線性代數解析--台大95資工所 廖亦德 解

參考章節使用簡稱，例如綜線CH3代表廖亦德著:「綜合線性代數」第3章,

題型代表廖亦德著:「線性代數題型剖析」書中的題型.

Each problem may have multiple answers. To get credits you must correctly select all answers.

For example, if answers are ABC, you get zero point no matter you choose A, BC, DE, or whatever else different from ABC.

Note that we consider only real-valued matrices.

[1]. (3%) 【台大95資工】(複選題)

What of the followings are correct?

(A) The determinant of

$$\begin{bmatrix} 2 & 1 & 3 & 4 & 0 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

is 130.

(B) The determinant of

$$\begin{bmatrix} 2 & 1 & 3 & 4 & 0 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

is 140.

(C) The determinant of

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

is 65.

(D) The determinant of

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

is 70.

(E) The determinant of

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{bmatrix}$$

is 80.

【分析】本題屬於題型04B. 相關類題請參閱綜線CH4範例12.

【解】AC.

【討論】

$$\begin{vmatrix} 2 & 1 & 3 & 4 & 0 \\ 1 & 5 & 2 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{vmatrix}$$ (第5列降階展開, 綜線CH4定理11)

而 $$\begin{vmatrix} 2 & 1 & 3 & 4 \\ 1 & 5 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} -6 & 1 & 3 & 4 \\ 1 & 5 & 2 & 0 \\ -5 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$ (第4行的–2倍加入第一行)

$$= \begin{vmatrix} -6 & 1 & 3 \\ 1 & 5 & 2 \\ -5 & 0 & 0 \end{vmatrix}$$ (第4列降階展開)

$$=-5\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix}$$ (第3列降階展開)

$=(-5)(2-15)= 65$

所以原5階行列式=130.

[2]. (3%) 【台大95資工】(複選題)

Let

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \quad (1)$$

Which of the followings are correct?

(A) $u_{11} = 1$

(B) $u_{12} = 2$

(C) $l_{21} = 2$

(D) It is impossible to have (1), so no way to have u_{11}, u_{12} , etc.

(E) $u_{33} = 0$

【分析】本題屬於題型03E. 相關類題請參閱綜線CH3範例28.

【解】D.

【討論】

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 5 & 3 \end{bmatrix} \begin{matrix} (-2) \; (-3) \\ \leftarrow \\ \leftarrow \end{matrix} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & -1 & -6 \end{bmatrix}$$

由此可知A不能做LU分解, 只能經列對調做P^TLU分解. (綜線CH3定理29)

我們可依矩陣乘法求算A的前兩行, 證明未知數無解:

$1 = 1 \cdot u_{11}$, $2 = 1 \cdot u_{12}$,

$2 = l_{21} \cdot u_{11}$, $4 = l_{21} \cdot u_{12} + 1 \cdot u_{22}$,

$3 = l_{31} \cdot u_{11}$, $5 = l_{31} \cdot u_{12} + l_{32} \cdot u_{22}$.

由前五式得出 $u_{11}=1$, $u_{12}=2$, $l_{21}=2$, $u_{22}=0$, $l_{31}=3$

再代入第六式得出矛盾: $5 = 3 \cdot 2 + 0$.

[3]. (3%) 【台大95資工】(複選題)

Let det(A) be the determinent of the matrix $A \in \mathbb{R}^{n\times n}$. Which of the followings are correct?

(A) $\det(A) \neq 0 \Leftrightarrow A$ invertible

(B) $\det(A^T)=\det(A)$

(C) $\det(AB)=\det(A)\det(B)$

(D) If A is invertible, then $\det(A^{-1})=\det(A)^{-1}$

(E) $\det(cA)=c^{n-1}\det(A)$

【分析】本題屬於題型04A.

【解】A, B, C, D.

【討論】E應該是 $\det(cA) = c^n\det(A)$

[4]. (3%) 【台大95資工】(複選題)

What is the solution of

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 3 & 1 & 1 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 \end{bmatrix}+\begin{bmatrix} 1 & -2 & -3 & -1 \\ -2 & 2 & 1 & -2 \\ -3 & 1 & 1 & 3 \\ -1 & -2 & 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

(A) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 24 & 28 & 32 \\ 64 & 72 & 80 & 88 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 24 & 28 & 32 \\ 64 & 72 & 88 & 80 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (C) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 24 & 28 & 32 \\ 64 & 72 & 88 & 80 \\ 0 & 0 & 4 & 0 \end{bmatrix}$

(D) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 24 & 28 & 32 \\ 64 & 76 & 88 & 80 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (E) $\begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 20 & 28 & 32 \\ 60 & 76 & 88 & 88 \\ 0 & 0 & 4 & 0 \end{bmatrix}$

【分析】本題屬於題型02A.

