線性代數解析--交大90資科所 廖亦德 解

1. (7%) 【交大90資料】

Let $B_1=\left\{\begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix},\begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix},\begin{bmatrix}a_3\\b_3\\c_3\end{bmatrix}\right\}$ and $B_2=\left\{\begin{bmatrix}r_1\\s_1\\t_1\end{bmatrix},\begin{bmatrix}r_2\\s_2\\t_2\end{bmatrix},\begin{bmatrix}r_3\\s_3\\t_3\end{bmatrix}\right\}$ be two distinct ordered bases of R^3.

(a) Write down the change of basis matrix (from B_1 to B_2).

(b) Let T be a linear transformation from R^3 to R^3 defined by

$$T(\begin{bmatrix}x\\y\\z\end{bmatrix})=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

Write down the matrix representation of T with respect to the bases B_1 and B_2.

【分析】本題(a)屬於題型06D, (b)屬於題型06D07C.

【解】(a)

設所求爲 $\begin{bmatrix}p_{11}&p_{12}&p_{13}\\p_{21}&p_{22}&p_{23}\\p_{31}&p_{32}&p_{33}\end{bmatrix}$, 則

$$\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = p_{11} \begin{bmatrix} r_1 \\ s_1 \\ t_1 \end{bmatrix} + p_{21} \begin{bmatrix} r_2 \\ s_2 \\ t_2 \end{bmatrix} + p_{31} \begin{bmatrix} r_3 \\ s_3 \\ t_3 \end{bmatrix}$$

$$\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = p_{12} \begin{bmatrix} r_1 \\ s_1 \\ t_1 \end{bmatrix} + p_{22} \begin{bmatrix} r_2 \\ s_2 \\ t_2 \end{bmatrix} + p_{32} \begin{bmatrix} r_3 \\ s_3 \\ t_3 \end{bmatrix}$$

$$\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = p_{13} \begin{bmatrix} r_1 \\ s_1 \\ t_1 \end{bmatrix} + p_{23} \begin{bmatrix} r_2 \\ s_2 \\ t_2 \end{bmatrix} + p_{33} \begin{bmatrix} r_3 \\ s_3 \\ t_3 \end{bmatrix}$$

(綜線CH6定義33)

$$\therefore \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{21} \\ p_{31} \end{bmatrix}$$

$$\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} p_{12} \\ p_{22} \\ p_{32} \end{bmatrix}$$

$$\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} p_{13} \\ p_{23} \\ p_{33} \end{bmatrix}$$

(綜線CH2定理6)

$$\therefore \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

(綜線CH2定理6)

$$\therefore \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix}^{-1} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

(b)設所求為 $\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$，則

$$T\left(\begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}\right) = m_{11}\begin{bmatrix} r_1 \\ s_1 \\ t_1 \end{bmatrix} + m_{21}\begin{bmatrix} r_2 \\ s_2 \\ t_2 \end{bmatrix} + m_{31}\begin{bmatrix} r_3 \\ s_3 \\ t_3 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix}$$

$$T\left(\begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}\right) = m_{12}\begin{bmatrix} r_1 \\ s_1 \\ t_1 \end{bmatrix} + m_{22}\begin{bmatrix} r_2 \\ s_2 \\ t_2 \end{bmatrix} + m_{32}\begin{bmatrix} r_3 \\ s_3 \\ t_3 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} m_{12} \\ m_{22} \\ m_{32} \end{bmatrix}$$

$$T\left(\begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix}\right) = m_{13}\begin{bmatrix} r_1 \\ s_1 \\ t_1 \end{bmatrix} + m_{23}\begin{bmatrix} r_2 \\ s_2 \\ t_2 \end{bmatrix} + m_{33}\begin{bmatrix} r_3 \\ s_3 \\ t_3 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} m_{13} \\ m_{23} \\ m_{33} \end{bmatrix}$$

(綜線CH7定義9)

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{21} \\ m_{31} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} m_{12} \\ m_{22} \\ m_{32} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} m_{13} \\ m_{23} \\ m_{33} \end{bmatrix}$$

(依T之定義)

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

(綜線CH2定理6)

$$\therefore \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \\ t_1 & t_2 & t_3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

【另解】(此法較快, 但需能靈活運用定理. 請參閱CH7定理19證法二)

將B_1的三個向量拼成P, 並將B_1的三個向量拼成Q. 並設S爲R^3的標準基底.

