Section 3.1 Graphing using the first derivative 使用一階導函數繪圖

本節 summary 中提供繪圖的程序：

1. 找出定義域 (domain)：排除任何 x 值使得函數 $f(x)$ 無定義
2. 找出 $f(x)$ 所有漸近線
   A. 垂直漸近線 (極限值趨近於正負無限大)
   B. 水平漸近線 (x 趨近於正負無限大時，$f(x)$ 趨近於常數)
3. 找出所有**臨界數** (critical numbers)：f' 為零或 f'未定義，但 f 有定義。
4. 所有臨界數將定義域分割成多個區間，每個區間以**一階導數檢定** 來判斷 $f(x)$ 的遞增或遞減，也可判斷臨界數為極大值或極小值。
5. 依照上述資訊繪出函數圖形。

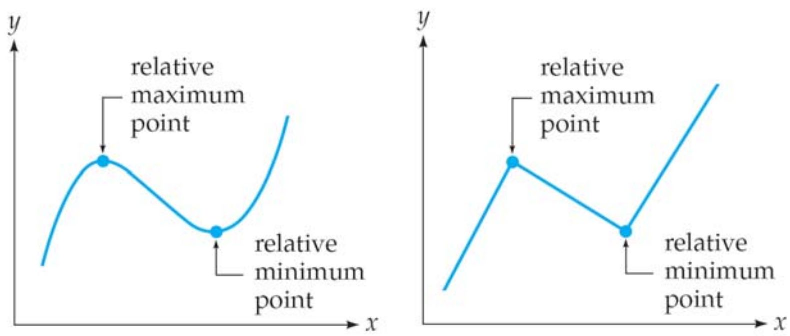
下一節會教到，配合**二階導數檢定，找出反曲點與函數凹性。**

【Topic 1.相對極值與臨界點】Relative Extreme Points and Critical Numbers

1. 相對極大值與極小值，可以非正式的定義為函數圖形的山峰 (peak) 與 山谷 (valley)

2. 相對極值的定義：對一個函數 f，相對極值 (relative extreme points) 由該點的函數值而定。
   A. 如果對 c 點附近所有的點，$f(c) \geq f(x)$ 都成立，就稱函數 $f(x)$ 在 $x = c$ 點上為相對極大值 (relative maximum value)。
   B. 如果對 c 點附近所有的點，$f(c) \leq f(x)$ 都成立，就稱函數 $f(x)$ 在 $x = c$ 點上為相對極小值 (relative minimum value)。

3. 相對極值會發生在斜率為 0，或者是斜率未定義 (undefined) (或可說：導數不存在) 的點，如下圖所示。這些點的 x 座標稱為臨界數 (***critical numbers***)。



4. 臨界數的定義

Critical Number

A *critical number* of a function f is an x-value in the domain of f at which either

$f'(x) = 0$

or $f'(x)$ is undefined

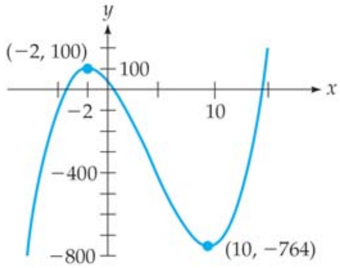
Derivative is zero or undefined

※ Example 1. Graph the function $f(x) = x^3 - 12x^2 - 60x + 36$.

<sol>： 步驟 1. 找出臨界數 (critical numbers)，所有臨界數會將定義域分割成多個區間。

步驟 2. 各區間內找任意一個測試點帶入導函數後，可得到導數的正負符號表。<u>正號代表函數遞增</u>，<u>負號代表函數遞減</u>。

步驟 3. 繪出函數圖形。此步驟請將臨界點 (臨界數與<u>極值</u>)、<u>函數與 x 軸的交點</u>、<u>函數與 y 軸的交點</u>也求出。



【Topic 2. 一階導數測定】First-Derivative Test for Relative Extreme Values

1. 導數正負號，代表函數的遞增或遞減。某個區間內，若 $f'>0$，則函數 f <u>遞增 ↗</u>；若 $f'<0$，則函數 f <u>遞減 ↘</u>。

2. 所有臨界數將定義域分割成多個區間；每個區間內找出(任意)一個**檢測點**，求出**檢測點**導數的正負號。在正負符號表中，↗ → ↘ (上、水平、下) 代表一個<u>相對極大值</u>，↘ → ↗ (下、水平、上) 代表一個<u>相對極小值</u>。相對極值可使用下列正式的一階導數測定來檢測。

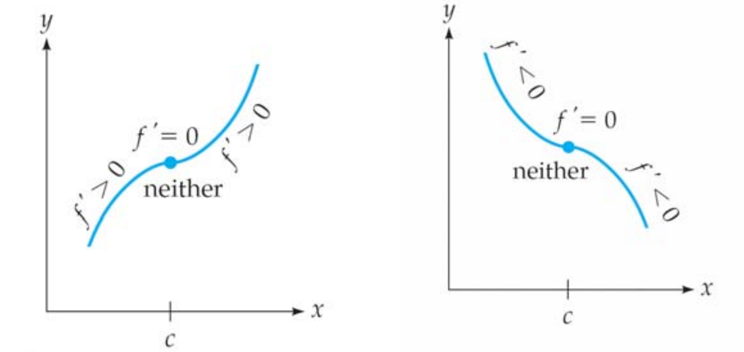
First-Derivative Test

If a function f has a critical number c, then at $x = c$ the function has a

relative maximum if $f' > 0$ just before c and $f' < 0$ just after c.

relative minimum if $f' < 0$ just before c and $f' > 0$ just after c.

