線性代數解析--中央91資工所 廖亦德 解

[1]. (50%) 【中央91資工】

對錯申論題(一定要有說明或反例, 每小題答對得5分,答錯扣2分,不答0分, 本題總分≥0)

(a) All elementary row operations are reversible.

(b) The solusion set of $Ax=b$ is obtained by translating the solution set of $Ax=o$.

(c) Let T be a linear transformation. If $\{v_1, v_2, v_3\}$ is linearly dependent, then $\{T(v_1), T(v_2), T(v_3)\}$ is also linearly dependent.

(d) Let A be the square standard matrix of transformation T. T is one-to-one if and only if T is onto.

(e) If A and B are $n\times n$ matrices, then $(A+B)(A-B)=A^2-B^2$.

(f) A plane in R^3 is a two-dimensional subspace of R^3.

(g) A row replacement operation on matrix A doesn't change the determinent, and thus doesn't change the eigenvalues of A.

(h) Similar matrices always have the same eigenvectors.

(i) If P is an orthogonal vector set, then P is a linearly independent vector set.

(j) If $W=\text{span}\{v_1, v_2, ..., v_n\}$, then W and $W^{\perp}$ are always subspaces.

【分析】本題(a)(b)屬於題型03B, (c)屬於題型06A, (d)屬於題型08E, (e)屬於題型02A, (f)屬於題型05B, (g)(h)屬於題型12A, (i)屬於題型09A, (j)屬於題型05B及09A.

【解】(a) True. (綜線CH3定義3)

某列乘非零常數k時, 該列再乘$(1/k)$即還原.

某加入另一列的k倍時, 該列再加入另一列的$-k$倍即還原.

某兩列對調時, 該兩列再對調一次即還原.

(b) True. (綜線CH3定理11a)

設x_0爲一特解(即$Ax_0=b$), 則$\{b+u \mid Au=o\}$爲通解

(c) True. (綜線CH6定理11a)

若不全爲零的純量k_1, k_2, k_3使$k_1v_1+k_2v_2+k_3v_3=o$,

則由 $T(k_1v_1+k_2v_2+k_3v_3)=o$, 可得

$k_1T(v_1)+k_2T(v_2)+k_3T(v_3)=o$

(d) True. (綜線CH8定理11)

設A爲n階方陣.

T is one-to-one $\Longleftrightarrow$ $\ker T=0$ $\Longleftrightarrow$ $\text{rank}T=n$ $\Longleftrightarrow$ $\dim \text{Im}T=n$ $\Longleftrightarrow$ T is onto

(e) False. (綜線CH2範例5)

設 $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$, 則 $AB\neq BA$.

$(A+B)(A-B)=A^2+BA-AB-B^2\neq A^2-B^2$

(f) False. (綜線CH5定理11要訣)

必須通過原點才行.

(g) False.

設 $A=\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 2 \\ 2 & 3 \end{bmatrix}$,

則A將第一行的(3/2)倍加入第二行即變成B.

$\det(A)=\det(B)=-2$.

$\det(A-xI)=x^2-4=(x-2)(x+2)$, A的eigenvalue爲2, –2

$\det(B-xI)=x^2-3x-4=(x-4)(x+1)$, A的eigenvalue爲4, –1

(h) False. (但A, B的eigenvalue必定相同)

例如 $A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $D=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $P=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$,

則 $P^{-1}AP=D$.

A的eigenvector爲 $h[1, 1]^T$, 及 $k[1, -1]^T$, 其中h, k不爲0.

D的eigenvector爲 $p[1, 0]^T$, 及 $q[0, 1]^T$, 其中p, q不爲0.

(i) Flase. (綜線CH9定理15)

例如 {(0,0), (1,0)}在普通內積之下是orthononal set, 但卻不是independent set

(j) True.

證明W的封閉性: (綜線CH6定理4)

對$x, y\in W$, 及純量h, k.

令$x=\sum a_i v_i$, $y=\sum b_i v_i$, 則

$hx+ky=h\sum a_i v_i+k\sum b_i v_i=\sum(ha_i+kb_i)v_i$, 仍屬於$W$.

證明W的封閉性: (綜線CH11定理17)

對$x, y\in W^{\perp}$, 及純量h, k.

$\forall\, w\in W$, $\langle w, hx+ky\rangle=h\langle w, x\rangle+k\langle w, y\rangle=h\cdot 0+k\cdot 0=0$

$\therefore$ $hx+ky$仍屬於$W^{\perp}$.

[2]. (10%) 【中央91資工】

是非題(每小題答對得2分, 答錯扣2分, 不答0分, 本題總分≥0)

If A, B are two $n\times n$ matrices.

(a) $\det A^{-1}=(\det A)^{-1}$.

(b) $(A+B)^{-1}=A^{-1}+B^{-1}$

(c) $\det(A+B)=\det A+\det B$

(d) $(AB)^{T}=A^{T}B^{T}$.

(e) $\det(ABC)^{T}=\det A^{T}\det B^{T}\det C^{T}$

【分析】本題屬於題型04A.

