交大90資工所

1. (10%) 【交大90資工】

Let R be the set of real numbers and

$S=\{E\in R^{2\times 2} \mid \forall A\in R^{2\times 2}, \det(A+B)=\det(A)+\det(B), \text{ where } B=EA\}$.

Find S. (Hint: let $E=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and solve a, b, c, and d.)

【分析】本題屬於題型04A. 只用到綜線CH4定理6.

【解】

令 $E=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$E\in S$$

$$\iff \forall A\in R^{2\times 2}, \det(A+EA)=\det(A)+\det(EA)$$

$$\iff \forall A\in R^{2\times 2}, \det((I+E)A)=\det(A)+\det(EA)$$

$$\iff \forall A\in R^{2\times 2}, \det(I+E)\det(A)=\det(A)+\det(E)\det(A) \qquad \text{(綜線CH4定理6)}$$

$$\iff \forall A\in R^{2\times 2}\setminus\{O_{2\times 2}\}, \det(I+E)\det(A)=\det(A)+\det(E)\det(A)$$

$$\iff \det(I+E)=1+\det(E) \qquad \text{(兩邊同除}\det(A)\text{)}$$

$$\iff \begin{vmatrix} 1+a & b \\ c & 1+d \end{vmatrix}=1+\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$\iff (1+a)(1+d)-bc=1+ad-bc$$

$$\iff a+d=0$$

$$\therefore S=\left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \middle|\, a, b, c\in R \right\}$$

2. (8%)　　【交大90資工】

For each $f\in C[0,1]$ define $L(f)=F$, where

$$F(x)=\int_0^x f(t)dt,\ 0\le x\le 1$$

L is a linear transformation from $C[0,1]$ to $C[0,1]$. Are $L(1)$, $L(x)$, and $L(x^2)$ linearly independent? Prove your answer.

【分析】本題屬於題型06A. 請參閱綜線CH6.定義9, CH6範例12.

【解】令函數$F_1=L(1)$, $F_2=L(x)$, $F_3=L(x^2)$, 依定義:

對$0\le x\le 1$, $F_1(x)=\int_0^x 1dt=x$, $F_2(x)=\int_0^x t\,dt=x^2/2$, $F_3(x)=\int_0^x t^2\,dt=x^3/3$,

F_1, F_2, F_3爲線性獨立, 證明如下:

若 $aF_1+bF_2+cF_3=O$, (O爲$C[0,1]$上的零函數)

則 $\forall x\in[0,1]$, $aF_1(x)+bF_2(x)+cF_3(x)=0$, 即 $\forall x\in[0,1]$, $ax+bx^2/2+cx^3/3=0$

∴ $\forall x\in\{1,1/2,1/3\}$, $ax+bx^2/2+cx^3/3=0$

∴ $a+b/2+c/3=0$, $a/2+x^2/8+c/12=0$, $a/3+b/18+cx^3/81=0$

解方程式可得$a=0$, $b=0$, $c=0$.　　　(CH4定理18)

3. (12%)　　【交大90資工】

Prove or disprove the following statements:

(a) Let S be the subspace of R^2 spanned by e_1 and let T be the subspace of R^2 spanned by e_2. Then $S\cup T$ is a subspace of R^2. (4%)

(b) $\forall A\in R^{2\times 2}$ $AE=EA$ for some $E\in R^{2\times 2}$. Then either $E=O$ or $E=I$, the identity matrix. (4%)

(c) Let A and B be $n\times n$ matrices. $AB=O$ if and only if the column space of B is a subspace of the null space of A. (4%)

【分析】本題(a)屬於題型05B. 本題(b)屬於題型02A. 本題(c)屬於題型05C.

其中(a)小題的e_1, e_2依上下文判知爲(1,0), (0,1)

【解】(a) [Disprove] $e_1\in S\cup T$, $e_2\in S\cup T$, 但$e_1+e_2\notin S\cup T$,

(b) [Disprove] 例如$E=2I$合於條件“$\forall A\in R^{2\times 2}\ AE=EA$”，但並非$O$或$I$.

(c) [Prove] $AB=O$

$\Longleftrightarrow$ $\forall x\in R^{n\times 1}, ABx=o$

$\Longleftrightarrow$ $\forall z\in \mathrm{CSP}(B), Az=o$ (綜線CH5定理17)

$\Longleftrightarrow$ $\forall z\in \mathrm{CSP}(B), z\in \ker(A)$ (綜線CH5定義19)

$\Longleftrightarrow$ $\mathrm{CSP}(B)\subseteq \ker(A)$

【討論】若$n\times n$方陣E滿足條件“$\forall A\in R^{n\times n}\ AE=EA$”，
以$A=E_{pp}$, $p=1,2,...,n$，代入條件，依矩陣乘法可推得E爲對角矩陣
令$E=\mathrm{diag}(d_1, d_2, ..., d_n)$，再以$A=E_{pq}$代入條件，依矩陣乘法可推得$d_p=d_q$.
如此, $E=kI$.

4. (10%) 【交大90資工】

Consider $Ax=b$ where A is $m\times n$. Answer the following sub-problems with T(True) or F(False).