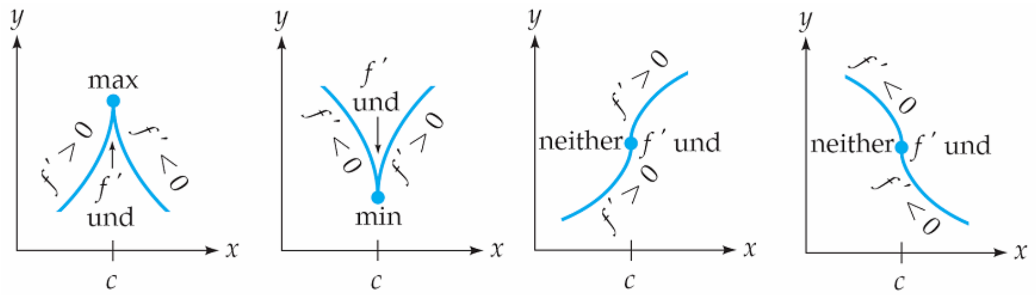
3. 如果臨界數 c 兩側的導函數值同號，則函數在 $x = c$ 時不具有相對極大或相對極小值。



f ′ is positive on both sides, so *f* has *neither* at $x = c$.

f ′ is negative on both sides, so *f* has *neither* at $x = c$.

4. 下列各圖顯示出一階導數測定也應用在導函數未定義的地方 (即圖中 f' und 的地方)。圖一與圖二 $f'(c)$ 未定義 (角點)，但 c 兩側的導函數值異號，故相對極值存在；圖三與圖四因為(反曲點) c 兩側的導函數值同號，因此相對極值不存在。



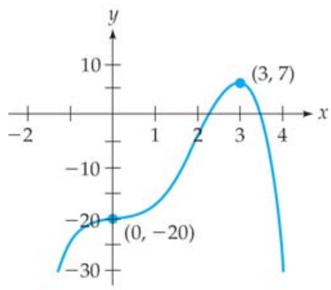
f ′ is positive then negative, so *f* has a relative *maximum* at $x = c$.

f ′ is negative then positive, so *f* has a relative *minimum* at $x = c$.

f ′ is positive on both sides, so *f* has *neither* at $x = c$.

f ′ is negative on both sides, so *f* has *neither* at $x = c$.

※ Example 2 Graph the function $f(x) = -x^4 + 4x^3 - 20$.



【Topic 3. 有理（分式）函式繪圖】Graphing Rational Functions

1. 分式函數為多項式的除法 $P(x)/Q(x)$。

2. 分式函數通常會有垂直漸近線 (vertical asymptotes)。如果分子次方與分母次方相同，則會有水平漸近線 (horizontal asymptotes)；如果分子次方比分母高一次，則有斜漸進線 (slant asymptotes)。 下列為垂直與水平漸近線的判斷方法：

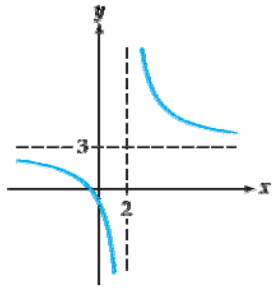
Vertical Asymptotes

A rational function $\frac{p(x)}{q(x)}$ has a **vertical asymptote** $x = c$ if $q(c) = 0$ but $p(c) \neq 0$.

Horizontal Asymptotes

A function $f(x)$ has a **horizontal asymptote** $y = c$ if $\lim_{x \to \infty} f(x) = c$ or $\lim_{x \to -\infty} f(x) = c$.

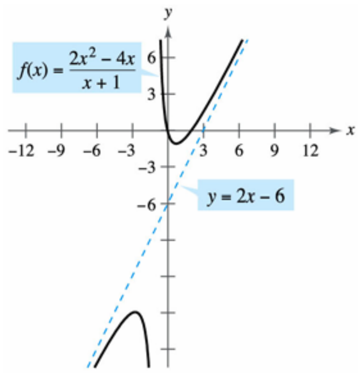
3. 以 $f(x) = \dfrac{3x+2}{x-2}$ 為例，當**分母為零**時 $(x = 2)$，函數值為正無限大或負無限大。當 x 趨近於正無限大或負無限大時，$f(x)$ 會趨近於 3 (why?)。因此圖中有垂直漸近線 $x = 2$ 與水平漸近線 $y = 3$。



Graph of $f(x) = \dfrac{3x+2}{x-2}$

※ 練習：請畫出 $f(x) = \dfrac{2x^2-4x}{x+1}$ 函數圖形。(課本未提及斜漸近線)

$$\text{Hint}: \lim_{x\to\infty} \frac{2x^2-4x}{x+1} = \lim_{x\to\infty}\left(2x - 6 + \frac{6}{x+1}\right)$$