(a) $\forall v \in R^3$, $[v]_S = P\,[v]_{B_1}$　　(綜線CH6定理33)

$\therefore \forall v \in R^3$, $[v]_S = Q\,[v]_{B_2}$　　(綜線CH6定理33)

$\therefore\quad \forall v \in R^3$,　$[v]_{B_2} = Q^{-1}\,[v]_S = Q^{-1}P\,[v]_{B_1}$

$\therefore$ 所求爲 $Q^{-1}P$　　(綜線CH6定理33)

(b) 設T相對於S, S的矩陣表示為M, 設T相對於S, S的矩陣表示為M',

$\forall v\in R^3$, $[T(v)]_{B_2} = M'[v]_{B_1}$ (綜線CH7定理15)

而 $[T(v)]_{B_2} = Q^{-1}[T(v)]_S$ (綜線CH6定理33)

$= Q^{-1}M\,[v]_S$ (綜線CH7定理15)

$= Q^{-1}M\,P\,[v]_{B_1}$ (綜線CH6定理33)

∴ 所求為 $Q^{-1}MP$ (綜線CH7定理15)

2. (7%) 【交大90資料】

Consider the vector space $C[-1, 1]$ with the inner product $\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx$

(a) Write down the Cauchy-Schwarz inequality (with respect to this particular inner product).

(b) Consider the subspace W=span$\{1, x, x^2\}$. Find an orthonormal basis for W.

(c) Find the orthogonal projection of x^3 onto the subspace W.

【分析】本題屬於題型09C. 請參閱綜線CH9定理12, CH9定理16.

【解】(a) $\left(\int_{-1}^{1} f(x)g(x)dx\right)^2 = \left(\int_{-1}^{1} f(x)^2dx\right)\left(\int_{-1}^{1} g(x)^2dx\right)$

(b) Let $u_0=1$, $u_1=x$, $u_2=x^2$.

$u_0'=u_0=1$.

$\langle u_0', u_0'\rangle = \int_{-1}^{1} 1dx = 2$. $\langle u_1,u_0'\rangle= \int_{-1}^{1} x\,dx = 0$

$u_1'=u_1-\dfrac{\langle u_1,u_0'\rangle}{\langle u_0',u_0'\rangle}u_0'=x-\dfrac{0}{2}\cdot 1 = x$.

$\langle u_1',u_1'\rangle = \int_{-1}^{1} x^2dx = 2/3$

$\langle u_2,u_0'\rangle = \int_{-1}^{1} x^2\,dx = 2/3$. $\langle u_2,u_1'\rangle = \int_{-1}^{1} x^3\,dx = 0$

$u_2' = u_2-\dfrac{\langle u_2,u_0'\rangle}{\langle u_0',u_0'\rangle}u_0'-\dfrac{\langle u_2,u_1'\rangle}{\langle u_1',u_1'\rangle}u_1' = x^2-\dfrac{2/3}{2}1-\dfrac{0}{2/3}\,x = x^2-1/3$

$\therefore$ { 1, x, $x^2-1/3$} is an orthogonal set .

$$\langle u_2',u_2'\rangle = \int_{-1}^{1}(x^2-1/3)^2dx = 8/45$$

$$\| u_0' \| = \langle u_0',u_0'\rangle^{1/2} = \sqrt{2}\ , \quad \| u_1' \| = \langle u_1',u_1'\rangle^{1/2} = \sqrt{2/3}$$

$$\| u_2' \| = \langle u_2',u_2'\rangle^{1/2} = \sqrt{8/45}\ .$$

$\therefore$ { $1/\sqrt{2}$, $\sqrt{3/2}\ x$, $\sqrt{45/8}\,(x^2-1/3)$ } is an orthonormal set.