【解】A.

【討論】本題應該利用分配律計算: (綜線CH2定理9)

原式

$$= \left(\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 2 & 1 & 2 \\ 3 & 1 & 1 & 3 \\ 1 & 2 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -2 & -3 & -1 \\ -2 & 2 & 1 & -2 \\ -3 & 1 & 1 & 3 \\ -1 & -2 & 3 & -2 \end{bmatrix} \right) \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 2 & 6 \\ 0 & 0 & 6 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \\ 9 & 10 & 11 & 12 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 20 & 24 & 28 & 32 \\ 64 & 72 & 80 & 88 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

[5]. (3%) 【台大95資工】(複選題, consider only real-valued matrices)

An $m \times n$ matrix A is full rank if rank(A)=min(m, n). Which of the followings are correct?

(A) A full rank $\Rightarrow$ AA^{T} invertible; AA^{T} invertible $\not\Rightarrow$ A full rank

(B) A full rank $\not\Rightarrow$ AA^{T} invertible; AA^{T} invertible $\Rightarrow$ A full rank

(C) A full rank $\not\Rightarrow$ AA^{T} invertible; AA^{T} invertible $\not\Rightarrow$ A full rank

(D) A full rank $\Rightarrow$ AA^{T} invertible; AA^{T} invertible $\Rightarrow$ A full rank

(E) All AA^{T}'s eigenvalues are strictly positive.

【分析】本題屬於題型09E.

本題的關鍵定理為 rank(A)=rank(AA^{T}) (綜線CH9定理20)

(E)小題用到正半定矩陣AA^{T}的特徵值. (綜線CH13定理18a, 定理17c)

【解】B.

【討論】若$m \times m$矩陣AA^{T}可逆, 則

$m = \text{rank}(AA^{\mathrm{T}})$ (綜線CH8定理17)

$\leq \text{rank}(A)$ (綜線CH8定理16)

$\leq m$ (綜線CH8定理15)

所以 rank(A)=m, 於是A是full rank.

反之, 當A是full rank時,

若rank(A)=m則AA^{T}可逆, 若rank(A)=n則$A^{\mathrm{T}}A$可逆.

以$A=\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$為例, A是full rank,

但$A^{\mathrm{T}}A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, 並不可逆.

另外, $A^{\mathrm{T}}A$的特徵值4, 1, 0並不是都大於0.

[6]. (3%) 【台大95資工】(複選題)

Which of the followings are correct?

(A) If A and B are invertible, then so is AB.

(B) If A is invertible, then so is A^{T}.

(C) Eigenvalues of a triangular matrix are its diagonal elements.

(D) It is impossible to have real-valued matrix A such that $A^2 = -I$.

(E) It is possible that $AB \neq BA$.

【分析】本題屬於題型04A.

【解】A, B, C, E.

【討論】(A) 取行列式並利用CH8定理17即得證.

(B) 取行列式並利用CH8定理17, 及CH4定理5即得證.

(C) 由特徵多項式的定義計算即得. (CH13定理8的證明)

(D) $\det(A^2)=\det(A)^2 \geq 0$, 而 $\det(-I)=(-1)^n$.

當A為奇數階方陣時, 確實不可能有$A^2=-I$.

但偶數階就可能, 例如

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$ (90度旋轉兩次)

(E) 此為矩陣乘法基本性質. (綜線CH2範例5)

[7]. (3%) 【台大95資工】(複選題)

We decide to use a matrix storing all web connections. If web i has n out-links and j is one site that it connects to, then we put the ij element to be $1/n$. Otherwise the ij element is zero. If n=0, then the ij element is zero. Which of the followings are incorrect?

(A) Zero rows are possible as some pages have no out-link

(B) Zero columns are possible as some pages are never linked

(C) The rank of the matrix > (# total web pages – 1).

(D) Sum of each row is 0 or 1

(E) Sum of each column is 1.

【分析】本題為應用性的閱讀測驗.

【解】incorrect的有C, E.

【討論】(C) 反例如下:

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$ 的 rank 只有 1.

(E) 反例如下:

$$\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$ 的行和不是1.

[8]. (3%) 【台大95資工】(複選題)

Which of the followings are correct?