(a) If the rank of A is n, then there is a solution. (2%)

(b) If the system $Ax=b$ has one solution, the it has the same solution as $A^TAx=A^Tb$ does. (2%)

(c) If rank(A)=min(n, m), then there exists a least squares solution given by $x=(A^TA)^{-1}A^Tb$. (2%)

(d) If rank(A)=n, then the least squares solution is given by $x=R^{-1}Q^Tb$ where matrices Q and R are the QR factorization of A with $A=QR$. (2%)

(e) If $b\in \mathrm{R}(A)^{\perp}$, the orthogonal complement of row space of A, then $AA^Tb=0$. (2%)

【分析】本題並未說明A是實數矩陣，但由(c)(d)之題目可研判命題者並不考慮複數矩陣.
小題(a)屬於題型08E. 請參閱綜線CH8定理8.
小題(b)(c)(d)屬於題型09E. 請參閱綜線CH9定理21a. 小題(b)中的it是指“the system $Ax=b$” .
小題(e)屬於題型11C. 請參閱綜線CH11定理23.

【解】(a) False.

例如 $m=3$, $n=2$, $A=\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$, $b=\begin{bmatrix}0\\0\\1\end{bmatrix}$, rank$A=n$, 但$Ax=b$無解,

[討論] 本題改成rank$A$=m就會是True. 理由如下:

$\mathrm{CSP}(A)=\{Ax \mid x\in K^{n\times 1}\}\subseteq K^{m\times 1}$ (綜線CH5定理17)

$\dim \mathrm{CSP}(A)= m \Longrightarrow \mathrm{CSP}(A)=K^{m\times 1}$ (綜線CH6定理22a)

(b) True.

當x_0爲$Ax=b$的解時, 由$Ax_0=b$推得$A^TAx_0=A^Tb$, 所以x_0也是$A^TAx=A^Tb$的一解.

$Ax=b$的解集合爲 $\{x_0+u \mid u\in \ker(A)\}$ (綜線CH3定理11a)

$A^TAx=A^Tb$的解集合爲 $\{x_0+u \mid u\in \ker(A^TA)\}$ (綜線CH3定理11a)

但$\ker(A^TA)=\ker(A)$ (綜線CH9定理20,實數版)

所以兩者解集合相同

(c) False.

例如 $m=2, n=3$, $A=\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}$, rank$A$=min($m$,$n$), 但 A^TA並不可逆.

[討論] 本題改成 rank$A$=n 就會是True. 理由如下:

rank$A=n \Longleftrightarrow \dim \ker A=0$ (綜線CH8定理8)

$\Longleftrightarrow \ker A=\{o\}$

$\Longleftrightarrow A$的行線性獨立 (綜線CH6定理15)

再套用最小平方定理即得. (綜線CH8定理21a)

(d) True.

rank$A$=n時, QR分解的R是可逆上三角矩陣. (綜線CH9定理19)

由$Rx=Q^Tb$再移項即得. (綜線CH9定理21a)

(e) False.

A爲$m\times n$矩陣, 則row space of $A\subseteq\mathbb{R}^{1\times n}$, 取complement仍在$\mathbb{R}^{1\times n}$中,

A^T爲$n\times m$矩陣, $b\in\mathbb{R}^{1\times n}$, 無法相乘並形成$A^Tb$

[討論] 本題改成 column space 就會是True. 理由如下:

$b\in(\mathrm{CSP}(A))^{\perp}=(\mathrm{lker}A)^T=\ker(A^T)$

$\therefore\ A^Tb=o, \qquad \therefore\ AA^Tb=o$

5. (10%) 【交大90資工】

Let A and B be two matrices and x be a column vector. Answer the following sub-problems with T(True) or F(False).

(a) A Symmetric matrix has real eigenvalues and is diagonalizable. (2%)

(b) If A is diagonalizable and invertible, then A^{-1} is also diagonalizable. (2%)

(c) If A and B are $n\times n$ matrices and A is invertible, then AB is similar to BA. (2%)

(d) If A and B are diagonalizable, then AB is also diagonalizable. (2%)

(e) If A is a symmetric $n\times n$ matrix whose entrices are all positive, then the quadratic form x^TAx is positive definite. (2%)

【分析】小題(a)屬於題型13B. 請參閱綜線CH13定理15

小題(b)(d)屬於題型12B.

小題(c)屬於題型07D. 請參閱綜線CH7定義21

小題(e)屬於題型10C. 請參閱綜線CH10定理28

【解】(a) True,

此敘述對實數矩陣會成立. (綜線CH13定理15)

本題並未說明A是實數矩陣, 但由上題之題目研判命題者並不考慮複數矩陣.

(b) True. (綜線CH16定理3)

(c) True.

$AB=A(BA)A^{-1}$ (綜線CH7習題22.2)

(d) False. 反例如下:

$$A=\begin{bmatrix}0&1\\1&0\end{bmatrix},\ B=\begin{bmatrix}1&0\\0&0\end{bmatrix},\ AB=\begin{bmatrix}0&0\\1&0\end{bmatrix}$$

(e) False. 反例如下:

$$A=\begin{bmatrix}1&2\\2&1\end{bmatrix},\ x=\begin{bmatrix}-2\\1\end{bmatrix},\ x^TAx<0$$