線性代數解析—交大91資工所

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[1]. (14%) 【交大91資工】

Given a transformation T frim R^4 to R^3 defined by

$$T(x) = \begin{bmatrix} x_1+2x_2+3x_3+4x_4 \\ x_1+3x_2+5x_3+7x_4 \\ x_1-x_3-2x_4 \end{bmatrix} = Ax$$

(a) Find a basis for im(A). (4%)

(b) Find a basis for ker(A) (4%)

(c) Find an orthonormal basis of im(A) (6%)

【分析】本題(a)(b)屬於題型08A. 相關類題請參閱綜線CH8範例6.

本題(c)屬於題型09C. 相關類題請參閱綜線CH9範例17.

【解】

$$\begin{bmatrix} x_1+2x_2+3x_3+4x_4 \\ x_1+3x_2+5x_3+7x_4 \\ x_1-x_3-2x_4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 0 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

(a) $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 7 \\ 1 & 0 & -1 & -2 \end{bmatrix} \sim \underset{\text{..........}}{\text{列運算}} \sim \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

im(A)= column space of A (綜線CH8定理5a)

可取 $\{ [1, 1, 1]^T, [2, 3, 0]^T \}$ 為im(A)之基底 (綜線CH6定理24)

(b) 接(a), ker(A)={ [t+2s, –2t–3s, t, s]^T | t, s爲純量 } (綜線CH3範例7)

={ t[1, –2, 1, 0]^T+s[2, –3, 0, 1]^T | t, s爲純量 }

可取 {[1, –2, 1, 0]^T, [2, –3, 0, 1]^T } 爲ker(A)之基底

(c) 接(a), 由Gram-Schmidt process可求得im(A)之正交基底(細節略)

{ [1, 1, 1]^T, [1/3, 4/3, –5/3]^T }

再單位化即得出正交單位基底

{ [$1/3^{1/2}$, $1/3^{1/2}$, $1/3^{1/2}$]^T, [$1/42^{1/2}$, $4/42^{1/2}$, $-5/42^{1/2}$]^T }

[2]. (7%) 【交大91資工】

Let P^2 be the vector space consisting of all polynomials of degree 2 with a basis of 1, t, t^2. Define a linear transformation $T(f(t)) = f(2)$ from P^2 to R.

Find a basis for the kernel of T.

【分析】本題屬於題型08A.

【解】Ker$T$={ $f(t)$ | $T(f(t))$=0 } (綜線CH8定義5)

={ $f(t)$ | $f(2)$=0 }

={ $a+bt+ct^2$ | $a+2b+4c=0$ }

={ $(-2b-4c)+bt+ct^2$ | $b, c\in$ R }

={ $b(-2+t)+c(-4+t^2)$ | $b, c\in$ R }=span{$-2+t$, $-4+t^2$}

而 { $-2+t$, $-4+t^2$ } 線性獨立,

∴ { $-2+t$, $-4+t^2$ }爲Ker$T$之基底.

[3]. (4%) 【交大91資工】

Consider the partitioned matrix

$$A=\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ O & O & A_{23} \end{bmatrix}$$

where A_{11} is invertible. Determine the rank of A in terms of the ranks of submatrices A_{11},

A_{12}, A_{13}, and A_{23}.

【分析】本題屬於題型08B.

【解】設A_{11}為$m\times m$矩陣, 則rank(A_{11})=m. (綜線CH8定理17)

再設A_{23}化為梯形矩陣R, 且R有n個非零列, 則rank(A_{23})=n. (綜線CH6定理23)

A_{11}可經列運算化為I_m. (綜線CH3定理16)

$$\therefore \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ O & O & A_{23} \end{bmatrix} \sim \begin{bmatrix} I_m & * & * \\ O & O & R \end{bmatrix}$$

$\therefore$ rank$A$=m+n (綜線CH6定理23)

=rank(A_{11})+rank(A_{23})

[4]. (11%) 【交大91資工】

Let L be the linear transformation on R^3 defined by

$L((x_1, x_2, x_3)^T)=[4x_1+2x_2-2x_3, 2x_1+x_2-x_3, -2x_1-x_2+x_3]^T$

(a) Find the matrix A such that $L(x)=Ax$. (2%)

(b) Now that matrix A is the matrix representing L with respect to $[e_1, e_2, e_3]$, find the transition matrix S corresponding to a change of basis from $[b_1, b_2, b_3]$ to $[e_1, e_2, e_3]$ where b_1, b_2, b_3 are the three eigenvectors of matrix A. (5%)

(c) Find the matrix representing L with respect to $[b_1, b_2, b_3]$. (4%)

【分析】本題(a)屬於題型07B, 本題(b)(c)屬於題型07C.

