線性代數解析--交大91資科所

廖亦德 解

[1]. (9%) 【交大91資科】

For each statement below, indicate whether it is true or not.

(a) The determinant of an idempotent matrix is either 0 or 1.

(b) The only vector space that contains a finite number of vectors is the zero vector space $Z=\{0\}$.

(c) Square matrices A and B are invertible if and only if AB is invertible.

(d) A square matrix is a nilpotent matrix of index k if and only if its transpose is a nilpotent matrix of index k.

(e) Let E and F be subsets of a vector space, then span(E)$\subseteq$span(F) if and only if $E\subseteq F$.

(f) The product of two orthogonal matrices is orthogonal, so is the sum of two orthogonal matrices.

(g) det: $M_{n,n}\to$R is a linear transformation.

(h) The column space of a 2 by 2 matrix has the same dimension as its row space.

(i) The only possible eigenvalues of a projection matrix are 0 and 1.

【分析】本大題全都是觀念題.

本題(a)屬於題型12A. 本題(b)屬於題型05A. 本題(c)屬於題型02B.

本題(d)屬於題型14A. 本題(e)屬於題型06B. 本題(f)屬於題型13A.

本題(g)屬於題型04A. 本題(h)屬於題型08C. 本題(a)屬於題型11A.

【解】(a) True.

idempotent的eigenvalue必為0或1. (綜線CH11定理14)

∴ 行列式為特徵值之乘積，必為0或1. (綜線CH13定理8)

(b) False.

若scalar field為finite field. 則有限維空間中的向量個數都是有限.

[編註]: 若在 real vector space或 complex vector space，則此題為True.

有的老師在教(考)線代時已假設只在real/complex 討論.
但本份試題第6題有提到real vector space, 而此題只說是vector space,
所以答False比較合理.

(c) True.

因det(AB)=det(A)det(B)　　(綜線CH4定理6)

AB可逆 $\Longleftrightarrow$ det(AB)≠0　　(綜線CH4定理17)

$\Longleftrightarrow$ det(A)≠0且det(B)≠0 $\Longleftrightarrow$ A可逆且B可逆.

(d) True.

$\because$ $(A^T)^i=(A^i)^T$　　(綜線CH2定理23)

$\therefore$ $\forall i$, $A^i=O \Longleftrightarrow (A^T)^i=O$

(e) False. 反例如下:

$E=\{(1, 1, 0)\}$, $F=\{(1, 0, 0), (0, 1, 0)\}$.

則span(E)⊆span(F) 但 $E \not\subseteq F$.

(f) False. 反例如下:

單位矩陣I是orthogonal matrix, 但 $I+I=2I$不再是orthogonal. (綜線CH13定義1)

(編註: product的部份正確, 見綜線CH13定理1b)

(g) False. 反例如下:

在 $n>1$時, det($2I$)=2^ndet(I), 與線性條件不符.

(h) True.

此為定理, 且對任意尺寸的矩陣都對　　(綜線CH8定理13)

(i) True.

Project matrix必為idempotent,　　(綜線CH11定理20)

故特徵值為0或1.　　(綜線CH11定理14)

[2]. (4%)　【交大91資料】

Consider the linear system

$ax+y+z=1$

$x+ay+z=0$

$x+y+az=0$

For what values of a does the linear system have a unique solution?

Also, use Cramer's rule to find the solution.

【分析】本題(a)屬於題型04D.

【解】

$$\Delta=\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix}=a^3-3a+2=(a+2)(a-1)^2$$

恰有一解 $\Longleftrightarrow$ 係數行列式不爲0　　(綜線CH4定理18)

$\Longleftrightarrow (a+2)(a-1)^2\neq 0 \Longleftrightarrow a\notin\{1,-2\}$

$$\Delta_x=\begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 1 \\ 0 & 1 & a \end{vmatrix}=a^2-1=(a+1)(a-1)$$

$$\Delta_y=\begin{vmatrix} a & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & a \end{vmatrix}=-(a-1)$$

$$\Delta_z=\begin{vmatrix} a & 1 & 1 \\ 1 & a & 0 \\ 1 & 1 & 0 \end{vmatrix}=-(a-1)$$

$\therefore\ x=\Delta_x/\Delta=(a+1)/((a+2)(a-1))$,

$y=\Delta_y/\Delta=-1/((a+2)(a-1))$,　$z=\Delta_z/\Delta=-1/((a+2)(a-1))$,

【補充】若題目要求對所有可能的a作答, 則須對$\Delta=0$的情形再做討論如下:

$a=-2$時, 原方程式爲

$(-2)x+y+z=1$

$x+(-2)y+z=0$

$x+y+(-2)z=0$

三式相加得 $0=1$, 故無解.

