線性代數解析--交大92資工所

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參考章節使用簡稱, 例如綜線CH3代表廖亦德著:「綜合線性代數」第3章.

題型代表廖亦德著:「線性代數題型剖析」書中的題型.

[1]--[5]. (50%) 【離散數學】

[6]. (10%) 【交大92資工】

Let A be an $m \times n$ matrix (where $m<n$) whose rank is r.

(a) What is the largest value r may be? (2%)

(b) How many vectors are in a basis for the row space of A. (2%)

(c) How many vectors are in a basis for the column space of A. (2%)

(d) Which vector space R^k has the row space as a subspace? (2%)

(e) Which vector space R^k has the column space as a subspace? (2%)

【分析】本題(a)屬於題型08C. 本題(b)(c)屬於題型06B. 本題(d)(e)屬於題型05C.
相關定義請參閱綜線CH5定義16, 定義19.

【解】(a) m , (綜線CH8定理15)

(b) r , (c) r , (綜線CH8定理13)

(d) R^n, (e) R^m (綜線CH5定義16)

[7]. (6%) 【交大92資工】

Let $f(x)=x$, and $g(x)=|\,x\,|$.

(a) Show that f and g are linearly independent in $C[-1, 1]$. (3%)

(b) Show that f and g are linearly dependent in $C[0, 1]$. (3%)

【分析】本題屬於題型06A. 相關類題請參閱綜線CH6範例12.

【解】(a) 若常數a, b使 $af(x)+bg(x)=0$ for all x in $[-1, 1]$.

以$x=1$代入得 $a+b=0$,

以$x=-1$代入得 $-a+b=0$,

由上兩式解得$a=b=0$

∴ f, g爲linearly independent. (綜線CH6定義9)

(b) 取$a=1$, $b=-1$, 則 $af(x)+bg(x)=0$ for all x in $[0, 1]$

∴ f, g爲linearly dependent. (綜線CH6定義9)

[8]. (9%) 【交大92資工】

Let $f(x)=x$, $g(x)=x^2$ and an inner product $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$ in $C[0, 1]$.

(a) Find $\|g\|$. (3%)

(b) Find $[d(f, g)]^2=\langle f-g, f-g\rangle$. (3%)

(c) Orthonormalize the set $B=\{f, g\}$. (3%)

【分析】本題(a)(b)屬於題型09A. $d(f, g)$代表f, g之間的抽象距離(綜線CH9定義4),

相關類題請參閱綜線CH9定義4之習題.

本題(c)屬於題型09C. 相關類題請參閱綜線CH9範例18.

【解】(a) $\langle g, g\rangle = \int_0^1 x^4 dx = 1/5$.

∴ $\|g\| = (1/5)^{1/2}$ (綜線CH9定義4)

(b) $\langle f-g, f-g\rangle = \int_0^1 (x-x^2)^2 dx = \int_0^1 (x^2-2x^3+x^4)dx = 1/3 - 1/2 + 1/5 = 1/30$

(c) $\langle f, f\rangle = \int_0^1 x^2 dx = 1/3$

$\langle f, g\rangle = \int_0^1 x^3 dx = 1/4$

$g_1 = g - (\langle f,g\rangle / \langle f,f\rangle) f = x^2 - (3/4)x$

$\langle g_1, g_1 \rangle = \int_0^1 (x^2 - (3/4)x)^2 dx = \int_0^1 (x^4 - (3/2)x^3 + (9/16)x^2) dx$

$= 1/5 - 3/8 + 3/16 = 1/80$

f 經normalize爲 $f / (1/3)^{1/2} = 3^{1/2}x$

g_1 經normalize爲 $g_1 / (1/80)^{1/2} = 5^{1/2}(4x^2 - 3x)$

∴ 所求爲 $\{3^{1/2}x, 5^{1/2}(4x^2 - 3x)\}$

[9]. (6%)　　【交大92資工】

Prove or disprove the following statements:

(a) If v is a nonzero vector in R^3, then v is an eigenvector of the matrix vv^T.　(3%)

(b) If v is an eigenvector for both matrix A and B then v is an eigenvector for matrix AB.

【分析】本題屬於題型12A. 相關類題請參閱綜線CH12定理26.

【解】(a) Prove:

$(vv^T)v = v(v^Tv)$　　(結合律: 綜線CH2定理9)

$= (v^Tv)v$　　(注意這不是交換律: 綜線CH2定理5a)

∴ v is an eigenvector of the matrix vv^T, with eigenvalue v^Tv

(綜線CH12定義1)

(b) Prive:

If v is an eigenvector for both matrix A and B .

