線性代數解析--中央90資工所 廖亦德 解

1. (10%) 【中央90資工】

Let A be an $n\times n$ matrix whose n column vectors are linearly independent. From the study of linear algebra, we know that the rank of A is n. Describe at least another 6 properties of A.

【分析】本題屬於題型08E.

【解】請參閱綜線CH8定理17

2. (10%) 【中央90資工】

Determine the plane : $ax+by+cz=1$ (i.e. find coefficients a, b, c) in R^3 that pass through points (–1, –2, 2), (2, 1, –1), and (3, –4, 2).

【分析】本題屬於題型01B. 請參閱綜線CH1定義19.

【解】三點分別代入得: $-a-2b+2c=1$, $2a+b-c=1$, $3a-4b+2c=1$

解聯立方程式(細節略), 得出 $a=1, b=2, c=3$.

3. (20%) 【中央90資工】

For each of the following two statements, if you think the statement is correct, then give a proof to prove that the statement is correct; otherwise give a counterexample to show that the statement is incorrect.

(a) If any three vectors v_1, v_2, v_3 in R^n are linearly independent, then the vectors $w_1=v_1+v_2$, $w_2=v_1+v_3$, $w_3=v_2+v_3$ are also linearly independent.

(b) If T is a linear transformation that maps R^n onto R^n (i.e., for any vector y in R^n, we can always find a vector x in R^n such that $T(x)=y$), then T must be one-to-one.

【分析】本題(a)屬於題型06A, 請參閱綜線CH6定理27c.

本題(b)屬於題型08E, 請參閱綜線CH8定理11.

【解】(a) True, 證明如下:

若 $aw_1+bw_2+cw_3=o$, 即 $a(v_1+v_2)+b(v_1+v_3)+c(v_2+v_3)=o$, (欲證$a=b=c=0$)

集項整理得 $(a+b)v_1+(a+c)v_2+(b+c)v_3=o$.

而已知v_1, v_2, v_3線性獨立,

∴ $a+b=0, a+c=0, b+c=0$. (綜線CH6定義9)

解方程式得 $a=0, b=0, c=0$.

(b) True, 請參閱綜線CH8定理11自證.

4. (5%) 【中央90資工】

If A is a 5 by 5 matrix with $|a_{ij}| \le 1$, then $\det A \le$? . Give the upper bound as tight as possible and give a reason to justify your answer.

【分析】本題屬於題型04B. 請參閱綜線CH4定義4.

【解】$|\det A| = |\sum \text{sgn}(\sigma) a_{1\sigma(1)}a_{2\sigma(2)}\ldots a_{n\sigma(n)}|$

$\le \sum |\text{sgn}(\sigma) a_{1\sigma(1)}a_{2\sigma(2)}\ldots a_{n\sigma(n)}| = \sum |a_{1\sigma(1)}a_{2\sigma(2)}\ldots a_{n\sigma(n)}| \le \sum 1 = 5!$

∴ $\det A \le 120$.

5. (5%) 【中央90資工】

Compute the determinent $\begin{vmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \\ -8 & -1 & 1 & 8 \end{vmatrix}$.

【分析】本題屬於題型04B. 請參閱綜線CH4範例12.

【解】

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -2 & -1 & 1 & 2 \\ 4 & 1 & 1 & 4 \\ -8 & -1 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 1 \\ -2 & 0 & 1 & 2 \\ 4 & 2 & 1 & 4 \\ -8 & 0 & 1 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 1 & 1 \\ -2 & 0 & 1 & 2 \\ 3 & 0 & 0 & 3 \\ -8 & 0 & 1 & 8 \end{vmatrix}$$

$$= -2\begin{vmatrix} -2 & 1 & 2 \\ 3 & 0 & 3 \\ -8 & 1 & 8 \end{vmatrix} = -2\begin{vmatrix} -2 & 1 & 2 \\ 3 & 0 & 3 \\ -6 & 0 & 6 \end{vmatrix} = 2\begin{vmatrix} 3 & 3 \\ -6 & 6 \end{vmatrix} = 72$$

6. (20%) 【中央90資工】

對錯申論題(一定要有說明，每小題答對給5分，答錯扣2分，不答0分)

(a) Two vector sets $\left\{ \begin{bmatrix} a+3b \\ a-b \\ 2a-b \\ 4b \end{bmatrix} \middle| \, a, b \in R \right\}$ and $\left\{ \begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix} \middle| \, a, b, c \in R \right\}$ are both subspace of R^4.

(b) If A is an $m \times n$ matrix and Col$A=R^{\mathrm{m}}$, then

(i) linear system $Ax=b$ has a unique solution for every b in R^{m}.

(ii) transformation $x \to Ax$ is one-to-one.

(c) If $n \times n$ matrix A has n linear independent eigenvectors, then A is invertible and A^{-1} also has n linear independent eignevectors.

(d) If matrix A is diagonalizable, then the columns of A are linearly independent and A has n distinct eignevalues.