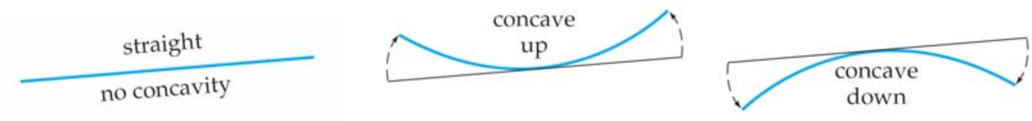


Section 3.2 Graphing using the first and second derivatives 使用一、二階導函數繪圖

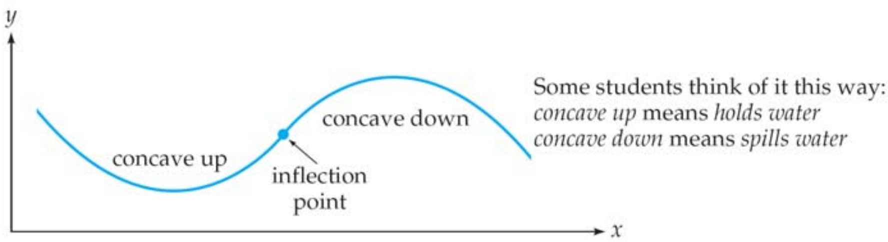
本節要使用二次導數來找出函數曲線的凹性 (concavity or curl)，並且定義重要的一個重要的觀念：反曲點 (inflection point)。二次導數也可以用來區別曲線的相對極值。

【Topic 1. 凹性與反曲點】Concavity and Inflection Points

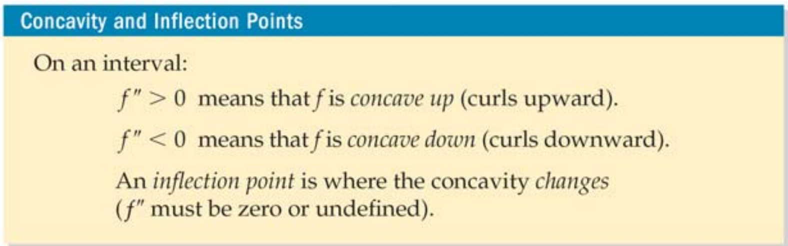
1. 直線不具有凹性 (concavity)。如果把直線的兩個端點向上扳，則會造成一個凹口向上 (concave up)；如果把直線的兩個端點向下扳，則會造成一個凹口向下 (concave down)。



2. 若某一個點的(前後)凹性 (concavity) 改變 (下變上或上變下) ，則我們稱這個點為反曲點 (inflection point)。



3. 凹性與反曲點：



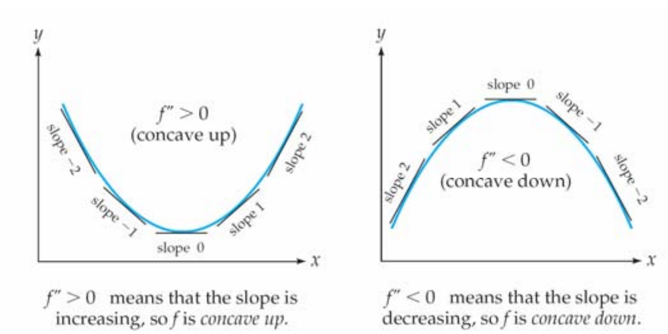
Concavity and Inflection Points

On an interval:

$f'' > 0$ means that f is *concave up* (curls upward).

$f'' < 0$ means that f is *concave down* (curls downward).

An *inflection point* is where the concavity *changes* (f'' must be zero or undefined).

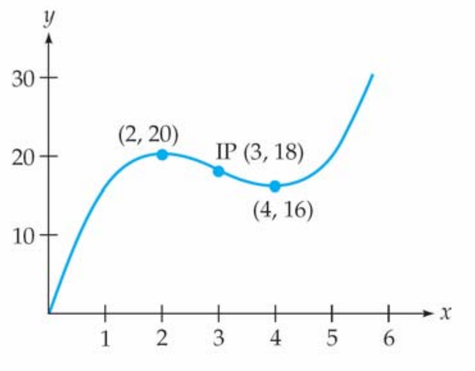


5. 上一節有提到過 (3-1. topic 2)，若 $f' > 0$，則函數 f 遞增；若 $f' < 0$，則函數 f 遞減。如果將 f' 視為一個新的函數 g，即 $g = f'$ (g 為斜率函數)，則
   A. 若 $g' > 0$，則(斜率)函數 g 遞增 (即：$f'' > 0$，f' 遞增，f 凹口向上)。[上圖左]
   B. 若 $g' < 0$，則(斜率)函數 g 遞減 (即：$f'' < 0$，f' 遞減，f 凹口向下)。[上圖右]

※ Example 1：Graphing and interpreting a company's annual profit function

A company's annual profit after x years is $f(x) = x^3 - 9x^2 + 24x$ million dollars (for $x \geq 0$). Graph this function, showing all relative extreme points and inflection points. Interpret the inflection points.

< sol >：

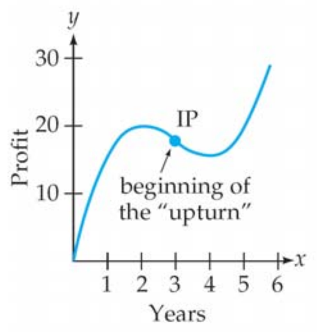


Interpretation of the inflection point:

Observe what the graph shows—that the company's profit increased (up to year 2), then decreased (up to year 4),and then increased again.

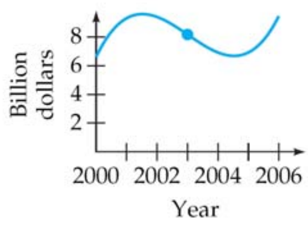
The inflection point at x = 3 is where the profit *first began to show signs of improvement.*

It marks the end of the period of increasingly steep (坡度) decline (下跌) and the first sign of an "upturn," where a clever investor might begin to "buy in."

