線性代數解析—清大91資應所 廖亦德 解

[1]. (10%) 【清大91資應】

If matrix A is similar to matrix B, show that

(a) $\det(A^c - cI) = \det(B^c - cI)$ (5%)

(b) The trace of A^T is equal to that of B^T

【分析】本題屬於題型07D.

本題A^c顯得奇怪，可能是筆誤. 原意可能是A^T或是A的正整數次方.

對正整數k, A^k是k個A相乘. A^0定義爲I.

若A可逆，則A^{-k}定義爲$(A^{-1})^k$. 但本題並未指出A可逆

【解】設$B=P^{-1}AP$. (綜線CH7定義21)

(a) $\det(B^c - cI) = \det((P^{-1}AP)^c - cI)$

$= \det(P^{-1}A^cP - cP^{-1}IP)$ (綜線CH7定理22)

$= \det(P^{-1}(A^c - cI)P)$ (提出左右因式)

$= \det(A^c - cI)$ (綜線CH4定理6a)

(b) $\mathrm{tr}(B^T) = \mathrm{tr}(B)$ (綜線CH2定理28⑤)

$= \mathrm{tr}(P^{-1}AP) = \mathrm{tr}(APP^{-1})$ (綜線CH2定理28③)

$= \mathrm{tr}(A) = \mathrm{tr}(A^T)$ (綜線CH2定理28⑤)

[2]. (15%) 【清大91資應】

Suppose a linear operator L transforms $(1, 0, -1)$ to $(0, 0, -2)$, $(1, -1, 2)$ to $(-1, 7, 1)$, and $(-1, -1, 1)$ to $(-1, 1, 2)$, respectively.

(a) Find the matrix A that represents L. (5%)

(b) Find the kernel of L. (5%)

(c) Find the determinant of the adjoint of matrix A, namely, $\det(\mathrm{adj}A)$. (5%)

【分析】本題(a)屬於題型07B. 相關類題請參閱綜線CH7範例11.

解矩陣方程式$XC=D$時, 對$[C^T|D^T]$列運算求解 (綜線CH3範例12a)

本題(b)屬於題型06C. 相關類題請參閱綜線CH5範例21.

本題(c)屬於題型04A及04B. 相關類題請參閱綜線CH4範例12.

【解】(a) 由題意得 (綜線CH7定理15)

$$A\begin{bmatrix}1\\0\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\-2\end{bmatrix},\quad A\begin{bmatrix}1\\-1\\2\end{bmatrix}=\begin{bmatrix}-1\\7\\1\end{bmatrix},\quad A\begin{bmatrix}-1\\-1\\1\end{bmatrix}=\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

$$A\begin{bmatrix}1&1&-1\\0&-1&-1\\-1&2&1\end{bmatrix}=\begin{bmatrix}0&-1&-1\\0&7&1\\-2&1&2\end{bmatrix}$$ (綜線CH2定理6:右直切)

可解得(細節略):

$$A=\begin{bmatrix}0&1&0\\2&-1&2\\-1&0&1\end{bmatrix}$$ (綜線CH3範例12a)

(b) A經列運算可化爲(細節略) I. (綜線CH3範例4b)

$\therefore\ \ker L=\{\,o\,\}$ (綜線CH5範例21)

(c) $\det A=$ ...(細節略) $=-4$ (綜線CH4範例12)

$\det(\operatorname{adj}A)=(\det A)^2$ (綜線CH4定理17要訣5)

$=16$

[3]. (5%) 【清大91資應】

Find the determinant of the matrix $\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$

【分析】本題屬於題型4B. 此題爲三線行列式, 相關類題請參閱綜線CH4範例13.

【解法一】直接降階求算可得det爲0. (細節略) (綜線CH4範例12)

【解法二】設此型之n階行列式爲d_n.

由降階可得 $d_n=d_{n-1}-d_{n-2}$. (綜線CH4範例13)

顯然 $d_1=1$, $d_2=0$,

∴ $d_3=0-1=-1$, $d_4=-1-0=-1$. $d_5=-1-(-1)=0$.

[4]. (12%) 【清大91資應】

Suppose matrix A equals $L\times U$, where L is a unit lower triangular matrix, and U is a unit upper triangular matrix. Prove or disprove the following:

(a) A shares the same row space with L. (3%)

(b) A shares the same column space with L. (3%)

(c) A shares the same row space with U. (3%)

(d) A shares the same column space with U. (3%)

【分析】本題屬於題型05C. 請參閱綜線CH5定理17⑤⑥.

