線性代數解析--成大91資工所

廖亦德 解

[1a]. (5%) 【成大91資工】

Find the sum of determinants. (5%)

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 2 & 3 & -1 & -2 \end{vmatrix}$$

【分析】本題屬於題型04B. 相關類題請參閱綜線CH4範例12.

【解】(a) 先求第一個行列式:

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 1 & -3 & -4 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 5 \\ 1 & -3 & -4 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 5 \\ 0 & -5 & -7 \end{vmatrix}$$

$$= -\begin{vmatrix} 5 & 5 \\ -5 & -7 \end{vmatrix} = -5\begin{vmatrix} 1 & 1 \\ -5 & -7 \end{vmatrix} = (-5)(-2) = 10$$

再求第二個行列式:

$$\begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 4 & 4 \\ 2 & 3 & -1 & -2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \end{vmatrix}$$ (第3,4列都減去第2列)

$$= -\begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 & 3 \\ -2 & -2 & 3 & 3 \\ 1 & 2 & -2 & -3 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$ (列對調)

=第一個行列式

∴ 所求爲20.

[1b]. (10%) 【成大91資工】

Let A be a nonsingular $n\times n$ matrix with a nonzero cofactor A_{nn} at (n, n) entry and set $c=\det(A)/A_{nn}$. Show that if we substract c from (n, n) entry of A then the resulting matrix will be singular.

【分析】本題屬於題型04A.

【解】(b) 爲證矩陣singular, 只須證明其行列式爲0即可: (綜線CH4定理17)

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots\dots & & \\ a_{n1} & a_{n2} & \dots & a_{nn}-c \end{vmatrix}$$

$$= \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots\dots & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \dots\dots & & \\ 0 & 0 & \dots & -c \end{vmatrix}$$ (拆開末列)

$= \det A + (-c)A_{nn}$ (第二個行列式末列降階)

$= \det A - \det A = 0$

[2]. (15%) 【成大91資工】

Consider the inner product space $C[0,1]$ with inner product defined by

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx .$$

Let S be the subspace spanned by the vectors 1 and $2x-1$. Find the best least squares approximation to $\sqrt{x}$ by a function from the space S.

【分析】本題屬於題型09B (不屬於題型09E).

【解】所求爲$x^{1/2}$對S的正投影. (綜線CH9定理13)

由積分計算(過程略)可發現 $\langle 1, 2x-1 \rangle = 0$.

$\langle x^{1/2}, 1 \rangle = \ldots = 2/3$, $\langle 1, 1 \rangle = \ldots = 1$,

$\langle x^{1/2}, 2x-1 \rangle = \ldots = 2/15$, $\langle 2x-1, 2x-1 \rangle = \ldots = 1/3$,

所求爲 $((2/3)/1)\cdot 1+((2/15)/(1/3))\cdot(2x-1) = 4/15+(4/5)x$. (綜線CH9定理12)

[3a]. (10%) 【成大91資工】

Compute e^A for the following matrix. (10%)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

【分析】本題屬於題型16B. 相關類題請參閱綜線CH16範例6.

【解】先對A做對角化(細節略):

$\det(A - xI) = \ldots = -x^3+x^2 = -x^2(x-1)$

解 $Av=o$得出0的eigenvector $[1, -1, 0]^T$, $[1, 0, -1]^T$.

解 $(A-I)v=o$得出1的eigenvector $[1, -1, 1]^T$

令 $P=\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, D=diag(0, 0, 1)

則 $A=PDP^{-1}$

$\exp(A)=P\text{diag}(e^0, e^0, e^1)P^{-1}=P\text{diag}(1, 1, e)P^{-1}$　　(綜線CH16定理3)

$$=\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} e & e-1 & e-1 \\ 1-e & 2-e & 1-e \\ e-1 & e-1 & e \end{bmatrix}$$

[3b]. (10%)　　【成大91資工】

Let L be the linear transformation mapping R^3 into R^3 defined by $L(x)=Ax$, where

$$A=\begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & -2 \\ 2 & -1 & -1 \end{bmatrix}, \text{ and let } v_1=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_2=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, v_3=\begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}.$$

Find the matrix representing L with respect to $[v_1, v_2, v_3]$.

【分析】本題屬於題型07B. 相關類題請參閱綜線CH7範例10.

【解】細節略, 答案爲

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$