(A) If $A, B \in R^{m\times k}$, then calculating $A+B$ costs $O(mk)$.

(B) If $A\in R^{m\times k}$, $B\in R^{k\times n}$, then calculating AB costs $O(nmk)$

(C) If $A\in R^{m\times m}$ is invertible, using Gaussian elimination to find A^{-1} costs $O(m^3)$.

(D) If $A\in R^{m\times k}$, $B\in R^{k\times m}$, $u\in R^{m\times 1}$, and $k \ll m$, then calculating $(AB)u$ costs less $A(Bu)$

(E) If $A\in R^{m\times k}$, $B\in R^{k\times m}$, $u\in R^{m\times 1}$, and $k \ll m$, then calculating $(AB)u$ needs more storage than $A(Bu)$.

【分析】本題屬於題型02A.

【解】A, B, C, E.

【討論】(A) $A+B$中共有mk個數, 每個數需要做一次‘加’, 所以是 $O(mk)$.

(B) AB中共有mn個數, 每個數需要做k次‘加乘’, 所以是 $O(mnk)$.

(C) 請參閱綜線CH3演算法4a及CH3定理11.

(D) 計算(AB)需 $O(m^2k)$, 再算 $(AB)u$ 需$O(m^2)$, 合計為$O(m^2k)$.

計算(Bu)需 $O(km)$, 再算 $A(Bu)$ 需 $O(mk)$, 合計為$O(km)$.

(E) 計算(AB)需用一個$m\times m$的陣列儲存答案, 再算$(AB)u$.

計算(Bu)需用一個$k\times 1$的陣列儲存答案, 再算$A(Bu)$.

所以在k極小於m時前者須耗費較多的記憶體.

以上是照一般的算法, 但本題會有爭議:

若用一個$1\times k$的陣列依序儲存A的第i列與B的乘積, 並存回A的第i列,

則兩種方法耗用的記憶體相同.

[9]. (3%) 【台大95資工】(複選題)

Given

$$A = \begin{bmatrix} 0.6 & 0.4 & 0.3 \\ 0.4 & 0.9 & 0.2 \\ 0.3 & 0.2 & 0.8 \end{bmatrix}$$

What is the sum of all A's eigenvalues?

(A) 2.7, (B) 2.1, (C) 2.3, (D) 2.4, (E) 2.5

【分析】本題屬於題型12A.

【解】C.

【討論】eignevalue的和=trace=0.6+0.9+0.8=2.3. (綜線CH12定理13要訣1)

[10]. (3%) 【台大95資工】(複選題)

Let $A\in \mathrm{R}^{n\times n}$, U, $V\in \mathrm{R}^{n\times k}$, What is $(A+UV^{\mathrm{T}})^{-1}$?

Assume A and $(I+V^{\mathrm{T}}A^{-1}U)$ are invertible.

(A) $A^{-1}U(I+V^{\mathrm{T}}A^{-1}U)^{-1}V^{\mathrm{T}}A^{-1}$

(B) $A^{-1}-A^{-1}U(I+V^{\mathrm{T}}A^{-1}U)^{-1}V^{\mathrm{T}}A^{-1}$

(C) $A^{-1}-A^{-1}(I+V^{\mathrm{T}}A^{-1}U)^{-1}V^{\mathrm{T}}A^{-1}$

(D) $A^{-1}-A^{-1}U(I+V^{\mathrm{T}}A^{-1}U)^{-1}UV^{\mathrm{T}}A^{-1}$

(E) $A^{-1}-A^{-1}UV^{\mathrm{T}}(I+V^{\mathrm{T}}A^{-1}U)^{-1}V^{\mathrm{T}}A^{-1}$

【分析】本題屬於題型02B. 相關類題請參閱綜線CH12定理1①.

本題要點是$(U+UV^{\mathrm{T}}A^{-1}U)$可提出左因子$U$, 也可提出右因子$U$.

【解】B.