【解】(a)

$$令\ A=\begin{bmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{bmatrix}.$$

則 $L((x_1, x_2, x_3)^T)=[4x_1+2x_2-2x_3, 2x_1+x_2-x_3, -2x_1-x_2+x_3]^T$

$= A[x_1, x_2, x_3]^T$

(編註: 綜線CH7定理15, A是L對標準基底的矩陣表示)

(b) $\det(A-xI)=-x^3+6x^2=-x^2(x-6)$, ∴ eigenvalues爲0, 0, 6 (細節略)

解 $(A-0I)v=o$ 得出0的eigenvectoer $b_1=[-1, 2, 0]^T$, $b_2=[1, 0, 2]^T$. (細節略)

解 $(A-6I)v=o$ 得出6的eigenvectoer $b_3=[-2, -1, 1]^T$. (細節略)

$$\therefore\ S=\begin{bmatrix} -1 & 1 & -2 \\ 2 & 0 & -1 \\ 0 & 2 & 1 \end{bmatrix}.$$ (綜線CH6定義33)

(c) $Ab_1=0b_1$, $Ab_2=0b_2$, $Ab_3=6b_3$.

$$\therefore\ 所求爲\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$ (綜線CH7定義9)

(編註: 讀者自行驗證此矩陣等於$S^{-1}AS$) (綜線CH7定理19)

[5]. (9%) 【交大91資工】

Consider the inner product space C[0,1] with inner product defined by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx$$

where $f(x)$ and $g(x)$ are two functions belonging to C[0, 1]. Let S be the subspace spanned by the vectors 1 and x. Find the best least squares approximation to $x^{1/3}$ on [0, 1] by a function from the subspace S.

【分析】本題屬於題型09B (不屬於題型09E), 相關類題請參閱綜線CH9範例11c.

【解】 先以Gram-Schmidt process將1, x正交化(過程略)爲 1, $x-1/2$,

再利用正投影公式求解: (綜線CH9定理12)

$\langle x^{1/3}, 1\rangle = ... = 3/4$, $\langle 1, 1\rangle = ... = 1$,

$\langle x^{1/3}, x-1/2\rangle = ... = 3/56$, $\langle x-1/2, x-1/2\rangle = ... = 1/12$,

所求爲 $((3/4)/1)\cdot 1+((3/56)/(1/12))\cdot(x-1/2) = 3/7+(9/14)x$.

【另解】設所求爲 $p(x)=a+bx$. 欲使 $\| x^{1/3} - p(x)\|$ 取極小,

亦即 $p(x)$爲$x^{1/3}$對span{1, x}的正投影 (綜線CH9定理13)

則 $\langle x^{1/3}-p(x), 1\rangle=0$, $\langle x^{1/3}-p(x), x\rangle=0$, (綜線CH9定理11b)

即 $\langle p(x), 1\rangle=\langle x^{1/3}, 1\rangle$, $\langle p(x), x\rangle=\langle x^{1/3}, x\rangle$, (左線性與移項)

$$\langle x^{1/3}, 1\rangle = \int_0^1 x^{1/3}\, dx = 3/4\ ,$$

$$\langle x^{1/3}, x\rangle = \int_0^1 x^{4/3}\, dx = 3/7\ ,$$

$$\langle p(x), 1\rangle = \int_0^1 (a+bx)\, dx = a\int_0^1 1\, dx + b\int_0^1 x\, dx = a + (1/2)b\ ,$$

$$\langle p(x), x\rangle = \int_0^1 (ax+bx^2)\, dx = a\int_0^1 x\, dx + b\int_0^1 x^2\, dx = (1/2)a + (1/3)b\ ,$$

由 $a+(1/2)b=3/4$, $(1/2)a+(1/3)b=3/7$ 可解得 $a=3/7$, $b=9/14$.

∴ 所求爲 $3/7+(9/14)x$.

[6]. (5%) 【交大91資工】

Find the subordinate matrix norm $\|A\|_\infty$, where

$$A=\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$$

【分析】本題屬於題型17A, 請參閱綜線附錄B定理22.

【解】所求爲 $\max\{|1|+|2|, |3|+|-4|\}=7$