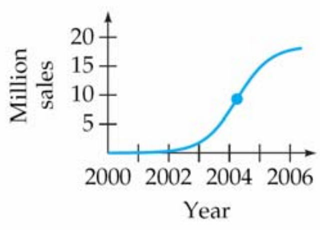


【Topic 2. 真實世界中的反曲點】Inflection Points in the Real World

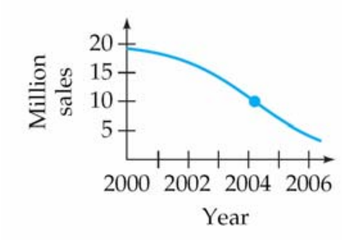
1. AT&T：The function in Example 1, while constructed for ease of calculation, is essentially the graph of net income for AT&T over recent years, shown below. The inflection point represents the first sign of the upturn in AT&T's net income, which occurred in 2003.



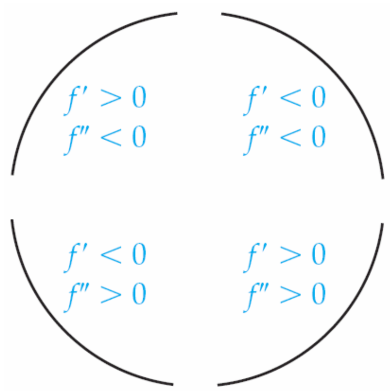
2. Annual sales of portable MP3：The graph below shows the annual sales of portable MP3 players. The inflection point, occurring between 2004 and 2005, marks the end of the period of increasingly rapid sales growth and first sign of approaching market saturation. This is when savvy manufacturers might begin to curtail new investment in MP3 production. *Source: Consumer USA* 2008.

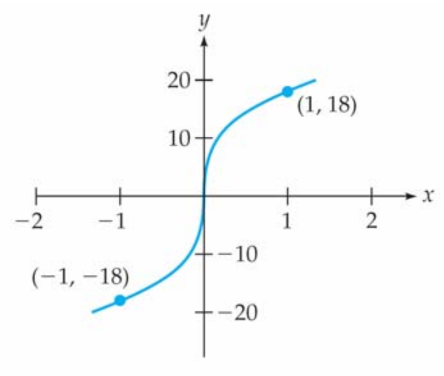


3. The graph below shows the annual sales of analog cameras a few years after the introduction of digital cameras. The inflection point, occurring in 2004, marks the end of the increasingly steep sales decline and the first sign of steadying but lower sales. *Source:* Euromonitor



4. Distinguish carefully between slope and concavity: *slope* measures *steepness*, whereas *concavity* measures *curl*. All combinations of slope and concavity are possible.

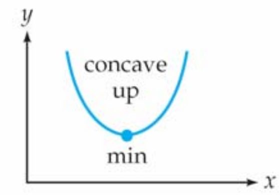




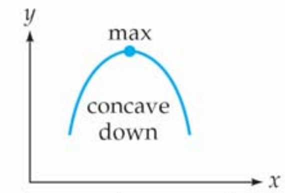
※ Example 2：Graph $f(x) = 18x^{1/3}$. (答案見右上方)

【Topic 3. 二階導數測定】Second-Derivative Test

1. 決定一個二階可微函數中的臨界數是否為相對極大或極小值，可由臨界數的凹性來判斷：如果臨界數所在區間為凹性向上，則為相對極小值；如果臨界數所在區間為凹性向下，則為相對極大值。如下圖所示。因為是利用二階導數決定凹性，因此又稱為二階導數測定。



Concave *up* at a critical number: relative *minimum*.



Concave *down* at a critical number: relative *maximum*.

2. 二階導數測定相對極值

Second-Derivative Test for Relative Extreme Points

If $x = c$ is a critical number of f at which f'' is defined, then

$f''(c) > 0$ means that f has a relative *minimum* at $x = c$.

$f''(c) < 0$ means that f has a relative *maximum* at $x = c$.

要使用二階導數測定，首先要先找出所有的臨界數，並將每一個臨界數代入二階導函數當中以決定二階導數的正負值，若二階導數為正，則有最小值；若二階導數為負，則有最大值。

※ Example 4：Use the second-derivative test to find all relative extreme points of

$f(x) = x^3 - 9x^2 + 24x.$ [可與 Example 1 比較]

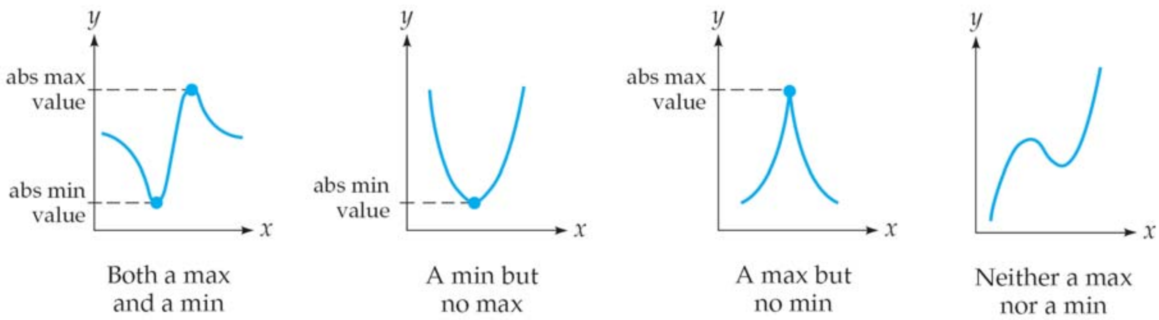
Section 3.3 Optimization 最佳化

許多問題要決定最大值或最小值，這些問題可歸類於解最佳化問題。微積分提供解最佳化問題的一般化技術。

【Topic 1. 絕對極值】Absolute Extreme Values

1. 函數的絕對極大值 (absolute maximum value) 為定義域中最大的函數值。函數的絕對極小值 (absolute minimum value) 為定義域中最小的函數值。絕對極值 (absolute extreme value) 為函數的絕對極大值或絕對極小值其中之一。函數的絕對極大值與極小值分別對應到函數圖形中的最高與最低點。