By definition, let $Av=pv$, $Bv=qv$ for scalars p, q.　　(綜線CH12定義1)

then　$(AB)v = A(Bv) = A(qv)$

$= q(Av) = q(pv) = (pq)v$

∴ v is an eigenvector of the matrix AB, with eigenvalue pq.　　(綜線CH12定義1)

[10]. (12%)　　【交大92資工】

Consider a linear transformation $T(x)=Ax$ with

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}$$

(a) Show how to find a basis for the kernel of T (i.e. the null space of A). (4%)

(b) Show how to find a basis for the column space of A. (No point will be given if you do not justify your answer. (4%)

(c) Describe the type of b that guarantees that $Ax=b$ has exact solution(s). And how many exact solutions are there? Why?

【分析】本題(a)(b)屬於題型05C. 相關類題請參閱綜線CH6範例23,24.
本題(c)屬於題型03C. 相關類題請參閱綜線CH3範例9. 所謂exact solution是相對於近似解說的(見綜線CH9定理21a). 就是先要找使$Ax=b$有解的條件.

【解】經列運算將A化爲row reduced form R: (過程略) (綜線CH3範例4b)

$$\begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$$

(a) 令 $x=[x_1, x_2, x_3, x_4, x_5]^T$, 求解$Ax=o$.

由前述列運算, 原方程式化爲

$$\begin{aligned} x_1 + x_3 + 2x_4 &= 0 \\ x_2 - 2x_3 - 3x_4 &= 0 \\ x_5 &= 0 \end{aligned}$$

解得kernel爲

$$\{ [-t_3-2t_4 , 2t_3+3t_4 , t_3 , t_4 , 0]^T \mid t_3, t_4 \in \mathrm{R} \}$$
$$=\{ t_3[-1 , 2 , 1 , 0 , 0]^T + t_4[-2 , 3 , 0 , 1 , 0]^T \mid t_3, t_4 \in \mathrm{R} \}$$ (綜線CH3範例7)

故取基底爲 $\{ [-1, 2, 1, 0, 0]^T, [-2, 3, 0, 1, 0]^T \}$

(b) A經列運算後第1, 2, 5行線性獨立　　(綜線CH6定理24)

∴ A的第1, 2, 5行線性獨立.

rank$A$=rank$R$=R的非零列數=3　　(綜線CH8定理13, CH6定理23)

∴ A的第1, 2, 5行形成column space的基底.　　(綜線CH6定理22)

取基底爲 $\{ [1, 1, 2, 1]^T, [0, 0, 1, 1]^T, [1, 2, 2, 0]^T \}$

(c) 令 $b=[b_1, b_2, b_3, b_4]^T$, 求解$Ax=b$.

經列運算: (過程略)(此運算之左半部和前面一樣)

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 1 & b_1 \\ 1 & 0 & 1 & 2 & 2 & b_2 \\ 2 & 1 & 0 & 1 & 2 & b_3 \\ 1 & 1 & -1 & -1 & 0 & b_4 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 1 & 2 & 0 & b_1-b_3+b_4 \\ 0 & 1 & -2 & -3 & 0 & -2b_1+b_3 \\ 0 & 0 & 0 & 0 & 1 & -b_1+b_3-b_4 \\ 0 & 0 & 0 & 0 & 0 & b_2-b_3+b_4 \end{array}\right]$$

$Ax=b$有exact solution $\Longleftrightarrow$ $b_2-b_3+b_4=0$　　(綜線CH3範例8)

在此條件下, 由

$$\begin{aligned} x_1 \quad + x_3 + 2x_4 \quad &= b_1-b_3+b_4 \\ x_2 - 2x_3 - 3x_4 \quad &= -2b_1+b_3 \\ x_5 &= -b_1+b_3-b_4 \end{aligned}$$

得通解爲

$x_1= -t_3-2t_4+b_1-b_3+b_4$,

$x_2=2t_3+3t_4 -2b_1+b_3$,

$x_3=t_3$,

$x_4=t_4$,

$x_5=-b_1+b_3-b_4$,　　$t_3, t_4 \in \mathrm{R}$　　(綜線CH3範例7)

∴ 有無限多個exact solution

[11]. (7%) 【交大92資工】

Let $A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

(a) Find the eigenvalues of A. (3%)

(b) Prove or disprove that matrix A is diagonalizable.

【分析】本題屬於題型12B. 相關類題請參閱綜線CH12範例17.

【解】(a) $\det(A - xI) = -x^2(x-3)$,

A的eigenvalue為0, 0, 3.

(b) 0的algebraic multiplicity為2, (綜線CH12定義18)

而 $\text{rank}(A - 0I)=\text{rank}(\ [1, 1, 1]\ )= 1$, (綜線CH8範例14a)

∴ $\dim \text{Ker}(A - 0I)=3-1=2$ (綜線CH8定理8)

另 3的algebraic multiplicity為1, (綜線CH12定義18)

此即強迫其geometric multiplicity也是1. (綜線CH12定理19)

∴ A可對角化. (綜線CH12定理21)

【討論】其實這個A一眼就可判定特徵值與可對角化.

因為$\text{rank}(A)=1$且$\text{tr}(A)=3$. (綜線CH12定理26)