$a=1$時, 原方程式爲

$x+y+z=1$

$x+y+z=0$

$x+y+z=0$

由前兩式得 0=1, 故無解.

【另解】若題目禁用Cramer's rule或要求用列運算作答,

則作答如下: (參閱綜線CH3範例9)

$$\begin{matrix}(1)\\(1)\\ \end{matrix}\left[\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & 0 \\ 1 & 1 & a & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & 0 \\ a+2 & a+2 & a+2 & 1 \end{array}\right]$$

若 $a=-2$, 此時最後一列為 $0x+0y+0z=1$. ∴ 無解.

若 $a\neq-2$, 則繼續計算, 第三列除以$a+2$之後,

$$\sim \begin{matrix}\\ \\(-1)\end{matrix}\left[\begin{array}{ccc|c} a & 1 & 1 & 1 \\ 1 & a & 1 & 0 \\ 1 & 1 & 1 & 1/(a+2) \end{array}\right] \sim \left[\begin{array}{ccc|c} a-1 & 0 & 0 & (a+1)/(a+2) \\ 0 & a-1 & 0 & -1/(a+2) \\ 1 & 1 & 1 & 1/(a+2) \end{array}\right]$$

若 $a=1$, 則第二列為 $0x+0y+0z=-1/(a+2)$. 亦無解.

若 $a\neq-1$, 則繼續計算. 第一、二列都除以$a-1$,

$$\sim \begin{matrix}(-1)\\(-1)\\ \end{matrix}\left[\begin{array}{ccc|c} 1 & 0 & 0 & (a+1)/((a+2)(a-1)) \\ 0 & 1 & 0 & -1/((a+2)(a-1)) \\ 1 & 1 & 1 & 1/(a+2) \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & (a+1)/((a+2)(a-1)) \\ 0 & 1 & 0 & -1/((a+2)(a-1)) \\ 0 & 0 & 1 & -1/((a+2)(a-1)) \end{array}\right]$$

∴ $x=(a+1)/((a+2)(a-1)),\quad y=z=-1/((a+2)(a-1))$.

[3]. (4%) 【交大91資科】

Find the orthogonal complement of the following subspaces in R^3 .

(a) $\{(x, y, z) \mid x+2y+3z=0\}$

(b) $\{(x, y, z) \mid x+y+z=0 \text{ and } x-y+z=0\}$

【分析】本題屬於題型11C. 相關類題請參閱綜線CH11範例24.

【解】(a) 設$W=\{(x, y, z) \mid x+2y+3z=0\}$,

則 $W= (\ker([1, 2, 3]))^T=\text{lker}([1, 2, 3]^T)$ (綜線CH5定理19a)

$W^{\perp}=\text{RSP}([1, 2, 3])=\{(t, 2t, 3t) \mid t\in R\}$ (綜線CH11定理23)

(b) 設 $W=\{(x, y, z) \mid x+y+z=0 \text{ and } x-y+z=0\}$,

再設$A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

則 $W=(\ker A)^T=\text{lker}(A^T)$ (綜線CH5定理19a)

A經列運算可化爲$B=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$W^{\perp}=\text{RSP}(A)$ (綜線CH11定理23)

$=\text{RSP}(B)$ (綜線CH6定理17⑥)

$=\text{span}((1, 0, 1), (0, 1, 0))$

$= \{(t, s, t) \mid t, s\in R\}$

【另解】《幾何解法, 適用於R^3 》

(a) $x+2y+3z=0$的法向量爲(1, 2, 3).

∴ 所求之補空間爲直線 $\{(t, 2t, 3t) \mid t\in R\}$

(b) 原先的子空間爲兩平面的交線. 平行於兩個法向量的外積:

$(1, 1, 1) \times (1, -1, 1) = (2, 0, -2)$, (綜線CH1定義11)

平行於(1, 0, –1)

∴ 所求之補空間爲平面 $\{(x, y, z) \mid x-z=0\}$

[4]. (4%)　　【交大91資料】

Compute $A^{100}v$, where

$$A=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix},\quad v=\begin{bmatrix}1\\ 1\end{bmatrix}$$

【分析】本題屬於題型12A.