【討論】證明如下:

$$(A+UV^{\mathrm{T}})(A^{-1}-A^{-1}U(I+V^{\mathrm{T}}A^{-1}U)^{-1}V^{\mathrm{T}}A^{-1})$$

$$= (A+UV^{\mathrm{T}})A^{-1}-(A+UV^{\mathrm{T}})A^{-1}U(I+V^{\mathrm{T}}A^{-1}U)^{-1}V^{\mathrm{T}}A^{-1}$$

$= (I+UV^{T}A^{-1})-(U+UV^{T}A^{-1}U)(I+V^{T}A^{-1}U)^{-1}V^{T}A^{-1}$

$= (I+UV^{T}A^{-1})-U(I+V^{T}A^{-1}U)(I+V^{T}A^{-1}U)^{-1}V^{T}A^{-1}$

$= (I+UV^{T}A^{-1})-UV^{T}A^{-1}$

$= I$

[11]. (3%) 【離散】

[12]. (3%) 【離散】

[13]. (3%) 【離散】

[14]. (3%) 【離散】

[15]. (3%) 【離散】

[16]. (3%) 【離散】

[17]. (3%) 【離散】

[18]. (3%) 【離散】

[19]. (3%) 【離散】

[20]. (3%) 【離散】

[21]. (4%) 【台大95資工】(複選題)

Let k persons involve in m pairwise (i.e., two-person) games. We define an $m\times k$ game matrix G to show the setting: If the sth game involves persons i and j, then the sth row has the ith and jth entries to be one. Other entries are zeros. We assume that if two persons appear in one game, they do not appear together in another. Which of the followings are incorrect?

(A) Under this setting, there are at most $k(k-1)/2$ games

(B) Rank of $G \le k-1$

(C) The ith diagonal element of $G'G$=(#games the ith person involves – 1)

(D) The number of off diagonal elements of $G'G$ is $k(k-1)$.

(E) Any diagonal element of GG' is 1.

【分析】本題是閱讀測驗.

【解】incorrect的有 B, C, E. .

【討論】(A) 因任兩人不重覆比賽, 所以賽局數最多只能有k中取2的組合.

(B) rank可以到達k, 例如 m=3, k=3, rank=3

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(C) 由矩陣乘法的定義可得知

The ith diagonal element of $G'G$=(#games the ith person involves)

(D) $G'G$是$k\times k$矩陣, 共有k^2個元素, 扣除主對角線上的k個元素, 剩k^2-k個.

(E) 由矩陣乘法的定義可得知 GG'的對角線元素都是2.

[22]. (4%)　【台大95資工】(複選題)

Let $A\in\mathrm{R}^{n\times n}$ be any symmetric positive definite matrix and we write it as

$$A=\begin{bmatrix} \alpha & v^{\mathrm{T}} \\ v & B \end{bmatrix}$$

where α is a scalar, v is a vector, and $B\in\mathrm{R}^{n-1\times n-1}$. What of the followings are incorrect?

(A) A's principle sub-matrices are positive definite

(B) A's diagonal elements are all positive

(C) If $X\in\mathrm{R}^{n\times k}$ has rank k, then $B=X^{\mathrm{T}}AX \in\mathrm{R}^{k\times k}$ may not be positive definite

(D) $B-(1/\alpha)vv^{\mathrm{T}}$ may not be positive definite.

(E) We have

$$A=\begin{bmatrix} \beta & o^{\mathrm{T}} \\ v/\beta & I_{n-1} \end{bmatrix}\begin{bmatrix} 1 & o^{\mathrm{T}} \\ o & B-(1/\alpha)vv^{\mathrm{T}} \end{bmatrix}\begin{bmatrix} \beta & v^{\mathrm{T}}/\beta \\ o & I_{n-1} \end{bmatrix}$$

where $\beta=\alpha^{1/2}$, o is a zero vector and I_{n-1} is an $n-1\times n-1$ identity matrix.

【分析】本題屬於題型10C.

【解】incorrect的有C, D.

【討論】(A) 此為定理. (綜線CH10定理26a)

(B) 此為定理. (綜線CH10定理26a)

(C) 定理保證仍正定. (綜線CH10定理20)

(D) 由(E)(C)可知中間的矩陣仍正定, 再由(A)可知$B-(1/\alpha)vv^T$必須正定.

(E) 由矩陣塊狀乘法可驗證成立. (綜線CH2定理8)

[23]. (4%) 【台大95資工】(複選題)

Let A be any $n\times n$ symmetric positive definite matrix and $\mathbf{u}$ be any non-zero vector. Consider the following $(n+1)\times(n+1)$ matrix:

$$\begin{bmatrix} A & \mathbf{u} \\ \mathbf{u}^T & 0 \end{bmatrix} \tag{2}$$

Which of the followings is incorrect?

(A) If we solve the following linear system

$$\begin{bmatrix} A & \mathbf{u} \\ \mathbf{u}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathrm{y} \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ c \end{bmatrix} \tag{3}$$

then

$$y = (\mathbf{u}^T A^{-1}\mathbf{b})/(\mathbf{u}^T A^{-1}\mathbf{u})$$

(B) For (3), the solusion satisfies $\mathbf{x} = A^{-1}(\mathbf{b} - y\mathbf{u}\ )$

(C) The matrix (2) is positive semi-definite

(D) The matrix (2) is invertible

(E) The matrix (2) is symmetric

【分析】本題屬於題型10C.