本題看來像是LU分解, 但其實只用到列(行)運算的基本性質

【解】(a)(b)(c)(d) 都是prove. 統一說明如下:

單位上(下)三角的行列式不爲0, (綜線CH4定理4a)

所以是可逆矩陣. (綜線CH4定理17)

∴ A也是可逆矩陣. (綜線CH2定理12)

設A, L, U都是n階方陣,

∵ 這三個矩陣都可逆, ∴都列等價於I, (綜線CH3定理16)

∴row space都是$R^{1\times n}$ (參綜線CH5範例18)

∵ 這三個矩陣都可逆, ∴都行等價於I, (同理)

∴column space都是$R^{1\times n}$ (參綜線CH5範例18)

【討論】假設U只是upper triangular matrix (未必可逆),

則(c)仍成立, 但(a)不成立.

(c) prove:

∵ L可逆, ∴ A可經列運算化為U. (綜線CH3定理17)

∴ A與U的row space相同 (綜線CH5定理17)

(a) 舉反例如下:

$$設\ L=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}, U=\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}, 則\ A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$$

row space of $A = \{\, t[1, 1] \mid t\in\mathbb{R}\ \}$

row space of $L = \{\, [x, y] \mid x, y\in\mathbb{R}\ \}$

假設L只是lower triangular matrix (未必可逆),

則(b)仍成立, 但(d)不成立.

[5]. (8%) 【清大91資應】

Given $\boldsymbol{a}_1=[1, -1]^T$, $\boldsymbol{a}_2=[0, 1]^T$, $\boldsymbol{b}_1=[1, -1, 0]^T$, $\boldsymbol{b}_2=[1, 0, -1]^T$, $\boldsymbol{b}_3=[0, -1, -1]^T$, and the linear transformation L from R^2 into R^2 defined as: $L([x_1, x_2]^T)=x_1\boldsymbol{b}_2+x_2\boldsymbol{b}_1+(x_1+x_2)\boldsymbol{b}_3$.

(a) Find the matrix A representing L with respect to the bases $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ and $[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$ (2%)

(b) Find the matrix B representing L with respect to the bases $[\boldsymbol{e}_1, \boldsymbol{e}_2]$ and $[\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3]$ (2%)

(c) Find the matrix C representing L with respect to the bases $[\boldsymbol{a}_1, \boldsymbol{a}_2]$ and $[\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3]$ (2%)

(d) Find the matrix D representing L with respect to the bases $[\boldsymbol{a}_1, \boldsymbol{a}_2]$ and $[\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3]$ (2%)

【分析】本題屬於題型07B. 相關類題請參閱綜線CH7範例10.

【解】(a) $T(\boldsymbol{e}_1)=T([1, 0]^T)=\boldsymbol{b}_2+\boldsymbol{b}_3=[1, 0, -1]^T+[0, -1, -1]^T=[1, -1, -2]^T=\boldsymbol{e}_1-\boldsymbol{e}_2-2\boldsymbol{e}_3$

$T(\boldsymbol{e}_2)=T([0, 1]^T)=\boldsymbol{b}_1+\boldsymbol{b}_3=[1, -1, 0]^T+[0, -1, -1]^T=[1, -2, -1]^T=\boldsymbol{e}_1-2\boldsymbol{e}_2-\boldsymbol{e}_3$

$$\therefore\ A=\begin{bmatrix} 1 & 1 \\ -1 & -2 \\ -2 & -1 \end{bmatrix}$$ (綜線CH7定義9)

(b) $T(\boldsymbol{e}_1)=T([1, 0]^T)=\boldsymbol{b}_2+\boldsymbol{b}_3$

$T(\boldsymbol{e}_2)=T([0, 1]^T)=\boldsymbol{b}_1+\boldsymbol{b}_3$

$$\therefore\ A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$ (綜線CH7定義9)

(c) $T(\boldsymbol{a}_1)=T([1, -1]^T)=\boldsymbol{b}_2-\boldsymbol{b}_1=[1, 0, -1]^T-[1, -1, 0]^T=[0, 1, -1]^T=\boldsymbol{e}_2-\boldsymbol{e}_3$

$T(\boldsymbol{a}_2)=T([0, 1]^T)=\boldsymbol{b}_1+\boldsymbol{b}_3=[1, -1, 0]^T+[0, -1, -1]^T=[1, -2, -1]^T=\boldsymbol{e}_1-2\boldsymbol{e}_2-\boldsymbol{e}_3$

$$\therefore\ C=\begin{bmatrix} 0 & 1 \\ 1 & -2 \\ -1 & -1 \end{bmatrix}$$ (綜線CH7定義9)

(d) $T(\boldsymbol{a}_1)=T([1, -1]^T)=\boldsymbol{b}_2-\boldsymbol{b}_1$

$T(\boldsymbol{e}_2)=T([0, 1]^T)=\boldsymbol{b}_1+\boldsymbol{b}_3$

$$\therefore\ D=\begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$ (綜線CH7定義9)