【解】 Av

$$=\begin{bmatrix}1 & 1\\ 0 & 2\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}2\\ 2\end{bmatrix}=2v$$

$\therefore\ A^{100}v=2^{100}v$

【討論】若v不是特徵向量，就要先求A^{100}. 這就變成題型16B及16C.

[5]. (4%)　　【交大91資料】

Let $T:R^3\to R^3$ be the linear transformation given be the formula

$T(x, y, z)=(y+z, x+z, y+x)$

Compute the matrices A, B and P that satisfy $A=PBP^{-1}$, where

A is the matrix of T relative to the standard basis of R^3, and

B is the matrix of T relative to the basis {(1, 1, 1), (1, –1, 0), (1, 1, –2)}.

【分析】本題屬於題型07B與07C.

【解】1° $T(1, 0, 0)=(0, 1, 1)=0(1,0,0)+1(0,1,0)+1(0,0,1)$

$T(0, 1, 0)=(1, 0, 1)=1(1,0,0)+0(0,1,0)+1(0,0,1)$

$T(0, 0, 1)=(1, 1, 0)=1(1,0,0)+1(0,1,0)+0(0,0,1)$

$$\therefore\ A=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix}$$ (綜線CH7定義9)

2° 依關係式$A=PBP^{-1}$可知P為由新基底到標準基底的座標變換矩陣.

(1, 1, 1)=1(1,0,0)+1(0,1,0)+1(0,0,1)

(1, –1, 0)=1(1,0,0)+(–1)(0,1,0)+0(0,0,1)

(1, 1, –2)=1(1,0,0)+1(0,1,0)+(–2)(0,0,1)

$$\therefore\ P=\begin{bmatrix}1&1&1\\1&-1&1\\1&0&-2\end{bmatrix}$$ (綜線CH6定理33)

3° 依矩陣乘法,

$B=P^{-1}AP$

$$=\begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$ (綜線CH6定理33)

3° [另解]

T(1, 1, 1)=(2, 2, 2)=2(1,1,1)

T(1, –1, 0)=(–1, 1, 0)=(–1)(1,–1,0)

T(1, 1, –2)=(–1, –1, 2)=(–1)(1, 1, –2)

$$\therefore\ B=\begin{bmatrix}2&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$$ (綜線CH6定理33)

[6]. (10%) 【交大91資料】

Let V be the real vector space of polynomials of degree strictly less than 3 and let W be the real vector spacwe of polynomials of degree strictly less than 2. Define the transformation T from V to W by $T(v)=(v(t)-v(0))/t$.
Let the ordered basis for V be $B=\{1+t, t+t^2, 1+t^2\}$ and let the ordered basis for W be $C=\{1+t, 1-t\}$. Find the matrix representation of T.

【分析】本題屬於題型07B.

【解】 $T(1+t)= ((1+t)-1) / t = 1 = (1/2)(1+t) + (1/2)(1-t)$

$T(t+t^2)= ((t+t^2)-0) / t = 1+t = 1(1+t) + 0(1-t)$

$T(1+t^2)= ((1+t^2)-1) / t = t = (1/2)(1+t) + (-1/2)(1-t)$

$$\therefore \text{ 所求爲 } \begin{bmatrix} 1/2 & 1 & 1/2 \\ 1/2 & 0 & -1/2 \end{bmatrix}$$ (綜線CH7定義9)

[7]. (10%) 【交大91資料】

Consider the 5×5 matrix

$$A = \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 4 & 1 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

It is easy to see that $\lambda_1=7$ and $\lambda_2=4$ are the eigenvalues of A with algebraic multiplicities $m_1=2$ and $m_2=3$, respectively. Find the geometric multiplicities of λ_1 and λ_2.

【分析】本題(a)屬於題型15B.

【解】重排基底後可使矩陣轉成標準的Jordan form:

$$\begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

λ_1的幾何重數=dim Ker(A–7I)=2

λ_2的幾何重數=dim Ker(A–4I)=2

(詳情請參閱綜線CH14定理13b)

[8]. (5%)　【交大91資料】

Prove that λ=0 is an eigenvalue of A if and only if A is singular.

【分析】本題屬於題型12A.

【解】請參閱綜線CH14定理2b.