【解】incorrect的是A, C.

【討論】(A) 由矩陣的塊狀乘法可知 (綜線CH2定理8)

$A\mathbf{x}+y\mathbf{u}=\mathbf{b}$, (*1)

$\mathbf{u}^{\mathrm{T}}\mathbf{x}=c$ (*2)

由(D)可知 $\mathbf{x}$, y 必定有解.

(*1)兩邊左乘A^{-1}得出 $\mathbf{x}+yA^{-1}\mathbf{u}=A^{-1}\mathbf{b}$,

兩邊再左乘$\mathbf{u}^{\mathrm{T}}$得出 $\mathbf{u}^{\mathrm{T}}\mathbf{x}+y\mathbf{u}^{\mathrm{T}}A^{-1}\mathbf{u}=\mathbf{u}^{\mathrm{T}}A^{-1}\mathbf{b}$,

以(*2)代入得出 $c+y\mathbf{u}^{\mathrm{T}}A^{-1}\mathbf{u}=\mathbf{u}^{\mathrm{T}}A^{-1}\mathbf{b}$,

所以 $y = (\mathbf{u}^{\mathrm{T}}A^{-1}\mathbf{b} - c)/(\mathbf{u}^{\mathrm{T}}A^{-1}\mathbf{u})$.

(Note: 本小題若將題目的c改成0就會correct)

(B) 由 $A\mathbf{x}+y\mathbf{u}=\mathbf{b}$ 可知 $A\mathbf{x}=\mathbf{b}-y\mathbf{u}$

兩邊左乘A^{-1}即得 $\mathbf{x} = A^{-1}(\mathbf{b} - y\mathbf{u})$

(Note: 正定必可逆(綜線CH10定理23))

(C) 反例如下:

取n=1, A=5, u=1, 則 (2)的行列式<0

(Note: 正半定矩陣的行列式必≥0, (綜線CH10定理26a))

(D) 證明如下:

設原矩陣為A_1.

其中A為實數正定矩陣, 可取可逆實數矩陣P, 使

$P^{-1}AP=\mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_n)$, 且$P^{-1}=P^{\mathrm{T}}$

令$P_1=\begin{bmatrix} P & \mathbf{o} \\ \mathbf{o}^{\mathrm{T}} & 1 \end{bmatrix}$, 則 $P_1^{-1}=P_1^{\mathrm{T}}$ 且

$$P_1^{-1}A_1P_1=\begin{bmatrix} P^{\mathrm{T}} & \mathbf{o} \\ \mathbf{o}^{\mathrm{T}} & 1 \end{bmatrix}\begin{bmatrix} A & \mathbf{u} \\ \mathbf{u}^{\mathrm{T}} & 0 \end{bmatrix}\begin{bmatrix} P & \mathbf{o} \\ \mathbf{o}^{\mathrm{T}} & 1 \end{bmatrix}$$

$$=\begin{bmatrix} P^{-1}AP & P^{\mathrm{T}}\mathbf{u} \\ \mathbf{u}^{\mathrm{T}}P & 0 \end{bmatrix}$$

令 $P^T\mathbf{u}=[\ k_1, k_2, \ldots, k_n\ ]^T$,

以第i列的$(-k_i/\lambda_i)$倍加入第n+1列, i=1, 2, ..., n,

則第n+1列化成 $[\ 0, 0, \ldots, 0, -(k_1^2/\lambda_1+k_2^2/\lambda_2+\ldots+k_n^2/\lambda_n)]$

因P是可逆矩陣, $\mathbf{u}\neq\mathbf{o}$, 所以$P^T\mathbf{u}\neq\mathbf{o}$, (於是 $k_1, k_2, \ldots, k_n$ 不全為0)

又因$\lambda_1, \lambda_2, \ldots, \lambda_n$都大於0, (綜線CH13定理17c)

所以$(k_1^2/\lambda_1+k_2^2/\lambda_2+\ldots+k_n^2/\lambda_n)>0$,

於是 $\text{rank}(P_1^{-1}A_1P_1)=n+1$, (綜線CH6定理23)

所以 $P_1^{-1}A_1P_1$ 可逆, (綜線CH8定理17)

所以 $A_1=P_1(P_1^{-1}A_1P_1\ )P_1^{-1}$ 也可逆. (綜線CH2定理12)

(E) 顯然成立.