(b)

$$\langle x^3,u_0'\rangle = \int_{-1}^{1} x^3\,dx = 0\ . \qquad \langle x^3,u_1'\rangle = \int_{-1}^{1} x^4\,dx = 2/5$$

$$\langle x^3,u_2'\rangle = \int_{-1}^{1} x^3(x^2-1/3)\,dx = 0$$

所求之投影爲　　(綜線CH9定理12)

$$\frac{\langle x^3,u_0'\rangle}{\langle u_0',u_0'\rangle}u_0' + \frac{\langle x^3,u_1'\rangle}{\langle u_1',u_1'\rangle}u_1' + \frac{\langle x^3,u_2'\rangle}{\langle u_2',u_2'\rangle}u_2' = \frac{2/5}{2/3}\ x = (3/5)x$$

3. (7%)　　【交大90資料】

Let T be the linear transformation from R^3 to R^3 defined as follows.

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \text{rotating} \begin{bmatrix} x \\ y \end{bmatrix} \text{ counter clockwise by an angle } \theta .$$

(a) Write down the matrix representation A of T with respect to the standard basis.

(b) Find the eigenvalues of A and the corresponding eigenvectors.

【勘誤】本題題目中的兩個R^3都應該改成R^2.

【分析】本題屬於題型12C. 請參閱綜線CH12範例9. CH12習題9.4.

【解】

(a) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

(b) $\det(A-\lambda I) = \ldots = \lambda^2-2(\cos\theta)\lambda+1$,

$\lambda = \cos\theta \pm (\cos^2\theta - 1)^{1/2} = \cos\theta \pm i\sin\theta$

當$\sin\theta=0$時, $\cos\theta=\pm1$, $A=I$或$-I$,

對$A=I$, eigenvalue為1,1, eigenvector為所有非零向量.

對$A=-I$, eigenvalue為$-1,-1$, eigenvector為所有非零向量.

當$\sin\theta\neq0$時(θ不為π的整倍數),

$$A-\lambda I = \begin{bmatrix} \cos\theta-\lambda & -\sin\theta \\ \sin\theta & \cos\theta-\lambda \end{bmatrix}$$

$$= \begin{bmatrix} \mp i\sin\theta & -\sin\theta \\ \sin\theta & \mp i\sin\theta \end{bmatrix} \sim \begin{bmatrix} \mp i & -1 \\ 1 & \mp i \end{bmatrix} \sim \begin{bmatrix} \mp i & -1 \\ 0 & 0 \end{bmatrix}$$

解得eigenvector $[\ \pm i, 1\ ]^T$.

4. (7%) 【交大90資料】

Find the values of k so that the vectors $[3-k \quad -1 \quad 0]^T$, $[-1 \quad 2-k \quad -1]^T$, and $[0 \quad -1 \quad 3-k]^T$ span a two-dimensional space.

【分析】本題屬於題型06A. 請參閱綜線CH6定理14.

【解】先做列運算:

$$\begin{bmatrix} 3-k & -1 & 0 \\ -1 & 2-k & -1 \\ 0 & -1 & 3-k \end{bmatrix} \sim \begin{bmatrix} 0 & k^2-5k+5 & k-3 \\ -1 & 2-k & -1 \\ 0 & -1 & 3-k \end{bmatrix} \sim \begin{bmatrix} 0 & k^2-5k+4 & 0 \\ -1 & 2-k & -1 \\ 0 & -1 & 3-k \end{bmatrix}$$

$$= \begin{bmatrix} 0 & (k-4)(k-1) & 0 \\ -1 & 2-k & -1 \\ 0 & -1 & 3-k \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & (k-4)(k-1)(3-k) \\ -1 & 2-k & -1 \\ 0 & -1 & 3-k \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 2-k & -1 \\ 0 & -1 & 3-k \\ 0 & 0 & (k-4)(k-1)(3-k) \end{bmatrix}$$

