線性代數解析--成大90資工所

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1. (10%) 【成大90資工】

Determine all values of the b_i that make the following linear system consistent.

$$x_1+x_2-x_3=b_1$$
$$2x_2+x_3=b_2$$
$$x_2-x_3=b_3$$

【分析】本題屬於題型03C. 請參閱綜線CH3範例9.

【解】
$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & b_1 \\ 0 & 2 & 1 & b_2 \\ 0 & 1 & -1 & b_3 \end{array}\right] \overset{\text{列運算}}{\sim \cdots\cdots \sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \quad -b_3 \\ 0 & 1 & 0 & (1/3)b_2+(1/3)b_3 \\ 0 & 0 & 1 & (1/3)b_2-(2/3)b_3 \end{array}\right]$$

∴ 任何b_1, b_2, b_3都使此方程式有解(consistant).

2(a). (5%) 【成大90資工】

Explain and determine whether the matrices A and B are diagonalizable. (5%)

$$A=\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \quad B=\begin{bmatrix} 3 & 2 & 5 & 1 \\ 2 & 0 & 2 & 6 \\ 5 & 2 & 7 & -1 \\ 1 & 6 & -1 & 3 \end{bmatrix}$$

【分析】本題屬於題型12B及題型13B. 請參閱綜線CH12定理21, CH13定理15.

【解】1° $\det(A-xI)=(3-x)^3$

$$A-3I=\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix},$$

解得特徵子空間

$$\left\{\begin{bmatrix}t\\0\\0\end{bmatrix}\middle|\, t\in\mathbb{R}\right\}$$

特徵值3的幾何重數1小於代數重數3,

∴ A不可對角化 (綜線CH12定理21)

2° B為實數對稱矩陣,

∴ B可(正交)對角化. (綜線CH13定理15)

2(b). (10%) 【成大90資工】

Find the unique solution (assuming that it exists) of the system of equations expressed by the partitioned matrix. (10%)

$$\left[\begin{array}{cccc|c}a_1&b_1&c_1&d_1&3b_1\\a_2&b_2&c_2&d_2&3b_2\\a_3&b_3&c_3&d_3&3b_3\\a_4&b_4&c_4&d_4&3b_4\end{array}\right]$$

【分析】本題屬於題型04D. 請參閱綜線CH4定理18.

【解】設此唯一解為 $[x_1, x_2, x_3, x_4]^T$, 則

$$x_1=\frac{1}{D}\begin{vmatrix} 3b_1 & b_1 & c_1 & d_1 \\ 3b_2 & b_2 & c_2 & d_2 \\ 3b_3 & b_3 & c_3 & d_3 \\ 3b_4 & b_4 & c_4 & d_4 \end{vmatrix}, \quad x_2=\frac{1}{D}\begin{vmatrix} a_1 & 3b_1 & c_1 & d_1 \\ a_2 & 3b_2 & c_2 & d_2 \\ a_3 & 3b_3 & c_3 & d_3 \\ a_4 & 3b_4 & c_4 & d_4 \end{vmatrix}$$

$$x_3=\frac{1}{D}\begin{vmatrix} a_1 & b_1 & 3b_1 & d_1 \\ a_2 & b_2 & 3b_2 & d_2 \\ a_3 & b_3 & 3b_3 & d_3 \\ a_4 & b_4 & 3b_4 & d_4 \end{vmatrix}, \quad x_4=\frac{1}{D}\begin{vmatrix} a_1 & b_1 & c_1 & 3b_1 \\ a_2 & b_2 & c_2 & 3b_2 \\ a_3 & b_3 & c_3 & 3b_3 \\ a_4 & b_4 & c_4 & 3b_4 \end{vmatrix}$$

其中 $D=\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$ 必不為0. (綜線CH4定理18)

3(a). (5%) 【成大90資工】

Supply a third column vector so that the following matrix is orthogonal. (5%)

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{3} & 0 \\ 1/\sqrt{3} & -1/\sqrt{2} \end{bmatrix}$$

【分析】本題屬於題型13A. 請參閱綜線CH1定義11.

【解】 $[\,1, 1, 1\,] \times [1, 0, -1]$

$$= \left[\ \begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix}, -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\ \right]$$

$= [\,-1, 2, -1\,]$

∴ 第三行可取為 $[\,-1/\sqrt{6}\,, 2/\sqrt{6}\,, -1/\sqrt{6}\,]^T$

【討論】也可取前兩行, 再補第三個向量$[\,0, 0, 1]^T$, 然後進行Gram-Schmidt正交化.

但比較麻煩.

3(b). (10%) 【成大90資工】

Find the least-squares fit of the data points (–3, 8), (–1, 5), (1, 3), and (3, 0) by a straight line, i.e. by a linear function $y=r_0+r_1x$. (10%)

【分析】本題屬於題型09E. 請參閱綜線CH9範例21c.

【解】對方程式求最小平方解:

$$\begin{bmatrix} 1 & -3 \\ 1 & -1 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 3 \\ 0 \end{bmatrix}$$

兩邊左乘 $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -3 & -1 & 1 & 3 \end{bmatrix}$, 得

$$\begin{bmatrix} 4 & 0 \\ 0 & 20 \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix} = \begin{bmatrix} 16 \\ -26 \end{bmatrix}$$

解得 $r_0=4$, $r_1= -13/10$,

所求爲 $y = 4 - (13/10)x$.

4. (10%) 【成大90資工】

Consider the vector space P_2 of polynomials of degree at most 2, and let B' be the ordered basis $(1, x, x^2)$ for P_2 . Let $T:P_2\to P_2$ be the linear transformation such that $T(1)=3+2x+x^2$, $T(x)=2$, $T(x^2)=2x^2$. Find $T^4(x+2)$.

【分析】本題屬於題型071. 請參閱綜線CH7範例4.

【解】$T(x+2)=T(x)+2T(1)=2+2(3+2x+x^2)=8+4x+2x^2$.　　(線性條件: 綜線CH7定義1)

$T^2(x+2)=T(8+4x+2x^2)=8T(1)+4T(x)+2T(x^2)=8(3+2x+x^2)+4(2)+2(2x^2)=32+16x+12x^2$. .

$T^3(x+2)=T(32+16x+12x^2)=32(3+2x+x^2)+16(2)+12(2x^2)=128+64x+56x^2$.

$T^4(x+2)=T(128+64x+56x^2)=128(3+2x+x^2)+64(2)+56(2x^2)=512+256x+240x^2$.

【另解】

T對B'的矩陣表示 $[\,T\,]=\begin{bmatrix}3&2&0\\2&0&0\\1&0&2\end{bmatrix}$, $x+2$的B'座標爲 $\begin{bmatrix}2\\1\\0\end{bmatrix}$

(綜線CH7定義9, CH6定義28)

$$[\,T\,]^2=\ldots=\begin{bmatrix}13&6&0\\6&4&0\\5&2&4\end{bmatrix},\quad [\,T\,]^4=[\,T\,]^2[\,T\,]^2=\ldots=\begin{bmatrix}205&102&*\\102&52&*\\97&46&*\end{bmatrix}.$$

$$[\,T^4(x+2)]=[\,T^4][\,x+2\,]=\begin{bmatrix}205&102&*\\102&52&*\\97&46&*\end{bmatrix}\begin{bmatrix}2\\1\\0\end{bmatrix}=\begin{bmatrix}512\\256\\240\end{bmatrix}.$$

(綜線CH7定理15)

$\therefore\ T^4(x+2)=512+256x+240x^2$.　　(CH6定義28)