2. 一個函數的絕對極值可能兩者都存在，或一個或兩個都不存在，如下圖所示。



3. 連續函數 (continuous function) 在閉區間 (closed interval) 中的極值

Optimizing Continuous Functions on Closed Intervals

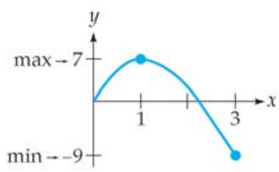
A continuous function *f* on a closed interval $[a, b]$ has both an *absolute maximum value* and an *absolute minimum value.* To find them:

1. Find all critical numbers of *f* in $[a, b]$.
2. Evaluate *f* at the critical numbers and at the endpoints *a* and *b*.

The largest and smallest values found in step 2 will be the absolute maximum and minimum values of *f* on $[a, b]$.

1. 在 (a, b) 上找出 f 的臨界數
2. 求這些臨界數的函數值
3. 求 $[a, b]$ 端點 $f(a)$ 與 $f(b)$
4. 上述函數值中<u>最大的</u>就是絕對極大值；<u>最小的</u>就是絕對極小值。

※ 例 1.　Find the absolute extreme values of $f(x) = x^3 – 9x^2 + 15x$ on $[0, 3]$.



練習：Find the absolute extreme values of $f(x) = 3x^4 – 4x^3$ on $[-1, 2]$. (max: $f(2) = 16$; min: $f(1) = -1$)

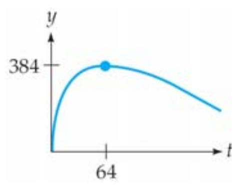
【Topic 2. 二階導數測定絕對極值】Second-Derivative Test for Absolute Extreme Values

我們可以應用二階導數測定相對極值的方法來找絕對極值，找出所有相對極值與端點函數值後，決定出絕對極值，方法與上一個主題相同。

【Topic 3. 最佳化的應用】Applications of Optimization

1. *OPTIMIZING THE VALUE OF A TIMBER FOREST*：If a timber forest (林場) is allowed to grow for t years, the value of the timber increases <u>in proportion to</u> (正比於) the square root (平方根) of t, while maintenance costs are proportional to t. Therefore, the value of the forest after t years is of the form $V(t) = a\sqrt{t} - bt$, where a and b are constants.

※ 例 2. The value of a timber forest after t years is $V(t) = 96\sqrt{t} - 6t$ thousand dollars (for $t > 0$). Find when its value is maximized.



2. *MAXIMIZING PROFIT*：

A. 三大經濟要素 (Three economic ingredients)：

▶ 利潤等於收入減掉支出。The first is that profit (利潤) is defined as *revenue* (收入) *minus cost* (費用): Profit = Revenue–Cost.

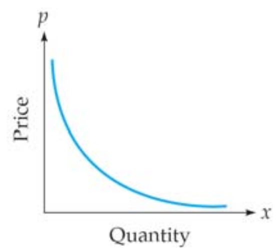
▶ 收入等於產品售價乘上銷售的產品數量。The second ingredient is that revenue is <u>*price*</u> times <u>*quantity*</u>. For example, if a company sells 100 toasters for \$25 each, the revenue will obviously be 25 * 100 = \$2500.

▶ 供應價格與供應數量成反比。The third economic ingredient reflects the fact that, in general, price and quantity are <u>inversely related</u>: increasing the price decreases sales, while decreasing the price increases sales.

B. 右圖中的某一點表示在價格 p 時，消費者將購買的數量 x，我們稱為此關係圖為價格函數 (**price function**)

Price Function

$p(x)$ gives the price p at which consumers will buy exactly x units of the product.

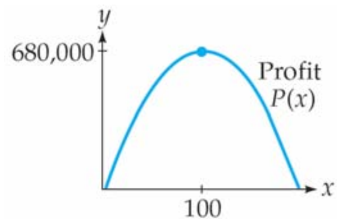


※ 例 3. It costs the American Automobile Company \$8000 to produce each automobile, and fixed costs (rent and other expenses that do not depend on the amount of production) are \$20,000 per week. The company's price function is $p(x) = 22{,}000 - 70x$, where p is the price at which exactly x cars will be sold.

a. How many cars should be produced each week to maximize profit?

b. For what price should they be sold?

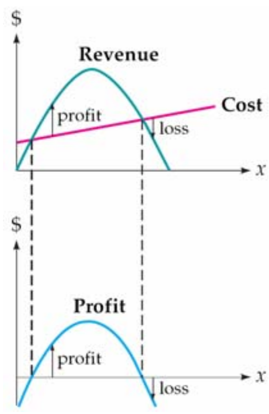
c. What is the company's maximum profit?



C. Graphs of the Revenue (收入), Cost (支出), and Profit (利潤) Functions

收入與支出函數如右上圖所示。當收入函數值大於支出函數值時，則產生利潤，否則產生損失。

我們將利潤函數 $P(x)$ 表示成收入函數 $R(x)$ 減去支出函數 $C(x)$，如右下圖所示，則某一點 x 利潤函數的高度代表該點的利潤，如果 $R(x) > C(x)$ or $P(x) > 0$.



