 & 6 \\ -1 & 5 & -1 & -3 \\ 3 & 4 & 1 & 3 \end{bmatrix}$$

【分析】本題屬於題型03C. 請參閱綜線CH3範例9, 本題與CH3習題9.2完全相同.

【解】設$b=[p, q, r]^T$, 依列運算解$Ax=b$:

$$\left[\begin{array}{cccc|c} 4 & -1 & 2 & 6 & p \\ -1 & 5 & -1 & -3 & q \\ 3 & 4 & 1 & 3 & r \end{array}\right] \sim \left[\begin{array}{cccc|c} 0 & 19 & -2 & -6 & p+4q \\ -1 & 5 & -1 & -3 & q \\ 0 & 19 & -2 & -6 & 3q+r \end{array}\right]$$

$$\sim\left[\begin{array}{cccc|c} 0 & 19 & -2 & -6 & p+4q \\ -1 & 5 & -1 & -3 & q \\ 0 & 0 & 0 & 0 & -p-q+r \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -5 & 1 & 3 & -q \\ 0 & 19 & -2 & -6 & p+4q \\ 0 & 0 & 0 & 0 & p+q-r \end{array}\right]$$

∴ 有解 $\Longleftrightarrow$ $p+q-r=0$ $\Longleftrightarrow$ $r=p+q$.

有解時繼續計算:

$$\sim\left[\begin{array}{cccc|c} 1 & -5 & 1 & 3 & -q \\ 0 & 19 & -2 & -6 & p+4q \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -5 & 1 & 3 & -q \\ 0 & 1 & -2/19 & -6/19 & p/19+4q/19 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim\left[\begin{array}{cccc|c} 1 & 0 & 9/19 & 27/19 & 5p/19+q/19 \\ 0 & 1 & -2/19 & -6/19 & p/19+4q/19 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

通解為 $x_1=5p/19+q/19-9s/19-27t/19$

$x_2=p/19+4q/19+2s/19+6t/19$

$x_3=s$

$x_4=t$

7. (5%)　　【交大89資料】

Suppose that $B=\{v_1; v_2; ...;v_p\}$ and $B'=\{v_p; v_{p-1}; ...; v_1\}$ are two ordered bases for R^P. Find the matrix M that translates between coordinates with respect to these two bases.

【分析】本題屬於題型06D. 請參閱綜線CH6定理33.

【解】$v_p\ =0v_1+0v_2+....+0v_{p-1}+1v_p$,

$v_{p-1}=0v_1+0v_2+....+1v_{p-1}+0v_p$,

....

$v_1\ =1v_1+0v_2+....+0v_{p-1}+0v_p$,

$$M=\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ & & & & \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$ (綜線CH6定義33)

8. (10%) 【交大89資科】

Suppose that T is a linear transformation from V to W, both p-dimensional. Prove that the null space of T equals $\{\boldsymbol{o}\}$ if and only if the image space of T equals W.

【分析】本題屬於題型08E. 請參閱綜線CH8定理11.

【解】$\because$ $\dim V=\dim(\mathrm{Ker}T)+\dim(\mathrm{Im}T)$ (綜線CH8定理8)

$\therefore$ $\mathrm{Ker}T=\{\boldsymbol{o}\} \Longleftrightarrow \dim(\mathrm{Ker}T)=0$ (維度的定義)

$\Longleftrightarrow \dim V=\dim(\mathrm{Im}T)$ (綜線CH8定理8)

$\Longleftrightarrow \dim W=\dim(\mathrm{Im}T)$ (已知)

$\Longleftrightarrow W=\mathrm{Im}T.$ (綜線CH6定理22a)

9. (10%) 【交大89資科】

Find the eigenvalues of matrix

$$A=\begin{bmatrix} 2 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

together with their algebraic and geometric multiplicities, and find as many eigenvectors (in a linearly independent set) as possible.

【分析】本題屬於題型12C. 請參閱綜線CH12範例10.

【解】$\det(A-xI)=(2-x)^2(3-x)$,

∴ eigenvalues 爲2, 3,

2的algebraic multiplicity爲2, 3的algebraic multiplicity爲1.

$$A-2I=\begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ 2的geometric multiplicity爲 $\dim(\ker(A-2I))=3-\mathrm{rank}(A-2I)=1$

又可解得2的eigenvector $[\,1, 0, 0\,]^T$.

$$A-3I=\begin{bmatrix} -1 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

∴ 3的geometric multiplicity爲 $\dim(\ker(A-3I))=3-\mathrm{rank}(A-3I)=1$

又可解得3的eigenvector $[\,0, 1, 1\,]^T$.

所求爲 $\{\, s[\,1, 0, 0\,]^T, t[\,0, 1, 1\,]^T \,\}$, 其中$s, t$皆不爲0.