$[3-k \quad -1 \quad 0]^T$, $[-1 \quad 2-k \quad -1]^T$, and $[0 \quad -1 \quad 3-k]^T$ span a two-dimensional space.

$$\Longleftrightarrow \text{rank} \begin{bmatrix} 3-k & -1 & 0 \\ -1 & 2-k & -1 \\ 0 & -1 & 3-k \end{bmatrix} = 2$$ (綜線CH8定義13)

$\Longleftrightarrow (k-4)(k-1)(3-k)=0$

$\Longleftrightarrow k \in \{1, 3, 4\}$

5. (7%) 【交大90資科】

Find an orthonormal basis for the solution space of the following homogeneous system.

$x + y - z + w = 0$

$2x - y + z + 2w = 0$

【分析】本題屬於題型09C. 請參閱綜線CH3範例7, CH9定理16.

【解】先解方程式(細節略) (參綜線CH3範例7)

得出解集合 $\{ [-t, s, s, t]^T \mid s, t \in \mathbb{R} \}$

$= \{ s[0, 1, 1, 0]^T + t[-1, 0, 0, 1]^T \mid s, t \in \mathbb{R} \}$

$\therefore$ 可取基底 $\{ [0, 1, 1, 0]^T, [-1, 0, 0, 1]^T \}$

其中兩向量已正交,

$\therefore$ 可取基底 $\{ [0, 1/\sqrt{2}, 1/\sqrt{2}, 0]^T, [-1/\sqrt{2}, 0, 0, 1/\sqrt{2}]^T \}$

6. (7%) 【交大90資科】

Find the matrix P such that $P^{-1}AP$ is a diagonal matarx, where

$$A=\begin{bmatrix}1&1&0\\0&2&1\\0&0&3\end{bmatrix}$$

【分析】本題屬於題型12C. 請參閱綜線CH12範例17.

【解】$\det(A-xI)=(1-x)(2-x)(3-x)$,

$$A-I=\begin{bmatrix}0&1&0\\0&1&1\\0&0&2\end{bmatrix}\sim\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$$，解得eigenvector $\begin{bmatrix}1\\0\\0\end{bmatrix}$

$$A-2I=\begin{bmatrix}-1&1&0\\0&0&1\\0&0&1\end{bmatrix}\sim\begin{bmatrix}1&-1&0\\0&0&1\\0&0&0\end{bmatrix}$$，解得eigenvector $\begin{bmatrix}1\\1\\0\end{bmatrix}$

$$A-2I=\begin{bmatrix}-2&1&0\\0&-1&1\\0&0&0\end{bmatrix}\sim\begin{bmatrix}1&0&-1/2\\0&1&-1\\0&0&0\end{bmatrix}$$，解得eigenvector $\begin{bmatrix}1\\2\\2\end{bmatrix}$

令$P=\begin{bmatrix}1&1&1\\0&1&2\\0&0&2\end{bmatrix}$，則 $P^{-1}AP=\mathrm{diag}(1,2,3)$

7. (8%)　【交大90資科】

Determine whether the given set, together with the standard operations, is a vector space.

(是非題，答錯一題倒扣2分，扣完爲止)

(a) The set of all 2×2 singular matrices.

(b) The set of all 2×2 nonsingular matrices.

(c) The set of all 2×2 diagonal matrices.

(d) The set of all 2×2 singular matrices of the form

$$\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$$

【分析】本題屬於題型05B. 請參閱綜線CH5範例12--15b.

【解】(a) 非, (b) 非, (c) 是, (d) 非

【討論】singular matrix相加未必仍爲singular, 例如

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

non-singular matrix相加未必仍爲non-singular, 例如

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}$$

diagonal matrix的線性組合一定是diagonal, 因爲

$$x\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + y\begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} xa+yc & 0 \\ 0 & xb+yd \end{bmatrix}$$