因為 $P(x) = R(x) - C(x)$，對此方程式微分得 $P'(x) = R'(x) - C'(x)$. 當設定 $P'(x) = 0$ (求最大化收入)，等同於 $R'(x) = C'(x)$，即邊際收入 (marginal revenue) 等於邊際支出 (marginal cost)。

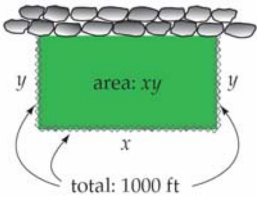
Classic Economic Criterion for Maximum Profit

At maximum profit:

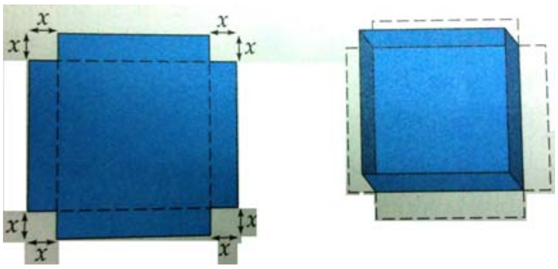
$$\begin{pmatrix}\text{Marginal}\\ \text{revenue}\end{pmatrix} = \begin{pmatrix}\text{Marginal}\\ \text{cost}\end{pmatrix}$$

3. MAXIMUM SOMETHING：

※ 例 4. A farmer has 1000 feet of fence and wants to build a rectangular enclosure along a straight wall. If the side along the wall needs no fence, find the dimensions that make the enclosure as large as possible. Also find the maximum area.



※ 例 5. An open-top box is to be made from a square sheet of metal 12 inches on each side by cutting a square from each corner and folding up the sides, as in the diagrams below. Find the volume of the largest box that can be made in this way.



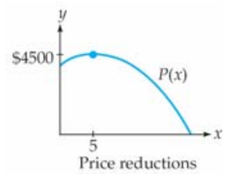
Section 3.4 Further applications of optimization 更進一步的最佳化應用

※ 例 1. *FINDING PRICE AND QUANTITY FUNCTIONS*

A store can sell 20 bicycles per week at a price of $400 each. The manager estimates that for each $10 price reduction she can sell two more bicycles per week. The bicycles cost the store $200 each. If x stands for *the number of $10 price reductions*, express the price p and the quantity q as functions of x.

※ 例 2. *MAXIMIZING PROFIT*

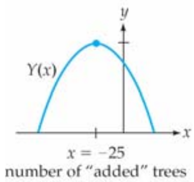
Using the information in Example 1, find the price of the bicycles and the quantity that maximize profit. Also find the maximum profit.



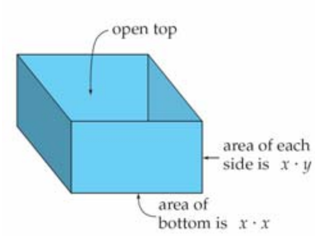
p.s. In example 2 we choose x to be *the number of $10 price reductions.* We chose this x because from it we could easily calculate both the new price and the new quantity. Other choices for x are also possible, but in situations where a price change will make one quantity rise and another fall, it is often easiest to choose x to be the *number of such changes*.

※ 例 3. *MAXIMIZING PROFIT*

An orange grower finds that if he plants 80 orange trees per acre, each tree will yield 60 bushels (容量單位，約 36 公升) of oranges. He estimates that for each additional tree that he plants per acre (畝), the yield of each tree will decrease by 2 bushels. How many trees should he plant per acre to maximize his harvest?



※ 例 4. *MINIMIZING PACKAGE MATERIALS*



A moving company wishes to design an open-top box with a square base whose volume is exactly 32 cubic feet. Find the dimensions of the box requiring the least amount of materials.

【註】如果底部與側邊的花費不同，則需重新計算材料花費的公式。假設底部所需的成本為$4 / feet^2，側邊所需的成本為$2 / feet^2，則總成本為 $\text{Cost} = (x^2)(4) + (4)(xy)(2)$。

【Topic 1. 最大化稅收】Maximizing Tax Revenue

政府經由收稅來獲得金錢。

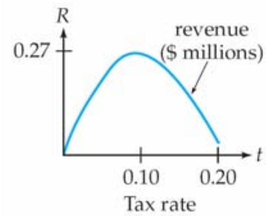
(1) 如果銷售稅太高，則交易會被抑制，稅收會因此減少。

(2) 如果銷售稅太低，則交易會非常興盛，但政府的稅收仍會減少。

因此經濟學家會想要定出稅率來最大化政府的稅收，如例 5 所示。

※ 例 5. *MAXIMIZING TAX REVENUE*

Economists estimate that the relationship between the tax rate t on an item and the total sales S of that item (in millions of dollars) is $S(t) = 9 - 20\sqrt{t}$, for $0 \le t \le 0.20$. Find the tax rate that maximizes revenue to the government. [Hint: tax revenue $R(t) = t \cdot S(t)$]



Section 3.5 Optimizing lot size and harvest size 最佳化訂單數量與最大可承受量

本節討論兩個重要的最佳化應用，第一個為經濟的問題，第二個為生態的問題。如何找到最佳的訂貨量 (order size r lot size) 可使得庫存成本 (inventory cost) 最低；同時討論人類在補獵動物時，要如何保持原有物種的族群數量。上列兩種問題是彼此獨立的。