【分析】本題(a)屬於題型05C. 本題(b)屬於題型08E. 本題(c)(d)屬於題型12B.

【解】(a) True,

$$\left\{\begin{bmatrix} a+3b \\ a-b \\ 2a-b \\ 4b \end{bmatrix} \middle|\, a, b\in R \right\} = \left\{ a\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + b\begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \middle|\, a, b\in R \right\} = \text{CSP}\begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 2 & -1 \\ 0 & 4 \end{bmatrix}$$

$$\left\{\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix} \middle|\, a, b, c\in R \right\} = \text{CSP}\begin{bmatrix} 4 & 3 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ -2 & 0 & 1 \end{bmatrix}$$

定理保證$4\times n$矩陣的column space爲$\mathbb{R}^{4\times 1}$的subspace.　　(綜線CH5定理17)

(b) False.

(i)只成立一半: 必有解, 但未必唯一　　(綜線CH5定理17)

不唯一之反例與(ii)同:

(ii)不成立: 反例如下:

$$m=2,\ n=3,\ A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\quad \text{Col}(A)=\mathbb{R}^{2\times 1}$$

$$A\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} = A\begin{bmatrix} 5 \\ 6 \\ 0 \end{bmatrix}$$

(c) False.　反例如下:

$$A=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, A\text{有三個線性獨立之特徵向量}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

但A不可逆.

(d) False. 反例如下:

$$A=\begin{bmatrix}1&0&0\\0&1&0\\0&0&2\end{bmatrix}$$, A可對角化, 但相異特徵值只有2個.

7. (10%) 【中央90資工】

Find a base for the orthogonal complement of column space of matrix $A=\begin{bmatrix}-2&2&-3\\4&-6&8\\-2&-3&2\\-4&1&-3\end{bmatrix}$

【分析】本題屬於題型11C. 請參閱綜線CH11範例24.

【解】$(CSP(A))^{\perp}=\ker(A^T)$ (綜線CH11定理23)

解方程式 $A^Tx=o$:

$$\begin{bmatrix}-2&4&-2&-4\\2&-6&-3&1\\-3&8&2&-3\end{bmatrix}\sim\underset{\text{列運算}}{\cdots\cdots\cdots\cdots}\sim\begin{bmatrix}1&0&6&5\\0&1&5/2&3/2\\0&0&0&0\end{bmatrix}$$

通解為 $[-6s-5t, -5s/2-3t/2, s, t]^T$, s,t為任意常數.

即 $[-12p-10q, -5p-3q, 2p, 2q]^T$, p, q為任意常數.

即 $p[-12, -5, 2, 0]^T+q[-10, -3, 0, 2]^T$, p, q為任意常數.

∴ 基底可取為 $\{[-12, -5, 2, 0]^T, [-10, -3, 0, 2]^T\}$

8. (10%) 【中央90資工】

Find a least-squares solution of $Ax=b$ and the least squares error associated with the solution, where $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$.

【分析】本題屬於題型09E. 請參閱綜線CH9定理21a.

【解】解 $A^TAx=A^Tb$, 即解:

$$\begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

解得 $x_1=1, x_2=1$.

$\| Ax-b \|^2 = \| [-1, -0, 2]^T \|^2 = 5$

9. (10%)　【中央90資工】

Find a singular value decomposition of matrix $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$

【分析】本題屬於題型13F. 請參閱綜線CH13範例29.

【解】1° 先對A^TA做正交對角化:

$$A^TA= \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$$

$\det(A^TA-xI) = \quad \ldots \quad = -x(x-9)(x-25)$

解 $(A^TA-25I)\boldsymbol{v}=\boldsymbol{o}$得出25的特徵向量 $[1, 1, 0]^T$

解 $(A^TA-9I)\boldsymbol{v}=\boldsymbol{o}$得出9的特徵向量 $[1, -1, 4]^T$

解 $(A^TA-0I)\boldsymbol{v}=\boldsymbol{o}$得出0的特徵向量 $[-2, 2, 1]^T$

令$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 0 & 4/\sqrt{18} & 1/3 \end{bmatrix}$

則 Q爲正交矩陣，且$Q^{T}A^{T}AQ=\begin{bmatrix} 25 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

2° 取奇異值矩陣

$\Sigma=\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}$.　　(並且 $S=\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$)

3°

$AQ=\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{18} & -2/3 \\ 1/\sqrt{2} & -1/\sqrt{18} & 2/3 \\ 0 & 4/\sqrt{18} & 1/3 \end{bmatrix}$

$= \left[\begin{array}{cc|c} 5/\sqrt{2} & 3/\sqrt{2} & 0 \\ 5/\sqrt{2} & -3/\sqrt{2} & 0 \end{array}\right]$　　$(= [\,B \mid C\,])$

將AQ的非零行單位化，此時已是正交矩陣P :

$P=\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

則得出奇異值分解式 $A=P\Sigma Q^{T}$.　　(讀者可自行驗算)