線性代數解析—政大90資科所

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計算機數學(離散數學部份共計60分)
計算機數學(線性代數部份共計40分)

1. (24%) 【政大90資科】

True or False: a brief proof or counterexample is needed.

(a) Every invertible matrix can be written as a product of elementary matrices.

(b) Similar matrices have the same eigenvalues and eigenvectors.

(c) If W is a subspace of a vector space of V, then every basis of W can be expanded into a basis for V. Conversely, every basis of V can be reduced to a basis of W.

【分析】本題(a)屬於題型03E. 請參閱綜線CH3定理16.
本題(b)屬於題型16A. 請參閱綜線CH16定理1a.
本題(c)屬於題型06B. 請參閱綜線CH6定理21.

【解】(a) True. (綜線CH3定理16)

可逆矩陣A可經由一連串的基本列運算化為I.

I也可經前述運算的逆運算化為A.

每個基本列運算都可用左乘一個基本列矩陣而完成.

∴ I可左乘一串適當的基本列矩陣而化為A.

∴ A就是這串基本列矩陣的乘積.

(b) False. (綜線CH16定理1a)

設$A=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $P=\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $B=P^{-1}AP$

則A, B相似, A的特徵向量為$[t, 0]^T$, $[0, t]^T$,

而B的特徵向量為$[t, t]^T$, $[-t, t]^T$,

(c) False. (只對前半) (綜線CH6定理21)

設$V=\mathbb{R}^2$, $W=\{(t, t)|\ t\in \mathbb{R}\}$.

V的基底{(1,0), (0,1)}不能reduce成W的基底.

2. (8%) 【政大90資料】

Find all solutions to the following linear system by reducing the associated matrix to row reduced echelon form.

$$\begin{aligned} 2x-y-3z+w&=2\\ x\quad -2z+w&=1\\ -3x+y+z+2w&=3 \end{aligned}$$

【分析】本題屬於題型03A. 請參閱綜線CH3範例7.

【解】分隔矩陣經列運算(細節略)可化爲

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & -1 & -3/2 \end{array}\right]$$

通解爲 $x=-2+t$, $y=-3/2$, $z=-3/2+t$, $w=t$. t爲任意參數.

3. (8%) 【政大90資料】

Let V be the space of all 2×2 matrices with real entries. Let T: $V\to V$ be defined by $T(A)=A'$ where A' is the transpose matrix of A. Show that T is a linear transformation. Is T diagonalizable? Explain your answer.

【分析】本題屬於題型12C. 請參閱綜線CH12範例17a.

【解】1° $\forall A, B\in V$, $\forall h, k\in \mathbb{R}$,

$$\begin{aligned} T(hA+kB)&=(hA+kB)'\\ &=hA'+kB' \qquad \text{(綜線CH2定理23)}\\ &=hT(A)+kT(B) \end{aligned}$$

2° 可對角化.

$$T\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\right)=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad T\left(\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right)=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},$$

$$T\left(\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\right)=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad T\left(\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right)=-\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix},$$

T擁有以eigenvector組成之基底:

$$\left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\right\}$$

∴ T可對角化.　　(綜線CH12定義15)

【討論】本題本質上與下列映射相同

$T:\ \mathbb{R}^4\longrightarrow\mathbb{R}^4,\ T(x, y, z, w)=(y, x, z, w)$

特徵向量利用幾何直覺即可取得.