[6]. (15%)　【清大91資應】

Let S=Span([1, 3, 1, 1]^T, [1, 1, 1, 1]^T, [–1, 5, 2, 2]^T) be a subspace of R^4,
and let b=[4, –1, 5, 1]^T

(a) Find an orthonormal basis for S.　(5%)

(b) Use your answer in (a) to find the projection p of b onto S.　(5%)

(c) Given $A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 5 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix}$,

Use your answer in (b) to solve the least squares problem $Ax=b$.　(5%)

【分析】本題(a)屬於題型09C. 相關類題請參閱綜線CH9範例17.
本題(b)屬於題型09B. 請參閱綜線CH9定理12.
本題(c)屬於題型09E. 請參閱綜線CH9定理21a.

【解】(a) 由Gram-Schmidt process可求得S的正交基底 (過程略，參綜線CH9範例17)

{ [1, 3, 1, 1]^T, [1/2, –1/2, 1/2, 1/2]^T, [–2, 0, 1, 1]^T }

再單爲化即得出正交單位基底

{ [1/(12)$^{1/2}$, 3/(12)$^{1/2}$, 1/(12)$^{1/2}$, 1/(12)$^{1/2}$]^T,
[1/2, –1/2, 1/2, 1/2]^T,　[–2/6$^{1/2}$, 0, 1/6$^{1/2}$, 1/6$^{1/2}$]^T }

(b) 利用正投影公式可求得b對S的正投影向量 (過程略，參綜線CH9定理12)

proj(b)=[4, –1, 3, 3]^T.

(c) $Ax=b$的最小平方問題應解 Ax=proj(b)　(綜線CH9定理21a)

經列運算可求得x=[–3/2, 31/6, –1/3]^T.　(過程略，參綜線CH3範例7)

[7]. (18%)　【清大91資應】

Let A be a diagonalizable matrix.

(a) Show that the number of nonzero eigenvalues (counted according to multiplicity) of A equals the rank of A.　(6%)

(b) Show that e^A is nonsingular. (6%)

(c) Let $p(\lambda)$ be the characteristic polynomial of A:

$p(\lambda)=a_n\lambda^n+a_{n-1}\lambda^{n-1}+...+a_1\lambda+a_0$.

Show that $p(A)=O$. (6%)

【分析】本題(a)屬於題型12A.

本題(b)屬於題型16B. 請參閱綜線CH16定理3.

本題(c)屬於題型16C. 請參閱綜線CH16範例18a.

【解】設$A=SDS^{-1}$. 其中$D=\text{diag}(\lambda_1, ..., \lambda_n)$

(a) 設$\lambda_1, ..., \lambda_r$不爲0, 而$\lambda_{r+1}, ..., \lambda_n$皆爲0.

再設$\{ v_1, v_2, ..., v_n \}$爲$K^{n\times 1}$的基底, 且滿足$Av_i=\lambda_i v_i,\ i=1,2,...,n$.

$CSP(A)=\{Ax \mid x\in K^{n\times 1} \}$ (綜線CH5定理17)

$=\{A(t_1v_1+t_2v_2+ ...+t_nv_n) \mid t_1, ..., t_n\in K\}$

$=\{t_1\lambda_1v_1+t_2\lambda_2v_2+ ...+t_n\lambda_nv_n \mid t_1, ..., t_n\in K\}$

$=\{t_1\lambda_1v_1+ ...+t_r\lambda_rv_r \mid t_1, ..., t_r\in K\}$

$=\{k_1v_1+ ...+k_rv_r \mid k_1, ..., k_r\in K\}$

∴ $CSP(A)$可取$\{v_1, ..., v_r\}$爲基底.

∴ $\text{rank}A=\dim CSP(A) = r$ (綜線CH8定義13, CH6定理19)

(b) $e^{tA}=\exp(tSDS^{-1})=S\exp(tD)S^{-1}=S\,\text{diag}(\exp(t\lambda_1), ..., \exp(t\lambda_n))\,S^{-1}$

(綜線CH16定理3)

$e^Ae^{-A}=S\,\text{diag}(\exp(\lambda_1), ..., \exp(\lambda_n))\,S^{-1}S\,\text{diag}(\exp(-\lambda_1), ..., \exp(-\lambda_n))\,S^{-1}$

$=S\,\text{diag}(\exp(\lambda_1), ..., \exp(\lambda_n))\,\text{diag}(\exp(-\lambda_1), ..., \exp(-\lambda_n))\,S^{-1}$

$=S\,\text{diag}(\exp(\lambda_1)\exp(-\lambda_1), ..., \exp(\lambda_n)\exp(-\lambda_n)\,)\,S^{-1}$

$=S\,\text{diag}(\exp(0), ..., \exp(0)\,)\,S^{-1}$

$=S\,\text{diag}(1, ..., 1\,)\,S^{-1}$

$=S\,I\,S^{-1} = I.$

同理, $e^{-A}e^A=I$.