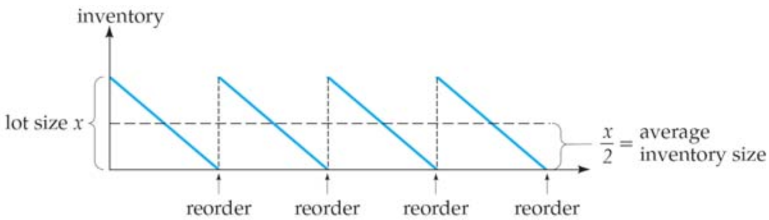
【Topic 1. 最小化庫存花費】Minimizing Inventory Costs

1. 商業行為會遭遇兩種維持存貨的花費(cost)，第一種為庫存花費 (**storage costs**)，係指對未銷售物品所需的倉儲與保險之費用；第二種為重訂花費 (**reorder costs**)，指的即為重新訂貨所需要的人力簿記與運送的費用。
2. 舉例而言，一個傢俱店每年銷售 250 組沙發，可以一次訂購所有沙發 (庫存花費最大)，或讓訂單量小而次數多。例如一張訂單 50 組沙發，一年之內平均訂購五次 (庫存花費降低，但重訂花費提高)。因此讓總庫存花費與重訂花費的總和最小化即為最佳化訂單數量問題。

※ 例 1. *MINIMIZING PACKAGE MATERIALS*

A furniture (傢俱) showroom expects to sell 250 sofas a year. Each sofa costs the store \$300, and there is a fixed charge of \$500 per order. If it costs \$100 to store a sofa for a year, how large should each order be and how often should orders be placed to minimize inventory costs?

<sol>： (1) Assume x = lot size, (2) storage costs = storage per item × average number of items, (3) reorder costs = cost per order × number of orders, and (4) $C(x)$ = storage costs + reorder costs.



※ 例 2. *MINIMIZING INVENTORY COSTS FOR A PUBLISHER* (解題方法與例 1 完全相同)

A publisher estimates the annual demand for a book to be 4000 copies. Each book costs \$8 to print, and setup costs are \$1000 for each printing. If storage costs are \$2 per book per year, find how many books should be printed per run and how many printings will be needed if costs are to be minimized.

【Topic 2. 最大可承受量】Maximum Sustainable Yield

1. 動物產出下一代，當出生率大於死亡率，族群個體總數就會增加；反之當死亡率大於出生率，則族群個體總數就會減少。在有限的空間與資源的情況下，當族群大到一個程度即達到飽和。因此適時的獵補 (harvest) 能讓族群個體總數變小，每個個體能獲得的食物與資源就相對增加；但大量的獵補可能造成某個物種滅絕。

2. 我們想要找出**最大可承受量(maximum sustainable yield)**，這個最大數量指的就是，如果每年都獵補這個數量，隔年物種就會恢復成去年的數量水準。因此對某些動物而言，可以決定他們的繁殖函數 (**reproduction function**) $f(p)$，這個函數表示，若現在的個體總數 (population) 為 p,一年之後的個體總數為 $f(p)$。

Reproduction Function

A reproduction function $f(p)$ gives the population a year from now if the current population is p.

3. 假設我們有一個繁殖函數 f 與目前的個體總數 p, 隔年個體總數會變成 $f(p)$，因此在這一年間，<u>族群成長數量</u> (*amount of growth*) 可表示成 $f(p) - p$。因此每年獵補此固定成長數量，就不會改變動物原本的數量 p，因此我們稱這個數量 $Y(p) = f(p) - p$ 為 **可承受量 (sustainable yield)** $Y(p)$。

$$\begin{pmatrix}\text{Amount}\\ \text{of growth}\end{pmatrix} = f(p) - p$$

($f(p)$: Next year's population; p: Current population)

Sustainable Yield

For reproduction function $f(p)$, the sustainable yield is

$$Y(p) = f(p) - p$$

4. 為了要最大化可承受量函數 $Y(p)$，可對上述方程式做一次微分並設 $Y'(p)$ 為 0 可得 $Y'(p) = f'(p) - 1 = 0$ 或 $f'(p) = 1$. 一但我們計算出這個數量 $Y(p)$ ，我們就可以年復一年的等待原始個體總數 p 變成 $f(p)$，並且狩獵 $Y(p)$ 數量，讓個體總數回復到 p。

Maximum Sustainable Yield

For reproduction function $f(p)$, the population p that results in the maximum sustainable yield is the solution to

$$f'(p) = 1$$

(provided that $f''(p) < 0$). The maximum sustainable yield is then

$$Y(p) = f(p) - p$$

※ 例 3. *FINDING MAXIMUM SUSTAINABLE YIELD*

The reproduction function for the American lobster in an East Coast fishing area is $f(p) = -0.02p^2 + 2p$ (where p and $f(p)$ are in thousands). Find the population p that gives the maximum sustainable yield and find the size of the yield.

Section 3.6 Implicit differentiation and related rates 隱微分與相關率

【Topic 1. 隱函數與顯函數】Implicit & Explicit functions

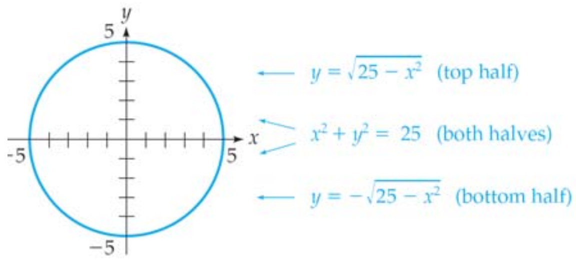
1. 隱函數(Implicit & Explicit functions) & 顯函數 (Implicit & Explicit functions) 的差異
   - 顯函數 : y 可用純 x 函數表示，即 $y = f(x)$。
   - 隱函數 : y 難以用純 x 函數表示，改用**關係方程式** (relation equation) 表示，例如 $f(x, y) = c$。
2. 隱函數範例：
   - 例 1. $xy = 1$
   - 例 2. $x^2+y^2 = 25$
   - 例 3. $x^2 - 2y^3 + 4y = 0.$