【解】(a) True, (b) False, (c) False, (d) False, (e) True.

【說明】(a)此爲定理(綜線CH4定理6a)

(b)例如取$A=2I_2$, $B=3I_2$. 顯然$(2I_2+3I_2)^{-1}=(5I_2)^{-1}\neq(2I_2)^{-1}+(3I_2)^{-1}$.

(c)例如取$A=2I_2$, $B=3I_2$. 顯然$\det(2I_2+3I_2)=\det(5I_2)\neq\det(2I_2)+\det(3I_2)$.

(d)應是$(AB)^{T}=B^{T}A^{T}$. 反例只須使$AB\neq BA$即可適用.

(e)$\det(ABC)^{T}=\det(C^{T}B^{T}A^{T})=\det(C^{T})\det(B^{T})\det(A^{T})=\det(A^{T})\det(B^{T})\det(C^{T})$

[3]. (10%) 【中央91資工】

Let $B=\left\{\begin{bmatrix}7\\5\end{bmatrix},\begin{bmatrix}-3\\-1\end{bmatrix}\right\}$, $C=\left\{\begin{bmatrix}1\\-5\end{bmatrix},\begin{bmatrix}-2\\2\end{bmatrix}\right\}$ be bases for R^2.

Find the change-of-coordinates matrices form "*B* to *C* "and " *C* to *B*"

【分析】本題屬於題型06D. 請參閱CH6定理33

【解】設

$$\begin{bmatrix}7\\5\end{bmatrix}=\mathrm{p}\begin{bmatrix}1\\-5\end{bmatrix}+\mathrm{q}\begin{bmatrix}-2\\2\end{bmatrix},\quad \begin{bmatrix}-3\\-1\end{bmatrix}=\mathrm{r}\begin{bmatrix}1\\-5\end{bmatrix}+\mathrm{s}\begin{bmatrix}-2\\2\end{bmatrix}$$

可解得 p= –3, q= –5, r=1, s=2.

令 $A=\begin{bmatrix}-3&1\\-5&2\end{bmatrix}$,

則A為the change-of-coordinates matrix form B to C. (即 $\forall v\in\mathrm{R}^2$, $[v]_C=A[v]_B$)

再求得$A^{-1}=\begin{bmatrix}-2&1\\-5&3\end{bmatrix}$, 此即 the change-of-coordinates matrix form C to B.

[4]. (10%) 【中央91資工】

Find the bases for Col$A$, Row$A$ and Nul$A$, where

$$A = \begin{bmatrix} 1 & 3 & 3 & 2 & -9 \\ -2 & -2 & 2 & -8 & 2 \\ 2 & 3 & 0 & 7 & 1 \\ 3 & 4 & -1 & 11 & -8 \end{bmatrix}$$

【分析】本題屬於題型06C. 請參閱CH6定理23a,定理24a, CH3範例7.

【解】經列運算, A可化為

$$\begin{bmatrix} 1 & 0 & -3 & 5 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ row space之基底可取為

$\{ [1 \ \ 0 \ \ -3 \ \ 5 \ \ 0], [0 \ \ 1 \ \ 2 \ \ -1 \ \ 0], [0 \ \ 0 \ \ 0 \ \ 0 \ \ 1] \}$

pivot 在第1,2,5行,

∴ column space之基底可取為

$\{ [1 \ \ -2 \ \ 2 \ \ 3]^T, [3 \ \ -2 \ \ 3 \ \ 4]^T, [-9 \ \ 2 \ \ 1 \ \ -8]^T \}$

null space

$= \{[3t_3-5t_4, -2t_3+t_4, t_3, t_4, 0]^T \mid t_3, t_4$為純量 $\}$

$= \{t_3[3, -2, 1, 0, 0]^T+t_4[-5, 1, 0, 1, 0]^T \mid t_3, t_4$為純量 $\}$

∴ null space之基底可取為$\{[3, -2, 1, 0, 0]^T, [-5, 1, 0, 1, 0]\}$

[5]. (10%) 【中央91資工】

Diagonalize the matrix

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix};$$

that is to find matrices P and D such that $A=PDP^{-1}$.

【分析】本題屬於題型12C. 請參閱CH12範例17.

【解】$\det(A-xI)= -x^3+12x^2-36x+32= -(x-2)^2(x-8)$

解 $\ker(A-2I)$得出2的eigenvector $[-1, 1, 0]^T$, $[-1, 0, 1]^T$

解 $\ker(A-8I)$得出8的eigenvector $[1, 1, 1]^T$

令 $P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, $D=\mathrm{diag}(2, 2, 8)$ 則 $A=PDP^{-1}$

[6]. (10%) 【中央91資工】

Find a QR factorization of matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix},$$

where columns of Q form an orthonormal basis.

【分析】本題屬於題型09D. 請參閱CH9範例19.

【解】(細節略)

$$Q = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}, R = \begin{bmatrix} 2 & 4 & 5 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$