∴ e^A 可逆. (且$(e^A)^{-1}=e^{-A}$) (綜線CH2定義10)

(c) $p(A)=p(SDS^{-1})=S\,p(D)S^{-1}$ (綜線CH16定理3)

$=S\,\text{diag}(p(\lambda_1), ..., p(\lambda_n))S^{-1}$ (綜線CH16定理3)

$=S\,\mathrm{diag}(0, ..., 0)S^{-1}$

$=O$.

[8]. (5%)　【清大91資應】

Let A be a symmetric matrix. Show that e^A is symmetric and positive definite.

【分析】本題屬於題型16A.

【勘誤】本題須補條件"A爲實數矩陣"才成立. 否則可舉反例如下:

若$A=(\quad i/2)I$, 則$\exp(A)=\exp(\quad i/2)I=iI$. 不會正定.

【解】[對稱] $(\exp(A))^T=(\sum (1/k!)A^k)^T$

$= (\sum (1/k!)(A^k)^T)$

$= (\sum (1/k!)(A^T)^k)$　　(綜線CH2定理23)

$= (\sum (1/k!)A^k) = \exp(A)$

[正定] ∵A爲實數對稱矩陣, 則A可正交對角化.　　(綜線CH13定理15)

令$A=SDS^{-1}$. 其中$D=\mathrm{diag}(\lambda_1, ..., \lambda_n)$, $\lambda_1, ..., \lambda_r$皆爲實數, 且$S^{-1}=S^T$.

$\exp(e^A)=\exp(SDS^{-1})=S\,\mathrm{diag}(\exp(\lambda_1), ..., \exp(\lambda_n))\,S^{-1}$　　(綜線CH16定理3)

∴ $\exp(e^A)$也可正交對角化.

而$\exp(e^A)$的特徵値$\exp(\lambda_1), ..., \exp(\lambda_n)$都大於0.

∴ $\exp(e^A)$爲正定矩陣.　　(綜線CH13定理17c)

[9]. (12%)　【清大91資應】

Give $A = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix}$.

(a) Find the Cholesky decomposision LL^T of A, where L is lower triangular with positive diagonal elements.　(6%)

(b) Find a unitary matrix U that diagonalizes A.

【分析】本題(a)屬於題型03E. 請參閱綜線CH3範例28a.

本題(b)屬於題型13C. 請參閱綜線CH13範例16.

【解】(a)

$$\begin{bmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & 5 \end{bmatrix} \xrightarrow{(1)\,(-1)} \sim \begin{bmatrix} 2 & 2 & -2 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{bmatrix} \xrightarrow{(2/3)} \sim \begin{bmatrix} 2 & 2 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & 5/3 \end{bmatrix}$$

$$\therefore\ A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -2 \\ 0 & 3 & -2 \\ 0 & 0 & 5/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5/3 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2^{1/2} & 0 & 0 \\ 0 & 3^{1/2} & 0 \\ 0 & 0 & (5/3)^{1/2} \end{bmatrix}^2 \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$令L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & -2/3 & 1 \end{bmatrix} \begin{bmatrix} 2^{1/2} & 0 & 0 \\ 0 & 3^{1/2} & 0 \\ 0 & 0 & (5/3)^{1/2} \end{bmatrix}$$

$$= \begin{bmatrix} 2^{1/2} & 0 & 0 \\ 2^{1/2} & 3^{1/2} & 0 \\ -2^{1/2} & -2/3^{1/2} & (5/3)^{1/2} \end{bmatrix}$$

則 $A=LL^T$

(b) $\det(A-xI) = \ldots = -x^3+12x^2-21x+10 = -(x-1)^2(x-10)$

解 $(A-I)v=o$ 可得出1的eigenvector $[-2, 1, 0]^T$, $[2, 0, 1]^T$

經Gram-Schmidt process可求得正交之eigenvector $[-2, 1, 0]^T$, $[2, 4, 5]^T$

解 $(A-10I)v=o$ 可得出10的eigenvector $[-1, -2, 2]^T$

(將已正交之eigenvector單位化並直排拼成U)

$$令U = \begin{bmatrix} -2/5^{1/2} & 2/45^{1/2} & -1/3 \\ 1/5^{1/2} & 4/45^{1/2} & -2/3 \\ 0 & 5/45^{1/2} & 2/3 \end{bmatrix}$$

則U爲unitary matrix, 且$U^{-1}AU$=diag(1, 1, 10)