[24]. (4%) 【台大95資工】(複選題)

For $A \in \mathrm{R}^{m\times n}$, define

$$\| A \|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}^2\right)^{1/2}$$

What of the followings are incorrect?

(A) $\| A \|_F = (\text{trace}(AA^T))^{1/2}$

(B) $\| A \|_F = (\text{trace}(A^TA))^{1/2}$

(C) For any scalar α, $\| \alpha A \|_F = |\alpha|\, \| A \|_F$

(D) $\| A+B \|_F = \| A \|_F + \| B \|_F$

(E) If $\mathbf{u}\in R^n$, $\mathbf{v}\in R^m$, and $A=\mathbf{u}\mathbf{v}^T$, then $\| A \|_F = 2\,\| \mathbf{u} \|_2 \| \mathbf{v} \|_2$

【分析】本題屬於題型17A. 相關類題請參閱綜線附錄B範例3.

【解】incorrect的有D, E.

【討論】(A)(B) 此為定理. (綜線附錄B定理3a)

(C) 此為定理. (綜線附錄B範例3)

(D) 應該是 $\leq$ (綜線附錄B範例3)

(E) 應該是 $\| A \|_F = \| \mathbf{u} \|_2 \| \mathbf{v} \|_2$, 證明如下:

令 $\mathbf{u}=[\ u_1\ u_2 \ldots u_n\ ]^T$, $\mathbf{v}=[\ v_1\ v_2 \ldots v_m\ ]^T$, 則

$$A = \begin{bmatrix} u_1v_1 & u_1v_2 & \ldots & u_1v_m \\ u_2v_1 & u_2v_2 & \ldots & u_2v_m \\ & \ldots\ldots & & \\ u_nv_1 & u_nv_2 & \ldots & u_nv_m \end{bmatrix}$$

$$\begin{aligned} \| A \|_F^{\,2} = \; & u_1^2v_1^2 + u_1^2v_2^2 + \ldots + u_1^2v_m^2 \\ & + u_2^2v_1^2 + u_2^2v_2^2 + \ldots + u_2^2v_m^2 \\ & + \quad \ldots\ldots \\ & + u_n^2v_1^2 + u_n^2v_2^2 + \ldots + u_n^2v_m^2 \\ = \; & u_1^2 (v_1^2 + v_2^2 + \ldots + v_m^2) \\ & + \; u_2^2 (v_1^2 + v_2^2 + \ldots + v_m^2) \\ & + \quad \ldots\ldots \\ & + \; u_n^2 (v_1^2 + v_2^2 + \ldots + v_m^2) \\ = \; & \\ = \; & (u_1^2 + u_2^2 + \ldots. + u_n^2)(v_1^2 + v_2^2 + \ldots + v_m^2) \\ = \; & \| \mathbf{u} \|_2^{\,2} \| \mathbf{v} \|_2^{\,2} \end{aligned}$$

兩邊開平方即得證.

[25]. (4%) 【台大95資工】(複選題)

Consider a function $f(W, H) = \| V - WH \|_F^{\,2}$, where $V \in \mathbb{R}^{n\times m}$ is a constant matrix, $W \in \mathbb{R}^{n\times r}$, $H \in \mathbb{R}^{r\times m}$ are two matrix variables. $\| \;.\; \|_F$ is defined in Problem 24.
If we define a matrix $\nabla_W f(W, H)$ so that

$$\nabla_W f(W, H)_{ij} \equiv \frac{\partial f(W, H)}{\partial W_{ij}}$$

then what is $\nabla_W f(W, H)$:

(A) $(WH - V)V^T W$

(B) $(WH - V)H^T$

(C) $VW^T(WH - V)H^T$

(D) WHH^T

(E) $W^T(WH-V)$

【分析】本題屬於向量分析的應用問題，超出一般線性代數的範圍.

【解】以上皆非.

【討論】為簡化問題，我們以 $n = r = m = 1$ 來觀察. 此時V, W, H都只是實數.

$$f(W, H) = (V-WH)^2$$

$$\nabla_W f(W, H) = \frac{\partial f(W, H)}{\partial W} = 2(V-WH)(-H)$$

$$= 2(WH-V)H$$

答案應是B，但漏了一個倍數2.

[26]. (4%) 【離散】

[27]. (4%) 【離散】

[28]. (4%) 【離散】

[29]. (4%) 【離散】

[30]. (4%) 【離散】