【Topic 2. 隱微分】Implicit Differentiation

1. 將隱函數轉換成顯函數：

方程式 $x^2 + y^2 = 25$ 定義出一個圓。然而此圓方程式不是一個 x 的函數*註 1，因此我們將此圓形拆成上下兩部份，則可分別表示成不同的函數。要找出這兩個函數，我們可以對 y 解關係方程式，可得

$$y^2 = 25 - x^2 \rightarrow y = \pm\sqrt{25 - x^2}$$



正的平方根代表是圓的上半部，負的平方根代表是圓的下半部。$x^2 + y^2 = 25$ 這個關係方程式同時定義了上述的兩個方程式。

註 1. 對 $y = f(x)$ 而言，唯一的 x 值要對應到唯一的 y 值，也就是說一個 x 值對應到一個 y 值 (one-to-one mapping 例如 $y = x$)，或多個 x 值對應到一個 y 值 (many-to-one mapping 例如 $y = x^2$ 或是 $y = 3$)。若有一個 x 值對應到兩個或兩個以上的 y (one-to-many mapping，如上例)，則不構成函數 (但仍可構成關係方程式)。

2. 對隱函數微分：

為了要找出任何一個 x 值在圓上的斜率，我們可以對上半圓或下半圓各別微分。但其實有更簡單的方法來對隱函數 (implicit equation) 微分，我們可以對方程式的兩邊對 x 做微分得：

$$x^2 + y^2 = 25 \rightarrow \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25) \rightarrow 2x + 2y\frac{dy}{dx} = 0 \rightarrow 2y\frac{dy}{dx} = -2x$$

$\rightarrow \dfrac{dy}{dx} = -\dfrac{x}{y}$，此為例題 1 的解答。

重點是使用「一般化指數規則」(連鎖規則, 第 2.6 節)：

$\dfrac{d}{dx}y^n = n \cdot y^{n-1} \cdot \dfrac{dy}{dx}$ << 非常重要，不要忘記。

3. 隱微分法 (Implicit differentiation) 包含下列三步驟：

Finding $\frac{dy}{dx}$ by Implicit Differentiation

1. Differentiate both sides of the equation *with respect to x*. When differentiating a y, include $\frac{dy}{dx}$.
2. Collect all terms involving $\frac{dy}{dx}$ on one side, and all others on the other side.
3. Factor out the $\frac{dy}{dx}$ and solve for it by dividing.

1. 將關係方程式 $f(x, y) = c$ 左右兩邊同時對 x 微分。
2. 將含有 dy/dx 的所有項都移到式子左邊，而把其他項都移到式子右邊
3. 左邊提出 dy/dx。再把式子左邊不含 dy/dx 的項除到右邊，解出 dy/dx。

※ 例 1 與 例 2. *DIFFERENTIATING IMPLICITLY*

For $x^2 + y^2 = 25$ **a.** find dy / dx **b.** find the slope at the points (3, 4) and (3, – 4).

※ 例 3. *FINDING DERIVATIVES – Implicit and explicit*

a. $\frac{d}{dx}x^3$ **b.** $\frac{d}{dx}y^3$ **c.** $\frac{d}{dx}(x^3y^5)$

※ 例 4. *FINDING AND EVALUATING AN IMPLICIT DERIVATIVE*

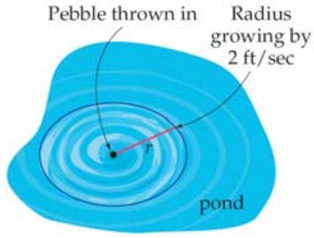
For $y^4 + x^4 - 2x^2y^2 = 9$ **a.** find dy / dx **b.** evaluate it at $x = 2, y = 1$.

【Topic 2. 相關變率】Related Rates

有時候方程式中的兩種變數，可以各別表示成第三種變數的函數，通常第三種變數為時間。舉例說明，如果一個季節性的產品，如冬天的外套，則它的需求方程式 (demand equation) 與價格函數 p 和銷售數量函數 x 相關，且這兩種函數將分別與時間函數 t 有關，其中 t 代表一年之中的季節 (time of year)。我們可以對需求方程式的等式兩邊對 t 微分，此方程式將與導函數 dp/dt 和 dx/dt 有關，我們稱為相關變率 (related rates)。

※ 例 6. *FINDING RELATED RATES*

A pebble (圓石) thrown into a pond (池塘) causes circular ripples (漣漪) to radiate outward (向外幅射). If the radius of the outer ripple is growing by 2 feet per second, how fast is the area of its circle growing at the moment when the radius is 10 feet?



To Solve a Related Rate Problem

1. Determine the quantities that are changing with time.
2. Find an equation that relates these quantities (a diagram may be helpful).
3. Differentiate both sides of this equation implicitly with respect to t.
4. Substitute into the new equation any given values for the variables and for the derivatives (interpreted as rates of change).
5. Solve for the remaining derivative and interpret the answer as a rate of change.

1. 先確定已知的量與待決定的量，將那些(變數)數量會隨著時間改變標示出。
2. 找出一個方程式跟這些量有相關，即找出關係方程式。
3. 對關係方程式的兩邊對時間 t 做隱微分
4. 將已知的變數值或者變數的導數，代入新的(隱微分後的)方程式之中，解出剩餘的變率。